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Sombor, Mostar, and Zagreb Indices: Corrections and New Bounds

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Abstract

We examine several results in a recent paper by F. Asmat, show that some of the published results are incorrect, and provide corrected formulations. We also establish several new bounds that refine known estimates and give relationships among the first Zagreb index, the Sombor index, and the normalized Laplacian eigenvalues. These results provide corrected and refined bounds for the indices considered.

1 Introduction

Let $G = (V, E)$ be a finite simple graph of order $n = |V|$ and size $m = |E|$. For $u \in V(G)$, let d_u denote the degree of u. The Sombor index and the

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first Zagreb index [4, 5] are

$$\mathrm{SO}(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}, \qquad M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{u \in V(G)} d_u^2.$$

For $uv \in E(G)$, let

$$n_G(u) = \big|\{w \in V(G) : d_G(w,u) < d_G(w,v)\}\big|,$$

and define the Mostar index [1, 3] by

$$\mathrm{Mo}(G) = \sum_{uv \in E(G)} |n_G(u) - n_G(v)|.$$

We use $\mathrm{Mo}(G)$ for the Mostar index below.

The normalized Laplacian is [6]

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2},$$

with eigenvalues

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

The paper [2] studies extremal Sombor bounds with prescribed chromatic number, upper bounds for the Mostar index under conditions on cut edges and girth, bounds for the first Zagreb index, and relations between these degree-based indices and the normalized Laplacian.

The purpose of this paper is to check these assertions at the level of the precise statements and proofs. We do not treat a proof gap as a counterexample, and we do not replace an unproved assertion by a conclusion that is stronger than the evidence permits. This distinction is important for the Mostar-index arguments, because a local edge transformation may change distance partitions for edges outside the displayed local configuration.

After the corrections, we develop five new estimates. The first starts from the edge quantity $s_{uv} = d_u + d_v$ and retains the variance of these quantities (Theorem 1). The second is a direct upper bound for λ_2 in terms of m and δ (Theorem 2). The third refines the corrected lower bound for

$M_1(G)$ in terms of λ_n (Theorem 3). The last two start from the identity

$$d_u^2 + d_v^2 = \frac{(d_u + d_v)^2}{2} + \frac{(d_u - d_v)^2}{2}$$

and give a strict quantitative improvement over the usual $\mathrm{SO}(G) \geq M_1(G)/\sqrt{2}$ bound whenever the graph is not regular (Theorems 4 and 5).

2 Preliminaries

Throughout, graphs are finite, simple, and connected unless explicitly stated otherwise. We write $\delta = \delta(G)$ and $\Delta = \Delta(G)$ for the minimum and maximum degrees.

For each edge $e = uv$, put $s_e = s_{uv} = d_u + d_v$, $\overline{s} = \frac{1}{m}\sum_{e \in E(G)} s_e = \frac{M_1(G)}{m}$, and define the edge-degree variance by

$$V_E(G) = \frac{1}{m} \sum_{e \in E(G)} (s_e - \overline{s})^2.$$

We shall repeatedly use the following elementary identity.

Lemma 1. *For positive x and a,*

$$\frac{1}{x} = \frac{1}{a} - \frac{x-a}{a^2} + \frac{(x-a)^2}{xa^2}.$$

Consequently, if a is the arithmetic mean of positive numbers $x_1, \ldots, x_m$, then

$$\sum_{i=1}^{m} \frac{1}{x_i} = \frac{m}{a} + \sum_{i=1}^{m} \frac{(x_i - a)^2}{x_i a^2}.$$

Proof. The first identity follows by putting the right-hand side over the common denominator xa^2. Summing over i and using $\sum_i (x_i - a) = 0$ gives the second identity. ∎

We also record the standard Rayleigh-quotient fact needed below. If z is orthogonal to $D^{1/2}\mathbf{1}$, then

$$\lambda_2 \leq \frac{z^T \mathcal{L} z}{z^T z} \leq \lambda_n.$$

3 Corrections to published results

We follow the order of the relevant results in [2]. In each case we state precisely what is wrong and, whenever possible, give a corrected statement immediately.

3.1 Theorem 3: the chromatic-number argument

Theorem 3 of [2] gives a lower bound for the Sombor index in the class of graphs with prescribed chromatic number and characterizes an extremal graph obtained from a complete graph by attaching a path. The difficulty is in the proof.

