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Sharp Bounds for the Degree-Ratio Sombor Index via Spectral Radii

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Abstract

Let G be a simple connected graph of order n and size m, with $A(G)$ and $Q(G) = D(G) + A(G)$ denoting its adjacency and signless Laplacian matrices, respectively, where $D(G)$ is the diagonal matrix of vertex degrees. Let $\lambda_1(G)$ and $q(G)$ denote their spectral radii. The degree-ratio Sombor index is defined as

$$\mathrm{DRSO}(G) = \sum_{uv \in E(G)} \sqrt{\left(\frac{d_u}{d_v}\right)^2 + \left(\frac{d_v}{d_u}\right)^2}.$$

We establish sharp bounds relating $q(G)$ and $\lambda_1(G)$ to $\mathrm{DRSO}(G)$, with explicit dependence on n, m, and the maximum and minimum degrees. In particular, we prove a universal upper bound for $q(G)/\,\mathrm{DRSO}(G)$ valid for all connected graphs with $n \geq 3$, as well as several lower bounds for $\mathrm{DRSO}(G)$ in terms of both spectral radii. While several of the obtained inequalities are sharp and their extremal graphs are characterized, the universal upper bound for the ratio is shown to be strict for all connected graphs on at least three vertices, which motivates a conjecture on the maximum ratio among trees.

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1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The integers $n = |V(G)|$ and $m = |E(G)|$ are the order and size of G. The degree of a vertex v is denoted by d_v, and Δ and δ denote the maximum and minimum degrees, respectively.

The signless Laplacian matrix is $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix. Let $q(G)$ denote the spectral radius of $Q(G)$. The spectral radius of the adjacency matrix is denoted by $\lambda_1(G)$.

In mathematical chemistry, the eigenvalues of $A(G)$ and $Q(G)$—respectively constituting the adjacency spectrum and the signless Laplacian spectrum—encode key structural features of molecular graphs and are closely tied to quantities such as total π-electron energy and molecular stability. Independently, scalar graph invariants, known as topological indices, have been extensively employed as molecular descriptors in quantitative structure–property and structure–activity relationships (QSPR/QSAR), offering computationally efficient correlations with diverse physicochemical and biological properties.

The Sombor index was introduced by Gutman [4] as a geometric-degree based topological index. Its degree-ratio variant, the degree-ratio Sombor index, was first introduced in [9] and is defined as

$$\mathrm{DRSO}(G) = \sum_{uv \in E(G)} \sqrt{\left(\frac{d_u}{d_v}\right)^2 + \left(\frac{d_v}{d_u}\right)^2}.$$

This particular invariant aggregates the local degree imbalance along each edge by measuring the relative deviation between the degrees of adjacent vertices. Extremal properties of trees with respect to the degree-ratio Sombor index have been recently investigated in [5, 7, 10].

The present work is devoted to establishing novel functional relationships between $\mathrm{DRSO}(G)$ and the spectral radii $q(G)$ and $\lambda_1(G)$. Specifically, we derive an unconditional upper bound on the quotient $q(G)/\mathrm{DRSO}(G)$, along with lower bounds for $\mathrm{DRSO}(G)$ that are expressed purely in terms of these spectral parameters.

2 Preliminaries

In this section, we recall some fundamental results and prove two elementary lemmas for the DRSO index.

The following classical results will be used throughout the paper.

Lemma 1. *([3]) For a connected graph G of order n and size m,*

$$q(G) \leq \frac{2m}{n-1} + n - 2,$$

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

Lemma 2. *([6]) For any connected graph G of order n and size m,*

$$\lambda_1(G) \leq \sqrt{2m - n + 1},$$

with equality if and only if $G \cong K_n$ or $G \cong S_n$.

Lemma 3. *([2]) For any non-empty graph G with maximum degree Δ,*

$$\lambda_1(G) \geq \sqrt{\Delta},$$

with equality if and only if every nontrivial component of G is a star, and at least one such component has maximum degree Δ.

Lemma 4. *([8]) For any connected graph G,*

$$q(G) \geq 2\lambda_1(G),$$

with equality if and only if G is regular.

The following two lemmas provide simple but crucial bounds for the DRSO index.

Lemma 5. *For any graph G of size m,*

$$\mathrm{DRSO}(G) \geq \sqrt{2}\, m.$$

Equality holds if and only if $d_u = d_v$ for every edge $uv \in E(G)$ (i.e., every connected component of G is regular).

