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On the Degree-Ratio Sombor Index of Cycle-Containing Graphs

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Abstract

We study the degree-ratio Sombor index ($DRSO$) on cycle-containing graphs from two complementary perspectives. First, we extend the $DRSO$–SO correlation analysis of Gutman et al. to c-cyclic molecular graphs (connected graphs with maximum degree $\Delta \leq 4$). Using exhaustive enumeration of 15 673 non-isomorphic molecular graphs with $n = 8, 9$ vertices and $c \leq 5$, and $n = 10$ with $c \leq 3$, we show that the cluster-and-line structure previously established for molecular trees persists for all c. A multivariate regression $DRSO = \beta_0 + \beta_1 SO + \beta_2 c + \beta_3 n + \beta_4 SDD$ achieves $R^2 = 0.998$, and separate regressions by cyclomatic class show that the SO–$DRSO$ slope decreases monotonically from 0.92 (trees) to 0.59 ($c = 5$). Second, we determine the maximum value of $DRSO$ among all connected n-vertex graphs in two natural settings. For $n \geq 4$, the star $K_{1,n-1}$ is the unique global maximizer. Among all connected graphs that contain at least one cycle, $J_n = K_2 \vee \overline{K}_{n-2}$ is the unique maximizer for $n \geq 6$.

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1 Introduction

Topological indices are numerical descriptors that capture structural features of molecular graphs and play a fundamental role in chemical graph theory. Among the numerous vertex-degree-based indices proposed in the literature, two have recently attracted particular interest: the Sombor index (SO) and the degree-ratio Sombor index ($DRSO$). While their behavior on trees is partially understood, the picture for graphs containing cycles remains largely unexplored. This paper investigates the $DRSO$–SO relationship in cycle-containing molecular graphs and determines extremal values of $DRSO$ under cyclomatic-number constraints.

Throughout this paper, $G = (V, E)$ denotes a simple connected graph with $n = |V|$ vertices and $m = |E|$ edges. The degree of a vertex v is written $d(v)$. The cyclomatic number (or first Betti number) of G is $c = m - n + 1$; trees correspond to $c = 0$, unicyclic graphs to $c = 1$, bicyclic to $c = 2$, etc. Let $\Delta = \Delta(G)$ denote the maximum degree.

The Sombor index [1] is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

Since its introduction in 2021, this vertex-degree-based topological index has attracted considerable attention, as documented in several review articles [3, 6, 8]. The complete graph K_n maximizes SO among n-vertex connected graphs [1].

Recently, Nyauli and Buragohain [7] introduced a novel geometric interpretation of vertex-degree-based topological indices, leading to the degree-ratio Sombor index (DRSO):

$$DRSO(G) = \sum_{uv \in E(G)} \sqrt{\left[\frac{d(u)}{d(v)}\right]^2 + \left[\frac{d(v)}{d(u)}\right]^2}.$$

The DRSO index applies the Euclidean metric to the degree-ratio pair $(\frac{d(u)}{d(v)}, \frac{d(v)}{d(u)})$ instead of the raw degree pair $(d(u), d(v))$. Among the fundamental properties established in [7] is the sharp lower bound $DRSO(G) \geq$

$\sqrt{2}\,m$, with equality if and only if G is regular.

Unlike the Sombor index, K_n is *not* the maximizer of DRSO. For $n \geq 4$,

$$DRSO(K_{1,n-1}) > DRSO(K_n),$$

a phenomenon observed by Gutman et al. [4]: DRSO rewards degree disparity—edges between vertices of very different degree contribute much more than edges between vertices of similar degree.

Gutman, Redžepović, Buragohain, and Nyauli [4] initiated a comparative study of DRSO and SO. For molecular trees with up to $n = 10$ vertices, they discovered that the data points in the $DRSO$–SO scatter plot group into clusters determined by the first Zagreb index M_1, and that within each cluster the data points form (approximately) straight lines, with the symmetric division deg index (SDD) governing the vertical dispersion. This behavior is explained by three facts: (i) a nearly perfect linear correlation holds between SO and M_1, as implied by the elementary bounds

$$\frac{1}{\sqrt{2}} M_1(G) \leq SO(G) < M_1(G)$$

from [2]; (ii) an equally strong linear correlation holds between $DRSO$ and SDD, following from

$$\frac{1}{\sqrt{2}} SDD(G) \leq DRSO(G) < SDD(G)$$

from [7], where the symmetric division deg index is

$$SDD(G) = \sum_{uv \in E} \left(\frac{d(u)}{d(v)} + \frac{d(v)}{d(u)} \right);$$

and (iii) M_1 and SDD are only weakly correlated. Together, (i) and (iii) imply that fixing M_1 confines SO to a narrow interval, creating the clusters; similarly, (ii) and (iii) imply that varying SDD within a fixed M_1 spreads $DRSO$ vertically, producing the lines.

A key algebraic result of [4] expresses M_1 of an n-vertex molecular tree

as

$$M_1(T) = 4n - 6 + 2(n_3 + 3n_4),$$

where n_3 and n_4 count vertices of degree 3 and 4, respectively. This implies that trees with different (n_3, n_4) combinations but equal $(n_3 + 3n_4)$ share the same M_1 and thus belong to the same cluster, while their distinct SDD ranges, inherited from the different degree distributions, produce distinct sub-lines within that cluster.