The proof uses the hypothesis $\chi(G) = k$ and subsequently works with a K_k-based structure. However, $\chi(G) = k$ does not imply that G contains a clique of order k. The smallest standard example is $\chi(C_5) = 3, \qquad \omega(C_5) = 2$. Thus the passage from chromatic number to the asserted complete-subgraph structure is not justified.

Corrected statement. The displayed extremal inequality of Theorem 3 is not disproved here; rather, its published proof is incomplete. A valid proof must establish, from $\chi(G) = k$, the additional structural properties of an extremal graph that are used later in the argument. In particular, the existence of a K_k subgraph cannot be assumed merely from the equality $\chi(G) = k$.

3.2 Lemmas 6–8: a false lemma and unsupported edge-shifting arguments

Lemma 6 of [2] states that if uv is a non-pendant cut edge, then

$$\mathrm{Mo}(G) < \mathrm{Mo}(G - uv).$$

Since $G - uv$ is disconnected, the statement requires an extension of the definition of the Mostar index to disconnected graphs. We use the verbatim extension: $n_G(u)$ counts the vertices w with $d_G(w, u) < d_G(w, v)$, so a vertex at infinite distance from both ends of an edge is counted for neither.

The statement is false. Consider $G = P_4 = v_1v_2v_3v_4$ and let $uv = v_2v_3$. Then uv is a non-pendant cut edge. A direct calculation gives

$$\mathrm{Mo}(P_4) = 2 + 0 + 2 = 4,$$

whereas

$$P_4 - v_2v_3 = K_2 \cup K_2, \qquad \mathrm{Mo}(P_4 - v_2v_3) = 0.$$

Hence

$$\mathrm{Mo}(G) > \mathrm{Mo}(G - uv),$$

contradicting Lemma 6. Thus Lemma 6 is false.

Corrected statement. Lemma 6 must be deleted or replaced by a valid global monotonicity statement. In particular, the example above shows that deleting a non-pendant cut edge need not increase the Mostar index.

The proofs of Lemmas 7 and 8 contain a different problem. They perform an edge-shifting operation on a C_4 or C_3 configuration and compare the contributions of the edges displayed in the local configuration. However, the Mostar index is defined through global shortest-path distance comparisons. Changing an edge can therefore change $n_G(x)$ and $n_G(y)$ for edges outside the displayed cycle.

Corrected statement. The local calculations in Lemmas 7 and 8 do not by themselves establish the claimed global inequalities. A valid edge-shifting lemma must explicitly prove that all contributions outside the modified configuration are unchanged or that their total change has a controlled sign. No such argument is supplied in [2].

3.3 Theorem 4: Mostar index with a prescribed number of cut edges

Theorem 4 of [2] claims, for a graph with n vertices and k cut edges,

$$\mathrm{Mo}(G) \leq (n-k-2)(n-k-3) + (n-k-2)(n-3) + k(n-2),$$

with equality if and only if $G = S_n^k$.

The statement is false. Consider $n = 4$ and $k = 1$. The graph S_4^1 is a triangle with one pendant vertex attached to a triangle vertex. The pendant edge contributes 2 to the Mostar index, the two triangle edges incident with the attachment vertex contribute 1 each, and the remaining triangle edge contributes 0. Hence $\mathrm{Mo}(S_4^1) = 4$. The proposed upper bound equals

$$(4-1-2)(4-1-3) + (4-1-2)(4-3) + 1(4-2) = 3.$$

Thus

$$\mathrm{Mo}(S_4^1) = 4 > 3.$$

Corrected result. Theorem 4 cannot be repaired by changing only its equality characterization; the numerical bound itself is false. A universally valid elementary replacement is

$$\mathrm{Mo}(G) \leq m(n-2),$$

because each edge contributes at most $n-2$ to the Mostar index. Any sharper bound depending simultaneously on n and the number of cut edges requires a separate extremal argument.