Proof. For each edge $uv \in E(G)$, let $x = d_u/d_v > 0$. Since $x^2 + x^{-2} \geq 2$, we have

$$\sqrt{\left(\frac{d_u}{d_v}\right)^2 + \left(\frac{d_v}{d_u}\right)^2} \geq \sqrt{2}.$$

Summing over all edges yields the result. Equality occurs exactly when $d_u = d_v$ for every edge, which means that every connected component of G is regular. ■

Lemma 6. *For any graph G of size m, maximum degree Δ, and minimum degree $\delta \geq 1$,*

$$\mathrm{DRSO}(G) \leq \sqrt{2}\,\frac{\Delta}{\delta}\,m.$$

Proof. Since $d_u, d_v \leq \Delta$ and $d_u, d_v \geq \delta$, we have

$$\sqrt{\left(\frac{d_u}{d_v}\right)^2 + \left(\frac{d_v}{d_u}\right)^2} \leq \sqrt{\left(\frac{\Delta}{\delta}\right)^2 + \left(\frac{\Delta}{\delta}\right)^2} = \sqrt{2}\,\frac{\Delta}{\delta}.$$

Summing over all edges gives the result. ■

3 Relationship between DRSO and the signless Laplacian spectral radius

The central theme of this section is to elucidate the precise analytic relations connecting the signless Laplacian spectral radius $q(G)$ with the degree-ratio Sombor index $\mathrm{DRSO}(G)$.

Theorem 1. *Let G be a connected graph of order $n \geq 3$. Then*

$$\frac{q(G)}{\mathrm{DRSO}(G)} < \frac{n}{\sqrt{2}(n-1)}.$$

Proof. From Lemma 1 and Lemma 5 we have

$$q(G) \leq \frac{2m}{n-1} + n - 2, \qquad \mathrm{DRSO}(G) \geq \sqrt{2}\,m.$$

Hence

$$\frac{q(G)}{\mathrm{DRSO}(G)} \leq \frac{\frac{2m}{n-1} + n - 2}{\sqrt{2}\, m} = \frac{1}{\sqrt{2}} \left(\frac{2}{n-1} + \frac{n-2}{m} \right).$$

Define the function $f(m) = \frac{2}{n-1} + \frac{n-2}{m}$ for $m > 0$. Since $n \geq 3$, we have $n - 2 > 0$ and therefore

$$f'(m) = -\frac{n-2}{m^2} < 0,$$

so f is strictly decreasing in m. Because $m \geq n - 1$, the maximum of f on this domain occurs at the smallest possible value $m = n - 1$. Thus

$$\frac{q(G)}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}} f(m) \leq \frac{1}{\sqrt{2}} f(n-1) = \frac{1}{\sqrt{2}} \left(\frac{2}{n-1} + \frac{n-2}{n-1} \right) = \frac{n}{\sqrt{2}(n-1)}.$$

To see that the bound is strict for every connected graph with $n \geq 3$, suppose equality were to hold. Then necessarily $m = n-1$ and both estimates used above must be equalities: $q(G) = \frac{2m}{n-1} + n - 2$ and $\mathrm{DRSO}(G) = \sqrt{2}\, m$. Lemma 1 forces G to be either K_n or S_n. Since $m = n - 1$ and $n \geq 3$, the only possibility is the star S_n. However, a direct calculation gives $\mathrm{DRSO}(S_n) = (n-1)\sqrt{(n-1)^2 + 1/(n-1)^2} > \sqrt{2}(n-1)$, so $\mathrm{DRSO}(G) > \sqrt{2}\, m$. This contradiction shows that equality cannot occur, hence the inequality is strict. ■

Corollary. *For any connected graph G of order $n \geq 2$,*

$$\mathrm{DRSO}(G) \geq \frac{\sqrt{2}(n-1)}{n}\, q(G).$$

For $n \geq 3$, the inequality is strict; for $n = 2$, equality holds only for K_2.

Proof. For $n \geq 3$, the result follows from Theorem 1 by taking reciprocals and observing strictness. For $n = 2$, the only connected graph is K_2 with $q(K_2) = 2$ and $\mathrm{DRSO}(K_2) = \sqrt{2}$, and the right-hand side becomes $\sqrt{2}$, so equality is attained. ■

Theorem 2. *Let G be a connected graph of order $n \geq 2$. Then*

$$\mathrm{DRSO}(G) \;\geq\; \frac{n-1}{\sqrt{2}}\, q(G) \;-\; \frac{(n-1)(n-2)}{\sqrt{2}}.$$

Equality holds if and only if $G \cong K_n$.

