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# Comments on "New Inequalities Linking the First and Second Zagreb Indices"

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**Abstract**

We comment on four inequalities presented by Mehiri in [MATCH Commun. Math. Comput. Chem. 97 (2027) 1139–1146]. The spectral radius inequality is an earlier result of Zhou, while the minimum degree inequality was already obtained by Réti. The claimed equality characterization for the universal inequality is incorrect, and the maximum degree inequality has its inequality sign reversed and is false as stated. We give the relevant references, corrected statements, and a small counterexample.

## 1 Introduction

Let $G = (V, E)$ be a finite simple graph, let $m = |E|$, and let $d(v)$ denote the degree of $v$. The first and the second Zagreb indices are defined as

$$M_1(G) = \sum_{v \in V} d(v)^2, \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E} d(u)d(v).$$

Let $\delta$, $\Delta$, and $\rho(G)$ denote the minimum degree, maximum degree, and adjacency spectral radius of $G$, respectively. Mehiri [2] presents four in-

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equalities relating these quantities. We record below two prior occurrences and two problems with the stated results.

## 2 The spectral radius inequality

Theorem 1 of [2] states

$$M_2(G) \leq \frac{\rho(G)}{2} M_1(G), \tag{1}$$

with equality, for connected $G$, precisely when the degree vector $d$ is a Perron eigenvector. Equivalently,

$$\frac{2M_2(G)}{M_1(G)} \leq \rho(G).$$

This inequality is not new, as it appears in Zhou's paper [5, Theorem 8]; it is also listed explicitly among previously known spectral radius estimates in [1], where it is attributed to Zhou.

There is, moreover, an internal contradiction in the equality discussion of [2, Theorem 2]. The theorem statement correctly allows connected irregular graphs satisfying equality, but the proof later claims that $Ad = \rho(G)d$ forces regularity. This is false. Graphs satisfying $Ad = \rho(G)d$ are usually called harmonic (or pseudo-regular, under a common convention), and connected irregular examples are well known; see, for example, [4]. The smallest standard example is the tree obtained from $K_{1,3}$ by subdividing each of its three edges. Its degree vector $d$ satisfies $Ad = 2d$, although the graph is not regular. Thus the Rayleigh-quotient proof of (1) is valid, but the final assertion that equality forces regularity is incorrect.

## 3 The other three inequalities

Theorem 2 of [2] states

$$M_2(G) \leq \frac{M_1(G)^2}{4}. \tag{2}$$

The displayed proof indeed establishes (2), but the equality characterization is incorrect. Setting $a_{uv} = \sqrt{d(u)d(v)}$, the first step is

$$\sum_{uv \in E} a_{uv}^2 \leq \left( \sum_{uv \in E} a_{uv} \right)^2 .$$

For nonnegative $a_{uv}$, equality here holds if and only if at most one of them is nonzero. Every edge of a simple graph contributes a positive $a_{uv}$, so if $G$ has at least two edges the inequality is strict already at this step. Consequently, equality in (2) holds exactly for graphs with at most one edge (including the edgeless graph), not for general regular graphs or disjoint unions of regular components of the same degree as claimed in [2].

Theorem 3 of [2] states

$$M_2(G) \geq \delta M_1(G) - m\delta^2. \tag{3}$$

For connected graphs this is already contained in Proposition 2 of Réti [3]. The extension to arbitrary simple graphs is immediate from the same edgewise argument and does not constitute a substantial new result.

Finally, Theorem 4 of [2] claims

$$M_2(G) \leq \Delta M_1(G) - m\Delta^2. \tag{4}$$

The sign is reversed. Indeed,

$$(\Delta - d(u))(\Delta - d(v)) \geq 0$$

implies

$$d(u)d(v) \geq \Delta\big(d(u) + d(v)\big) - \Delta^2,$$

and summing over all edges gives

$$M_2(G) \geq \Delta M_1(G) - m\Delta^2. \tag{5}$$

This lower bound, together with (3), is again part of Réti's Proposition 2 [3]. A four-vertex counterexample to (4) is obtained by attaching one

pendant vertex to a triangle. Its degree sequence is $(3, 2, 2, 1)$, so $m = 4$, $\Delta = 3$, $M_1 = 18$, and $M_2 = 19$, whereas

$$\Delta M_1 - m\Delta^2 = 3 \cdot 18 - 4 \cdot 9 = 18.$$

Thus (4) would require $19 \leq 18$.

# 4 Conclusion

Of the four main inequalities in [2], the spectral radius inequality was published by Zhou more than two decades earlier, the minimum degree inequality was published by Réti in 2012, the universal inequality has an incorrect equality characterization, and the maximum degree inequality is false as stated because its sign is reversed; the corrected inequality is likewise already present in Réti's work. Even if the elementary bound (2) has not previously been recorded in precisely that form, the paper in its published form therefore does not establish the claimed collection of new inequalities.

The mathematical checks, literature search, and most of the drafting of this commentary were carried out with substantial assistance from OpenAI's ChatGPT (GPT-5.6 Sol), with the final claims verified by the human author against the cited sources. The present example illustrates that current frontier large language models are already sufficiently capable to flag errors and prior overlap of this kind—including reversed inequality signs, incorrect equality cases, small counterexamples, and readily identifiable earlier results. We therefore recommend that authors use a strong LLM as an additional pre-submission audit of proofs and literature coverage before submitting a manuscript. Such a check is not a substitute for the author's own verification or for expert peer review, but it can already serve as an effective extra filter for issues of the type documented here.

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