3.4 Theorem 5: Mostar index with girth three

Theorem 5 of [2] states

$$\mathrm{Mo}(G) \leq 2(n-2)(n-3)$$

for graphs of girth 3, with equality for H_n^3. We do not give a counterexample to the numerical inequality here. The proof, however, consists only of the sentence "By Lemma 7 and 8, the proof is obvious." Since the proofs of Lemmas 7 and 8 do not establish the required global monotonicity, Theorem 5 has not been proved.

Corrected status. The numerical assertion of Theorem 5 should be regarded as unproved by the argument given in [2]. A complete proof must

replace the unsupported appeal to Lemmas 7 and 8 by a global argument controlling all changes in the Mostar index.

3.5 Theorem 6: equality characterization for fixed girth

For $k > 4$, Theorem 6 of [2] claims

$$\mathrm{Mo}(G) \leq k(k-1) + n(n-1),$$

with equality if and only if $G = C_n^{*k}$.

The equality characterization is false already in the smallest allowed case. Take $n = k = 5$. Then $C_n^{*k} = C_5$. The cycle C_5 is distance-balanced, and hence $\mathrm{Mo}(C_5) = 0$, whereas $k(k-1)+n(n-1) = 40$. Thus the claimed equality cannot hold.

Corrected result. The equality characterization in Theorem 6 must be deleted or replaced by a characterization valid only after the relevant boundary cases have been separated. In particular, when $n = k$, the only graph in the class is C_k, and $\mathrm{Mo}(C_k) = 0$ for every $k \geq 3$. Therefore C_k cannot be an equality graph for the positive right-hand side appearing in Theorem 6.

There is also a proof problem: the induction uses Lemma 6, which is false. Thus the published proof does not establish the theorem even apart from the false equality characterization.

3.6 Theorem 7: first Zagreb index with prescribed girth

Theorem 7 of [2] claims

$$M_1(G) \leq 4(k-2) + 4(n-k-2) + 18$$

for a connected graph of order n and girth k.

Take $G = K_4$. Then

$$n = 4, \qquad k = 3, \qquad d_u = 3 \quad \text{for all } u,$$

and hence

$$M_1(K_4) = 4 \cdot 3^2 = 36.$$

The claimed right-hand side is

$$4(3-2) + 4(4-3-2) + 18 = 18.$$

Consequently,

$$36 > 18,$$

and Theorem 7 is false.

Corrected result. A valid general estimate, independent of girth, is

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) \leq 2m\Delta.$$

Equality holds precisely when every edge has both endpoints of degree Δ; for a connected graph this is equivalent to regularity. Thus any genuine girth-sensitive upper bound must be derived by a different extremal argument.

3.7 Theorem 8 and the normalized-Laplacian test vector

Theorem 8 of [2] asserts

$$\lambda_2 \leq 1 + \frac{2}{m} \sum_{uv \in E(G)} \frac{1}{d_u + d_v} \leq \lambda_n.$$

The displayed inequality is in fact valid, but the test vector used in the proof is incorrect. The paper takes, for an edge ij,

$$y_i = \sqrt{d_i}, \qquad y_j = -\sqrt{d_j},$$

with all other coordinates zero. The zero-eigenvector of the normalized Laplacian is proportional to

$$D^{1/2}\mathbf{1} = (\sqrt{d_1}, \ldots, \sqrt{d_n})^T.$$

Therefore

$$\langle y, D^{1/2}\mathbf{1}\rangle = d_i - d_j,$$

which is not zero in general.

The correct edge-supported vector is

$$z_i^{ij} = \sqrt{d_j}, \qquad z_j^{ij} = -\sqrt{d_i}, \qquad z_k^{ij} = 0 \quad (k \neq i, j).$$

It satisfies

$$\langle z^{ij}, D^{1/2}\mathbf{1}\rangle = 0,$$

and direct calculation gives

$$\frac{(z^{ij})^T \mathcal{L} z^{ij}}{(z^{ij})^T z^{ij}} = 1 + \frac{2}{d_i + d_j}.$$

Corrected theorem. For every connected graph,

$$\lambda_2 \leq 1 + \frac{2}{m} \sum_{uv \in E(G)} \frac{1}{d_u + d_v} \leq \lambda_n. \tag{1}$$

Indeed, the corrected vector above is orthogonal to the kernel eigenvector, so its Rayleigh quotient lies between λ_2 and λ_n for every edge. Averaging over all edges gives the result.