Proof. By Lemma 1, the signless Laplacian spectral radius satisfies the upper bound

$$q(G) \leq \frac{2m}{n-1} + n - 2.$$

Rearranging this inequality provides a lower bound for the size m in terms of $q(G)$, namely

$$m \geq \frac{n-1}{2}\big(q(G) - (n-2)\big). \tag{1}$$

On the other hand, Lemma 5 ensures that

$$\mathrm{DRSO}(G) \geq \sqrt{2}\, m. \tag{2}$$

Combining the estimate from (1) with the inequality in (2) yields

$$\mathrm{DRSO}(G) \geq \sqrt{2}\, m \geq \sqrt{2} \cdot \frac{n-1}{2}\big(q(G) - (n-2)\big) = \frac{\sqrt{2}\,(n-1)}{2}\big(q(G) - (n-2)\big).$$

Since $\frac{\sqrt{2}\,(n-1)}{2} = \frac{n-1}{\sqrt{2}}$, the preceding chain reduces to

$$\mathrm{DRSO}(G) \geq \frac{n-1}{\sqrt{2}}\, q(G) - \frac{(n-1)(n-2)}{\sqrt{2}},$$

which establishes the desired inequality.

To characterize the extremal graphs, observe first that if $G \cong K_n$, including the case $n = 2$, then K_n is regular; consequently, equality is attained in Lemma 5, so that $\mathrm{DRSO}(K_n) = \sqrt{2}\, m$. Moreover, the complete graph also satisfies the equality condition in Lemma 1, since $q(K_n) = \frac{2m}{n-1} + n - 2$. Both estimates in the proof thus reduce to equalities, forcing the final bound to be sharp for complete graphs. Conversely, suppose that equality holds throughout the above argument. Then both auxiliary inequalities must be equalities simultaneously. Equality in $\mathrm{DRSO}(G) \geq \sqrt{2}\, m$ necessitates that G be regular, whereas equality in $q(G) \leq \frac{2m}{n-1} + n - 2$ forces G to be either the complete graph K_n or the star S_n (as stated in Lemma 1). Since the star S_n fails to be regular for every $n \geq 3$, the only remaining possibility is $G \cong K_n$, thereby completing the proof of the equivalence. ∎

Theorem 3. *Let G be a connected graph of order $n > 1$. Then*

$$\mathrm{DRSO}(G) \geq q(G) - n + \sqrt{2}(n-1),$$

with equality if and only if $G \cong K_2$.

Proof. Lemma 1 and Lemma 5 together imply

$$q(G) - \mathrm{DRSO}(G) \leq \frac{2m}{n-1} + n - 2 - \sqrt{2}m.$$

Define the auxiliary function $\phi(t) = \frac{2t}{n-1} + n - 2 - \sqrt{2}t$ for $t \geq n-1$. If $n \geq 3$, then $\phi'(t) = \frac{2}{n-1} - \sqrt{2} < 0$, so ϕ is strictly decreasing and attains its maximum at $t = n-1$. For $n = 2$ we have $m \geq 1$ and the only connected graph is K_2 with $m = 1$; in this case $\phi(t) = 2t + 0 - \sqrt{2}t = (2 - \sqrt{2})t$, which is increasing, but the restriction $t = m = 1$ forces the unique value, which coincides with $t = n - 1$. Thus in all cases the maximal possible value of the upper bound occurs at $m = n - 1$. Evaluating ϕ at $t = n - 1$ gives

$$q(G) - \mathrm{DRSO}(G) \leq (n-1)\Big(\frac{2}{n-1} - \sqrt{2}\Big) + n - 2 = n - \sqrt{2}(n-1).$$

Rearranging yields the claimed lower bound for $\mathrm{DRSO}(G)$.

Turning to the equality characterization, observe that equality in the above argument necessitates both $m = n - 1$ and equality in Lemma 1. For a connected graph, equality in Lemma 1 occurs precisely for the star S_n when $n \geq 3$ and for K_2 when $n = 2$. However, for S_n with $n \geq 3$, a direct computation gives $\mathrm{DRSO}(S_n) > \sqrt{2}(n-1)$, which renders the inequality strict and thus rules out equality for all stars. The remaining candidate is $G \cong K_2$, for which direct substitution yields $q(K_2) = 2$, $\mathrm{DRSO}(K_2) = \sqrt{2}$, and the bound reduces to an equality. This confirms that equality is attained if and only if $G \cong K_2$. ∎

Corollary. *For any connected graph G of order $n \geq 2$,*

$$\frac{q(G)}{\lambda_1(G)\,\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}(n-1)}.$$

For $n = 2$, equality holds for K_2; for $n \geq 3$, the inequality is strict.

Proof. Since G is non-empty, $\Delta \geq 1$ and by Lemma 3 we have $\lambda_1(G) \geq \sqrt{\Delta} \geq 1$. Thus

$$\frac{q(G)}{\lambda_1(G)\,\mathrm{DRSO}(G)} \leq \frac{q(G)}{\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}(n-1)},$$

where the last inequality follows from Theorem 1 (with strictness conditions as stated). The case $n = 2$ is readily verified for K_2. ■

Theorem 4. *Let G be a connected graph of order $n \geq 2$, size m, with minimum degree $\delta > 0$ and maximum degree Δ. Then*

$$q(G) \geq \frac{2\sqrt{2}\,\delta}{n\,\Delta}\,\mathrm{DRSO}(G).$$

Equality holds if and only if G is regular.

Proof. By Lemma 4, we have $q(G) \geq 2\lambda_1(G)$. Also, the classical bound $\lambda_1(G) \geq \frac{2m}{n}$ holds for every graph, with equality if and only if G is regular (see, e.g., [1]). Combining these estimates yields

$$q(G) \geq 2\lambda_1(G) \geq \frac{4m}{n}. \tag{3}$$

On the other hand, Lemma 6 gives

$$\mathrm{DRSO}(G) \leq \sqrt{2}\,\frac{\Delta}{\delta}\,m,$$

which, upon rearrangement, yields

$$m \geq \frac{\delta}{\sqrt{2}\,\Delta}\,\mathrm{DRSO}(G). \tag{4}$$

Combining (3) and (4), we obtain

$$q(G) \geq \frac{4}{n}\,m \geq \frac{4}{n} \cdot \frac{\delta}{\sqrt{2}\,\Delta}\,\mathrm{DRSO}(G) = \frac{2\sqrt{2}\,\delta}{n\,\Delta}\,\mathrm{DRSO}(G),$$

which is precisely the desired inequality.

It remains to characterize the extremal graphs. Equality in the final bound requires equality in both (3) and (4). Equality in (3) occurs if and only if both $q(G) = 2\lambda_1(G)$ and $\lambda_1(G) = \frac{2m}{n}$ hold simultaneously. The first equality is equivalent to G being regular (by Lemma 4). The second equality holds exactly when G is regular, as noted above. Thus equality in (3) is equivalent to the regularity of G. If G is regular, then $\delta = \Delta$ and $\mathrm{DRSO}(G) = \sqrt{2}m$ (by Lemma 5), so (4) becomes an equality as well. Therefore every regular graph attains the bound. Conversely, if equality holds in the final bound, then equality in (3) forces G to be regular. This completes the proof. ■

4 Relationship between DRSO and the adjacency spectral radius

The principal objective of this section is to investigate the relationship between the degree-ratio Sombor index and the adjacency spectral radius $\lambda_1(G)$.

Theorem 5. *Let G be a non-empty graph of order n and size m. Then*

$$\mathrm{DRSO}(G) \geq \frac{\sqrt{2}\,m}{\lambda_1(G)}.$$

Equality holds if and only if every component of G is either an isolated vertex or a K_2 (i.e., G is a disjoint union of copies of K_2 and isolated vertices, with at least one edge).

Proof. A direct application of Lemma 5 ensures that $\mathrm{DRSO}(G) \geq \sqrt{2}m$. Since G is non-empty, its maximum degree satisfies $\Delta \geq 1$; Lemma 3 then gives $\lambda_1(G) \geq \sqrt{\Delta} \geq 1$, whence $\frac{1}{\lambda_1(G)} \leq 1$. Multiplying the latter estimate by the positive quantity $\sqrt{2}m$ yields $\sqrt{2}m \geq \frac{\sqrt{2}m}{\lambda_1(G)}$. Combining these two estimates gives the chain

$$\mathrm{DRSO}(G) \geq \sqrt{2}m \geq \frac{\sqrt{2}m}{\lambda_1(G)},$$

thereby establishing the desired inequality.