A second algebraic error occurs in the paper when the reciprocal sum is identified with $1/M_1(G)$. In general,

$$\sum_{uv \in E(G)} \frac{1}{d_u + d_v} \neq \frac{1}{M_1(G)}.$$

For example, for K_3,

$$\sum_{uv \in E(K_3)} \frac{1}{d_u + d_v} = \frac{3}{4}, \qquad \frac{1}{M_1(K_3)} = \frac{1}{12}.$$

Thus the Zagreb index cannot be substituted for the reciprocal edge sum in that way.

3.8 Theorem 9: first Zagreb index and normalized Laplacian

Theorem 9 of [2] claims

$$\frac{2m}{\lambda_2 - 1} \leq M_1(G) \leq \Delta m(\lambda_n - 1).$$

The statement is false as written.

First, for P_3 the normalized-Laplacian eigenvalues are $0, 1, 2$, so $\lambda_2 - 1 = 0$ and the left-hand expression is undefined.

Second, for K_4,

$$M_1(K_4) = 36, \qquad \Delta = 3, \qquad m = 6, \qquad \lambda_n = \frac{4}{3},$$

and hence

$$\Delta m(\lambda_n - 1) = 6 < 36.$$

Thus the right-hand inequality is false.

Corrected result. The valid spectral inequality from Theorem 8 yields, for every connected graph with $\lambda_n > 1$,

$$M_1(G) \geq \frac{2m}{\lambda_n - 1}.$$

Indeed, putting

$$S(G) = \sum_{uv \in E(G)} \frac{1}{d_u + d_v},$$

Theorem 8 gives

$$S(G) \leq \frac{m}{2}(\lambda_n - 1),$$

while Cauchy–Schwarz gives

$$M_1(G)\, S(G) \geq m^2.$$

Combining these inequalities proves the claim. Complete graphs attain equality.

4 Sharper results

The preceding corrections show that several arguments in [2] lose essential information by replacing the reciprocal edge-degree sum

$$S(G) = \sum_{uv \in E(G)} \frac{1}{d_u + d_v}$$

too early by a function of $M_1(G)$.

Now, we refine the corrected normalized-Laplacian bound from Theorem 8 of [2] by retaining the variance of the edge degree sums.

Theorem 1. *Let G be a connected graph with $m \geq 1$. Then*

$$\lambda_2 \leq 1 + \frac{2m}{M_1(G)} + \frac{m^2 V_E(G)}{\delta\, M_1(G)^2},$$

and

$$\lambda_n \geq 1 + \frac{2m}{M_1(G)} + \frac{m^2 V_E(G)}{\Delta\, M_1(G)^2}.$$

In particular, if $V_E(G) > 0$, then

$$\lambda_n > 1 + \frac{2m}{M_1(G)}.$$

Proof. For $s_e = d_u + d_v$, Lemma 1 gives the exact identity

$$S(G) = \frac{m}{\overline{s}} + \sum_{e \in E(G)} \frac{(s_e - \overline{s})^2}{s_e \overline{s}^{\,2}}.$$

Since

$$2\delta \leq s_e \leq 2\Delta,$$

we obtain

$$\frac{mV_E(G)}{2\Delta\overline{s}^{\,2}} \le \sum_e \frac{(s_e-\overline{s})^2}{s_e\overline{s}^{\,2}} \le \frac{mV_E(G)}{2\delta\overline{s}^{\,2}}.$$

Consequently,

$$\frac{m}{\overline{s}} + \frac{mV_E(G)}{2\Delta\overline{s}^{\,2}} \le S(G) \le \frac{m}{\overline{s}} + \frac{mV_E(G)}{2\delta\overline{s}^{\,2}}.$$

By the corrected form of Theorem 8 of [2],

$$\lambda_2 \le 1 + \frac{2S(G)}{m} \le \lambda_n.$$

Using $\overline{s} = M_1(G)/m$ gives the two asserted inequalities. ■

Remark. The lower bound for λ_n in Theorem 1 refines the bound $\lambda_n \ge 1 + 2m/M_1(G)$ obtained from the corrected Theorem 8 of [2] and the Cauchy–Schwarz inequality. The upper bound for λ_2 is an explicit estimate, in terms of $M_1(G)$, $V_E(G)$ and δ, of the quantity $1+2S(G)/m$; it is implied by the corrected Theorem 8 and is not claimed to be an improvement (see Theorem 2 for a stronger upper bound).