To characterize equality, note that the final inequality becomes equality precisely when both $\mathrm{DRSO}(G) = \sqrt{2}m$ and $\lambda_1(G) = 1$ (because if either is strict, the chain is strict). The first equality, by Lemma 5, means that every connected component of G is regular, i.e., each component is either a regular graph of positive degree or an isolated vertex. The second equality, $\lambda_1(G) = 1$, implies that $\Delta = 1$ (otherwise $\sqrt{\Delta} > 1$), so every component with edges must consist entirely of vertices of degree 1, which forces those components to be copies of K_2. Conversely, any graph that is a disjoint union of copies of K_2 and isolated vertices, with at least one copy of K_2, has $\lambda_1(G) = 1$ and $\mathrm{DRSO}(G) = \sqrt{2}m$, thus attains equality. This completes the proof. ■

Corollary. *For any connected graph G of order n and size m,*

$$\mathrm{DRSO}(G) \geq \frac{\lambda_1^2(G) + n - 1}{\sqrt{2}}.$$

Equality holds if and only if $G \cong K_n$ (for $n = 2$ this is also the star S_2).

Proof. Lemma 5 gives $\mathrm{DRSO}(G) \geq \sqrt{2}m$, while Lemma 2 provides $m \geq \frac{\lambda_1^2(G)+n-1}{2}$. Substitution yields the result. Equality requires both bounds to be equal: G must satisfy $d_u = d_v$ for all edges (hence every component regular, and in a connected graph, regular) and attain equality in Lemma 2, which for a connected graph occurs only for K_n or S_n. Regularity eliminates S_n for $n \geq 3$, leaving K_n as the unique extremal graph (including the trivial case $n = 2$ where $K_2 \cong S_2$). ■

Theorem 6. *Let G be a connected graph of order $n > 1$. Then*

$$\mathrm{DRSO}(G) \geq \lambda_1(G) - \sqrt{n-1} + \sqrt{2}(n-1),$$

with equality if and only if $G \cong K_2$.

Proof. Lemma 5 ensures that $\mathrm{DRSO}(G) \geq \sqrt{2}m$, while Lemma 2 provides the classical estimate $\lambda_1(G) \leq \sqrt{2m - n + 1}$. Consequently, the difference $\lambda_1(G) - \mathrm{DRSO}(G)$ is bounded above by

$$\lambda_1(G) - \mathrm{DRSO}(G) \leq \sqrt{2m - n + 1} - \sqrt{2}m.$$

Define the auxiliary function $f(t) = \sqrt{2t-n+1} - \sqrt{2}t$ on the domain $t \geq n-1$. Differentiation with respect to t yields

$$f'(t) = \frac{1}{\sqrt{2t-n+1}} - \sqrt{2}.$$

Since $t \geq n-1$, we have $\sqrt{2t-n+1} \geq \sqrt{n-1}$. For $n \geq 2$, the inequality $\frac{1}{\sqrt{2t-n+1}} \leq \frac{1}{\sqrt{n-1}} < \sqrt{2}$ holds, whence $f'(t) < 0$ throughout the domain. The function f is therefore strictly decreasing, and its supremum is attained at the left endpoint $t = n-1$. Evaluating at this point gives

$$\lambda_1(G) - \mathrm{DRSO}(G) \leq \sqrt{2(n-1)-n+1} - \sqrt{2}(n-1) = \sqrt{n-1} - \sqrt{2}(n-1).$$

Rearranging the preceding inequality yields the claimed lower bound for $\mathrm{DRSO}(G)$.

It remains to characterize the extremal graphs. The above argument reveals that equality in the final bound necessitates two simultaneous conditions: first, $m = n-1$ (so that G is a tree), and second, equality in Lemma 2. For a connected graph, equality in $\lambda_1(G) \leq \sqrt{2m-n+1}$ is known to occur precisely for the star S_n when $n \geq 3$, and for K_2 when $n = 2$. If $G \cong S_n$ with $n \geq 3$, then $\lambda_1(S_n) = \sqrt{n-1}$ and $m = n-1$, but a direct computation gives $\mathrm{DRSO}(S_n) > \sqrt{2}(n-1)$; consequently, the difference $\lambda_1(S_n) - \mathrm{DRSO}(S_n)$ is strictly less than the upper bound $\sqrt{n-1} - \sqrt{2}(n-1)$, so equality fails. The remaining candidate is $G \cong K_2$, for which $n = 2$, $\lambda_1(K_2) = 1$, $\mathrm{DRSO}(K_2) = \sqrt{2}$, and the right-hand side evaluates to $1 - \sqrt{1} + \sqrt{2}(1) = \sqrt{2}$, thereby confirming that equality is indeed attained. This completes the proof. ■

Theorem 7. *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\frac{\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n-1}}.$$

For $n = 2$, equality is attained precisely when $G \cong K_2$; for every $n \geq 3$, the inequality is strict.

Proof. Since G is connected, we have $m \geq n-1$, which in particular ensures that $\mathrm{DRSO}(G) > 0$, so the ratio is well-defined. Lemma 2 gives $\lambda_1(G) \leq$

$\sqrt{2m-n+1}$, while Lemma 5 gives $\mathrm{DRSO}(G) \geq \sqrt{2}\,m$. Combining these estimates yields

$$\frac{\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{\sqrt{2m-n+1}}{\sqrt{2}\,m}.$$

Consider the function $h(t) = \frac{\sqrt{2t-n+1}}{\sqrt{2}\,t}$ defined for $t \geq n-1$. Differentiating with respect to t, we obtain

$$h'(t) = \frac{n-1-t}{\sqrt{2}\,t^2\sqrt{2t-n+1}}.$$

For $t > n-1$, the numerator $n-1-t$ is strictly negative, and hence $h'(t) < 0$. Thus h is strictly decreasing on the interval $t \geq n-1$, so its maximum is reached at the left endpoint $t = n-1$. Evaluating at this point gives

$$h(n-1) = \frac{\sqrt{2(n-1)-n+1}}{\sqrt{2}(n-1)} = \frac{\sqrt{n-1}}{\sqrt{2}(n-1)} = \frac{1}{\sqrt{2}\,\sqrt{n-1}},$$

which proves the desired bound.

It remains to examine the cases of equality. For $n = 2$, the only connected graph is K_2, and a direct substitution gives $\lambda_1(K_2) = 1$ and $\mathrm{DRSO}(K_2) = \sqrt{2}$, so the ratio equals $\frac{1}{\sqrt{2}}$, as required. Now suppose $n \geq 3$. To see that equality cannot occur, we distinguish two possibilities. If $m > n-1$, then from the strict decrease of h we have $h(m) < h(n-1)$, which immediately forces the inequality to be strict. If $m = n-1$, then G is a tree. In this case, equality in the upper bound $\lambda_1(G) \leq \sqrt{2m-n+1}$ would force G to be the star S_n (for $n \geq 3$). However, a straightforward computation for the star yields $\mathrm{DRSO}(S_n) > \sqrt{2}(n-1)$, so the lower bound $\mathrm{DRSO}(G) \geq \sqrt{2}m$ is strict. Consequently, the overall inequality is again strict. Therefore, equality is possible only when $G \cong K_2$, and the proof is complete. ■

5 Combined bounds involving both $q(G)$ and $\lambda_1(G)$

The preceding sections have treated the signless Laplacian spectral radius $q(G)$ and the adjacency spectral radius $\lambda_1(G)$ separately in relation to the degree-ratio Sombor index.

Theorem 8. *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\frac{q(G) + \lambda_1(G)}{\mathrm{DRSO}(G)} < \frac{3n}{\sqrt{2}(n-1)}.$$

Proof. Lemma 1 and Lemma 2 give

$$q(G) \leq \frac{2m}{n-1} + n - 2, \qquad \lambda_1(G) \leq \sqrt{2m - n + 1},$$

and Lemma 5 ensures that $\mathrm{DRSO}(G) \geq \sqrt{2}\, m$. Combining these estimates yields

$$\frac{q(G) + \lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}} \left(\frac{2}{n-1} + \frac{n-2}{m} + \frac{\sqrt{2m-n+1}}{m} \right).$$

Define $F(t) = \frac{n-2}{t} + \frac{\sqrt{2t-n+1}}{t}$ for $t \geq n-1$. Both terms are non-increasing on this interval, so F attains its maximum at $t = n-1$. Consequently,

$$\frac{q(G) + \lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}} \left(\frac{n}{n-1} + \frac{1}{\sqrt{n-1}} \right). \tag{5}$$

We now compare the right-hand side of (5) with the claimed bound. For a simpler, albeit slightly weaker, expression we further observe that

$$\frac{n}{n-1} + \frac{1}{\sqrt{n-1}} < \frac{3n}{n-1},$$

which, after multiplying by $\frac{1}{\sqrt{2}}$, yields the strict inequality in the statement. This completes the proof. ∎

Theorem 9. *Let G be a connected graph of order $n \geq 2$. Then*

$$\frac{q(G)\,\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}}.$$

For $n = 2$, equality holds precisely for $G \cong K_2$. For every $n \geq 3$, the inequality is strict.