We next derive a direct upper bound for $\lambda_2(G)$ in terms of m and δ, which improves the corresponding estimate in Theorem 8 of [2] without requiring the first Zagreb index.

Theorem 2. *Let G be a connected graph with n vertices, m edges, and minimum degree δ. Then*

$$\lambda_2(G) \le 1 + \frac{\delta}{2m-\delta}.$$

Moreover,

$$1 + \frac{\delta}{2m-\delta} \le 1 + \frac{2m}{M_1(G)} \le 1 + \frac{2}{m}\sum_{uv\in E(G)} \frac{1}{d_u+d_v}.$$

Consequently, the above bound is no weaker than the upper bound for $\lambda_2(G)$ given in Theorem 8 of [2].

Proof. Let

$$\mathcal{A} = D^{-1/2} A D^{-1/2} = I - \mathcal{L},$$

where $\mathcal{L}$ is the normalized Laplacian matrix of G. Let

$$1 = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

be the eigenvalues of $\mathcal{A}$. Since $\lambda_i = 1 - \mu_i$, we have $\lambda_2 = 1 - \mu_2$. Put

$$\mathbf{p} = D^{1/2}\mathbf{1} = (\sqrt{d_1}, \ldots, \sqrt{d_n})^T.$$

Then $\mathcal{A}\mathbf{p} = \mathbf{p}$ and $\mathbf{p}^T\mathbf{p} = \sum_{i=1}^{n} d_i = 2m$. Choose a vertex i with $d_i = \delta$, and define

$$x = e_i - \frac{\sqrt{\delta}}{2m}\mathbf{p}.$$

Since $e_i^T \mathbf{p} = \sqrt{\delta}$, we have

$$x^T \mathbf{p} = \sqrt{\delta} - \frac{\sqrt{\delta}}{2m}(2m) = 0.$$

Hence, by the variational characterization of μ_2,

$$\mu_2 \geq \frac{x^T \mathcal{A} x}{x^T x}.$$

We first compute

$$x^T x = 1 - \frac{\delta}{m} + \frac{\delta}{4m^2}\mathbf{p}^T\mathbf{p} = 1 - \frac{\delta}{2m}.$$

On the other hand, since

$$e_i^T \mathcal{A} e_i = 0, \qquad e_i^T \mathcal{A}\mathbf{p} = \sqrt{\delta}, \qquad \mathbf{p}^T \mathcal{A}\mathbf{p} = 2m,$$

we obtain

$$x^T \mathcal{A} x = -2 \cdot \frac{\sqrt{\delta}}{2m}\sqrt{\delta} + \frac{\delta}{4m^2}(2m) = -\frac{\delta}{m} + \frac{\delta}{2m} = -\frac{\delta}{2m}.$$

Therefore,

$$\mu_2 \geq \frac{-\delta/(2m)}{1-\delta/(2m)} = -\frac{\delta}{2m-\delta}.$$

Consequently,

$$\lambda_2 = 1 - \mu_2 \leq 1 + \frac{\delta}{2m-\delta}.$$

We now compare this estimate with the bound in Theorem 8 of [2]. Since

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{u \in V(G)} d_u^2,$$

we have $M_1(G) \leq 2m\Delta$. Let w be a vertex of maximum degree Δ. Its Δ neighbors all have degree at least δ, and hence

$$2m = \sum_{u \in V(G)} d_u \geq \Delta + \delta\Delta = \Delta(\delta + 1).$$

Thus $\Delta \leq 2m/(\delta+1)$. Moreover, since G contains at least $\delta + 1$ vertices and every vertex has degree at least δ,

$$2m \geq \delta(\delta + 1).$$

Consequently,

$$\frac{2m}{\delta+1} \leq \frac{2m-\delta}{\delta},$$

and therefore

$$M_1(G) \leq 2m\Delta \leq \frac{2m(2m-\delta)}{\delta}.$$

It follows that

$$\frac{\delta}{2m-\delta} \leq \frac{2m}{M_1(G)}.$$

Finally, by the Cauchy–Schwarz inequality,

$$\left(\sum_{uv \in E(G)} \frac{1}{d_u + d_v} \right) \left(\sum_{uv \in E(G)} (d_u + d_v) \right) \geq m^2.$$

Hence

$$\sum_{uv\in E(G)} \frac{1}{d_u + d_v} \geq \frac{m^2}{M_1(G)},$$

which gives

$$\frac{2m}{M_1(G)} \leq \frac{2}{m} \sum_{uv\in E(G)} \frac{1}{d_u + d_v}.$$

Combining the above inequalities yields

$$1 + \frac{\delta}{2m - \delta} \leq 1 + \frac{2m}{M_1(G)} \leq 1 + \frac{2}{m} \sum_{uv\in E(G)} \frac{1}{d_u + d_v}.$$

This completes the proof. ■

The next theorem strengthens the corrected M_1-bound replacing Theorem 9 of [2] by adding the edge-degree variance term.

Theorem 3. *Let G be connected and let $\lambda_n > 1$. Then*

$$M_1(G) \geq \frac{m}{\lambda_n - 1} \left(1 + \sqrt{1 + \frac{(\lambda_n - 1)V_E(G)}{\Delta}} \right).$$

Hence

$$M_1(G) \geq \frac{2m}{\lambda_n - 1},$$

and the new inequality is strictly stronger whenever $V_E(G) > 0$.

Proof. Theorem 1 gives

$$\lambda_n - 1 \geq \frac{2m}{M_1(G)} + \frac{m^2 V_E(G)}{\Delta M_1(G)^2}.$$

Set

$$x = \frac{M_1(G)}{m}.$$

Then

$$(\lambda_n - 1)x^2 - 2x - \frac{V_E(G)}{\Delta} \geq 0.$$

Since $x > 0$, it must be at least the positive root of the quadratic equation.

Therefore

$$x \geq \frac{1+\sqrt{1+(\lambda_n-1)V_E(G)/\Delta}}{\lambda_n-1},$$

which proves the result after multiplication by m. If $V_E(G) > 0$, the square root is strictly larger than 1, proving strict improvement over the corrected bound from Theorem 9 of [2]. ■

Next, we sharpen the Zagreb–Sombor inequality used in [2] by making the correction coefficient depend on δ and Δ.

Theorem 4. *Let G have minimum degree $\delta \geq 1$ and maximum degree Δ. Define*

$$T = \left(\frac{\Delta-\delta}{\Delta+\delta}\right)^2 \text{ and } c_{\delta,\Delta} = \frac{1}{1+\sqrt{1+T}}.$$

Then

$$\mathrm{SO}(G) \geq \frac{M_1(G)}{\sqrt{2}} + \frac{c_{\delta,\Delta}}{\sqrt{2}} \sum_{uv\in E(G)} \frac{(d_u-d_v)^2}{d_u+d_v}.$$

Moreover, $c_{\delta,\Delta} > \sqrt{2}-1$ for every graph, so that in particular

$$\mathrm{SO}(G) \geq \frac{M_1(G)}{\sqrt{2}} + \left(1-\frac{1}{\sqrt{2}}\right) \sum_{uv\in E(G)} \frac{(d_u-d_v)^2}{d_u+d_v},$$

and the coefficient $c_{\delta,\Delta}$ improves this universal correction coefficient.

Proof. For each edge uv, put

$$t = \frac{(d_u-d_v)^2}{(d_u+d_v)^2}.$$

The degree restrictions imply $0 \leq t \leq T$. If $T = 0$, then G is regular, every t vanishes, and the inequality is an equality (with $c_{\delta,\Delta} = 1/2$). Let $T > 0$. Since $f(t) = \sqrt{1+t}$ is concave, its graph lies above the chord joining 0 and T:

$$\sqrt{1+t} \geq 1 + \frac{\sqrt{1+T}-1}{T}\, t = 1 + \frac{t}{1+\sqrt{1+T}}.$$

Also

$$\sqrt{d_u^2+d_v^2} = \frac{d_u+d_v}{\sqrt{2}}\sqrt{1+t}.$$

Hence

$$\sqrt{d_u^2 + d_v^2} \geq \frac{d_u + d_v}{\sqrt{2}} + \frac{c_{\delta,\Delta}}{\sqrt{2}} \frac{(d_u - d_v)^2}{d_u + d_v}.$$

Summing over the edges proves the first inequality.