Proof. From Theorem 1 we have

$$\frac{q(G)}{\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}(n-1)},$$

and this inequality is strict for all $n \geq 3$ by the same theorem. Multiplying by $\lambda_1(G) > 0$ gives

$$\frac{q(G)\,\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}(n-1)}\,\lambda_1(G),$$

with strict inequality whenever $n \geq 3$. Moreover,

$$\lambda_1(G) \leq \sqrt{2m - n + 1} \leq \sqrt{n(n-1) - n + 1} = n - 1,$$

where the last estimate follows from $m \leq \frac{n(n-1)}{2}$. Substituting this bound yields

$$\frac{q(G)\,\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{n}{\sqrt{2}(n-1)} \cdot (n-1) = \frac{n}{\sqrt{2}},$$

and for $n \geq 3$ the first inequality is strict, so the final bound is strict as well.

For $n = 2$, $G \cong K_2$ gives $q(K_2) = 2$, $\lambda_1(K_2) = 1$, $\mathrm{DRSO}(K_2) = \sqrt{2}$, and the left hand side becomes $2 \cdot 1/\sqrt{2} = 2/\sqrt{2} = n/\sqrt{2}$, so equality is attained. ∎

Theorem 10. *Let G be a connected graph of order $n \geq 2$ with minimum degree $\delta > 0$ and maximum degree Δ. Then*

$$\frac{\lambda_1(G)}{q(G)\ \mathrm{DRSO}(G)} \leq \frac{n\,\Delta}{4\sqrt{2}\,\delta\,(n-1)^{3/2}}.$$

Equality holds if and only if $n = 2$ and $G \cong K_2$.

Proof. From Theorem 7 and Theorem 4 we have

$$\frac{\lambda_1(G)}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}\,\sqrt{n-1}}, \qquad q(G) \geq \frac{2\sqrt{2}\,\delta}{n\,\Delta}\,\mathrm{DRSO}(G).$$

The second inequality can be rewritten as

$$\frac{1}{q(G)} \leq \frac{n\,\Delta}{2\sqrt{2}\,\delta\,\mathrm{DRSO}(G)}.$$

Therefore,

$$\begin{aligned}\frac{\lambda_1(G)}{q(G)\,\mathrm{DRSO}(G)} = \left(\frac{\lambda_1(G)}{\mathrm{DRSO}(G)}\right)\cdot\frac{1}{q(G)} &\leq \frac{1}{\sqrt{2}\,\sqrt{n-1}}\cdot\frac{n\,\Delta}{2\sqrt{2}\,\delta\,\mathrm{DRSO}(G)} \\ &= \frac{n\,\Delta}{4\,\delta\,\sqrt{n-1}}\cdot\frac{1}{\mathrm{DRSO}(G)}. \qquad (6)\end{aligned}$$

By Lemma 5, we have

$$\mathrm{DRSO}(G) \geq \sqrt{2}\,m \geq \sqrt{2}\,(n-1).$$

Hence

$$\frac{1}{\mathrm{DRSO}(G)} \leq \frac{1}{\sqrt{2}\,(n-1)}.$$

Substituting this into (1) yields

$$\frac{\lambda_1(G)}{q(G)\,\mathrm{DRSO}(G)} \leq \frac{n\,\Delta}{4\,\delta\,\sqrt{n-1}}\cdot\frac{1}{\sqrt{2}\,(n-1)} = \frac{n\,\Delta}{4\sqrt{2}\,\delta\,(n-1)^{3/2}},$$

which is the claimed bound.

For equality to hold, we must have equality in both Theorem 7 and Lemma 5, as well as in the estimate $m \geq n-1$. By Theorem 7, equality occurs if and only if $n = 2$ and $G \cong K_2$. For $G \cong K_2$, we have $m = 1 = n-1$, $\Delta = \delta = 1$, and $\mathrm{DRSO}(K_2) = \sqrt{2}$, so that all intermediate inequalities become equalities and the final bound is attained.

For $n \geq 3$, Theorem 7 is strict for every connected graph; consequently, the product inequality (6) is also strict, and equality in the final bound cannot occur. Thus equality holds if and only if $n = 2$ and $G \cong K_2$. ■

6 A conjecture on extremal trees

In Theorem 1, we established that for every connected graph G of order $n \geq 3$,

$$\frac{q(G)}{\mathrm{DRSO}(G)} < \frac{n}{\sqrt{2}(n-1)}.$$

Consequently, the universal upper bound is never attained by any tree on $n \geq 3$ vertices. This naturally raises the question: among all trees of a given order n, which one maximizes the ratio $\frac{q(T)}{\mathrm{DRSO}(T)}$?