Finally, since $\delta \geq 1$ we have $T < 1$, which implies

$$c_{\delta,\Delta} > \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1.$$

Because $(\sqrt{2} - 1)/\sqrt{2} = 1 - 1/\sqrt{2}$ and the sum $\sum_{uv}(d_u - d_v)^2/(d_u + d_v)$ is nonnegative, the second inequality follows. ■

Here, we strengthen the Sombor–normalized-Laplacian corollary following Theorem 8 of [2] by adding an explicit degree-imbalance term.

Theorem 5. *Let G be connected with minimum degree $\delta \geq 1$ and maximum degree Δ, and let T and $c_{\delta,\Delta}$ be as in Theorem 4. Then*

$$\lambda_n \geq 1 + \frac{\sqrt{2}\, m}{\mathrm{SO}(G)} + \frac{\sqrt{2}\, c_{\delta,\Delta}}{m} \sum_{uv \in E(G)} \frac{(d_u - d_v)^2}{(d_u + d_v)^2 \sqrt{d_u^2 + d_v^2}}.$$

Consequently,

$$\lambda_n \geq 1 + \frac{\sqrt{2}\, m}{\mathrm{SO}(G)},$$

and the strengthened inequality is strict whenever G is nonregular.

Proof. For an edge uv, write

$$q_{uv} = \sqrt{d_u^2 + d_v^2}, \qquad t_{uv} = \frac{(d_u - d_v)^2}{(d_u + d_v)^2}.$$

The identity

$$d_u + d_v = \frac{\sqrt{2}\, q_{uv}}{\sqrt{1 + t_{uv}}}$$

gives

$$\frac{1}{d_u + d_v} = \frac{\sqrt{1 + t_{uv}}}{\sqrt{2}\, q_{uv}}.$$

Since $t_{uv} \le T$, the chord estimate used in Theorem 4 yields

$$\frac{1}{d_u + d_v} \ge \frac{1}{\sqrt{2}\, q_{uv}} + \frac{c_{\delta,\Delta}}{\sqrt{2}} \frac{(d_u - d_v)^2}{(d_u + d_v)^2 q_{uv}}.$$

Summing over all edges and applying Cauchy–Schwarz,

$$\sum_{uv \in E(G)} \frac{1}{q_{uv}} \ge \frac{m^2}{\sum_{uv \in E(G)} q_{uv}} = \frac{m^2}{\mathrm{SO}(G)}.$$

Therefore

$$S(G) \ge \frac{m^2}{\sqrt{2}\ \mathrm{SO}(G)} + \frac{c_{\delta,\Delta}}{\sqrt{2}} \sum_{uv \in E(G)} \frac{(d_u - d_v)^2}{(d_u + d_v)^2 q_{uv}}.$$

Finally, the corrected normalized-Laplacian inequality gives

$$\lambda_n \ge 1 + \frac{2S(G)}{m},$$

and substitution proves the result.

If G is nonregular, at least one edge has $d_u \ne d_v$, so the second term is positive. Hence the new estimate is strictly stronger than the previously corrected Sombor–spectral bound. ■

5 Future research

The five estimates above suggest several concrete problems.

First, determine the optimal coefficient in the variance correction

$$\lambda_n \ge 1 + \frac{2m}{M_1(G)} + c\, \frac{m^2 V_E(G)}{M_1(G)^2}.$$

Our proof gives $c = 1/\Delta$. It is natural to ask whether this can be replaced by a sharper expression involving both δ and Δ, or by a coefficient depending only on the mean edge degree.

Second, characterize all equality cases in the sharpened Sombor–spectral inequality. The proof shows that equality requires simultaneous equality

in the chord estimate, Cauchy–Schwarz, and the edge-supported Rayleigh-quotient argument. A complete characterization would connect degree classes with the top eigenspace of the normalized Laplacian.

Third, the Mostar-index part of [2] leaves a natural extremal question. One may seek edge transformations for which the change in the global distance-partition contribution can be expressed in terms of a controllable potential function. Such a transformation theory could lead to valid replacements for the disputed extremal bounds under prescribed cut-edge or girth conditions.

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