We have carried out an exhaustive computational investigation for orders up to $n = 10$. The results reveal an interesting pattern that leads to a new conjecture.

Computational evidence

For small orders the extremal tree is the path, while for larger orders a *spider* (a tree with exactly one vertex of degree at least three, the *center*, whose removal leaves only disjoint paths) takes over. Moreover, the optimal degree of the center increases gradually with n. Table 1 presents the maximal ratio and the corresponding tree for $n = 3, \ldots, 10$. All values are computed exactly or with high precision; the signless Laplacian spectral radius of each spider is obtained from the characteristic equation associated with an appropriate equitable partition (or, equivalently, from the corresponding quotient matrix after separating vertices by leg type and distance from the center). For balanced spiders with leg length 2, one has $q = \frac{k+3+\sqrt{k^2-2k+5}}{2}$; for unbalanced spiders the values are computed numerically.

In the table, a spider is denoted by the lengths of its legs; e.g. Spider $(2, 2, 2)$ is the tree consisting of a center of degree 3 to which three paths of length 2 are attached (overall $n = 1 + 3 \cdot 2 = 7$). The column k gives the degree of the center. For paths, k is simply the maximum degree.

Observations:

- For $n \leq 6$ the path P_n yields the largest ratio.
- For $n \geq 7$ a spider with a single high-degree vertex outperforms the path.

Table 1. Maximal ratios $\dfrac{q(T)}{\mathrm{DRSO}(T)}$ for trees of order $n \le 10$.

n	Extremal tree	k	$q(T)$	$\mathrm{DRSO}(T)$	$\frac{q(T)}{\mathrm{DRSO}(T)}$
3	P_3 (S_3)	2	3	$\sqrt{17}$	0.7276
4	P_4	2	$2+2\cos\frac{\pi}{4} = 2+\sqrt{2}$	$\sqrt{17}+\sqrt{2}$	0.6165
5	P_5	2	$2+2\cos\frac{\pi}{5}$	$\sqrt{17}+2\sqrt{2}$	0.5205
6	P_6	2	$2+2\cos\frac{\pi}{6} = 2+\sqrt{3}$	$\sqrt{17}+3\sqrt{2}$	0.4461
7	Spider $(2,2,2)$	3	$3+\sqrt{2}$	$\frac{\sqrt{97}+3\sqrt{17}}{2}$	0.3973
8	Spider $(3,2,2)$	3	≈ 4.438	≈ 12.523	0.3543
9	Spider $(2,2,2,2)$	4	$\frac{7+\sqrt{13}}{2}$	$4\sqrt{17}$	0.3215
10	Spider $(3,2,2,2)$	4	≈ 5.309	≈ 17.907	0.2965

- In all spider cases the legs are as balanced as possible (their lengths differ by at most one).
- The optimal center degree k increases slowly: for $n = 7, 8$ it is 3; for $n = 9, 10$ it becomes 4.
- The ratio continues to decrease with n, and the spider structure becomes increasingly favorable over the path.

These findings strongly suggest that the extremal tree is always a *balanced spider* whose center degree is chosen to maximize the ratio, and that the path dominates only for the smallest orders.

Conjecture 1. *Let T be a tree of order $n \ge 3$. Then*

$$\frac{q(T)}{\mathrm{DRSO}(T)} \le \max\left\{\frac{q(P_n)}{\mathrm{DRSO}(P_n)},\ \frac{q(\mathrm{Sp}_n(k))}{\mathrm{DRSO}(\mathrm{Sp}_n(k))}\right\},$$

where $\mathrm{Sp}_n(k)$ denotes a balanced spider on n vertices with center degree k (legs as equal as possible), and the maximum is attained precisely by

- *the path P_n for $n = 3, 4, 5, 6$;*
- *a balanced spider $\mathrm{Sp}_n(k_n)$ with an optimal center degree k_n for all $n \ge 7$.*

Moreover, k_n tends to infinity as $n \to \infty$.

Determining the exact value of k_n for every n and proving that a balanced spider is indeed optimal remain challenging open problems. The present

computational data give strong support for the conjecture and invite further spectral and structural investigations.

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