However, the analysis in [4] is restricted to trees and largely to $n \leq 10$. The authors explicitly noted that "the study of cycle-containing (molecular) graphs may remain a task for the future" [4], and they provided a generalization of the M_1 formula to c-cyclic molecular graphs:

$$M_1(G) = 4n + 6(c - 1) + 2(n_3 + 3n_4). \tag{1}$$

This naturally raises the question of whether the DRSO–SO clustering phenomenon extends to graphs containing cycles. In the present paper, we address this open problem and go further. Our work consists of two parts.

Part I: DRSO–SO correlation in cycle-containing molecular graphs. We perform exhaustive enumeration of all non-isomorphic molecular graphs (maximum degree $\Delta \leq 4$) with n vertices and cyclomatic number c, where $n \in \{8, 9\}$ with $c \leq 5$, and $n = 10$ with $c \leq 3$, yielding a comprehensive dataset of 15 673 graphs. Using this data, we (1) verify that the clustering predicted by the generalized M_1 formula holds, (2) determine whether the within-cluster splitting phenomenon generalizes to cycle-containing graphs, and (3) quantify the independent effect of c on the $DRSO$–SO relationship.

Part II: Maximum DRSO for cycle-containing graphs. We determine the maximum value of $DRSO$ among all connected n-vertex graphs in two natural settings: globally and for all cycle-containing graphs. The global maximizer is the star $K_{1,n-1}$. Among all connected graphs that contain a cycle, $J_n = K_2 \vee \overline{K}_{n-2}$ is the unique maximizer for $n \geq 6$.

The rest of this paper is organized as follows. Section 2 presents Part I: the computational enumeration and correlation analysis. Section 3 de-

velops the extremal theory (Part II), establishing the maximum DRSO globally and for all cycle-containing graphs. Section 4 concludes with a discussion and open problems.

2 Correlation analysis

In this section we present the computational investigation (Part I). All graphs considered here are molecular graphs (connected and $\Delta \leq 4$).

We generated all non-isomorphic molecular graphs with $n \in \{8, 9, 10\}$ and $c \leq 5$, subject to $\Delta \leq 4$. The enumeration proceeded by a bottom-up edge-addition scheme: starting from the set of molecular trees (generated via NetworkX's `nonisomorphic_trees` [5]), each successive cyclomatic class was obtained by adding a single edge to each graph of the previous class and filtering by $\Delta \leq 4$ and isomorphism. Table 1 summarizes the resulting dataset. Using this dataset, we examine the *DRSO–SO* relationship.

Table 1. Number of non-isomorphic molecular graphs in the dataset.

n	$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	Total
8	18	73	182	326	430	427	1456
9	35	185	573	1278	2161	2768	7000
10	75	475	1792	4875	—	—	7217
Total	128	733	2547	6479	2591	3195	**15673**

Gutman et al. [4] showed that for molecular trees, the *DRSO–SO* scatter plot exhibits natural clusters determined by M_1, with within-cluster lines governed by *SDD*. The mechanism rests on three correlations: *SO*–M_1 (strong), *DRSO–SDD* (strong), and M_1–*SDD* (weak).

Figure 1 demonstrates that all three relationships persist for the cycle-containing graphs in our dataset, using the $n = 10$, $c = 3$ class as a representative example.

When graphs of different cyclomatic numbers are plotted together (*DRSO* vs *SO*), the data points separate into distinct layers corresponding to different c values (Figure 2). This layering arises because M_1 depends on c through the $6(c - 1)$ term in Eq. (1), shifting the baseline of each

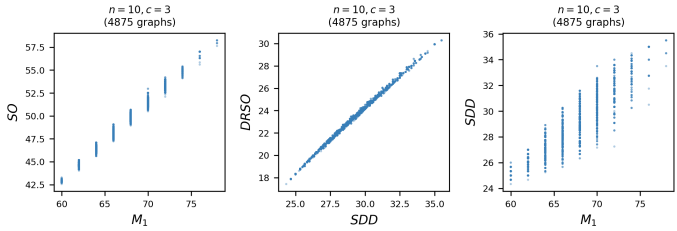

Figure 1. The three relationships underlying the clustering behavior, shown for $n = 10$, $c = 3$ (4875 graphs). Left: SO vs M_1, exhibiting strong linear correlation. Center: $DRSO$ vs SDD, also strongly linear. Right: M_1 vs SDD, displaying the weak correlation that permits within-cluster dispersion.

cyclomatic class.

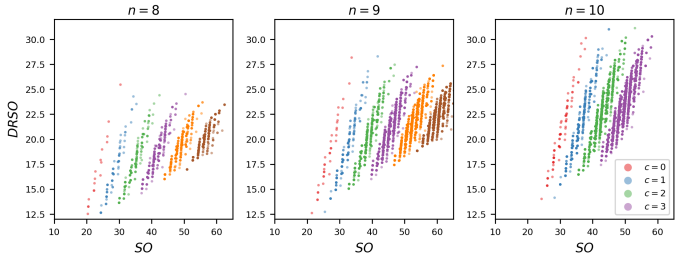

Figure 2. Global $DRSO$ vs SO scatter plot for all 15 673 graphs, color-coded by cyclomatic number c. Each c class forms a distinct cluster layer; layers are ordered monotonically by c. Left: $n = 8$; center: $n = 9$; right: $n = 10$.

To quantify the independent effect of c, we fitted a multivariate linear regression:

$$DRSO = \beta_0 + \beta_1 SO + \beta_2 c + \beta_3 n + \beta_4 SDD + \varepsilon. \tag{2}$$

The results (Table 2) show that each additional cycle reduces $DRSO$ by approximately 0.97 units, holding SO, n, and SDD fixed. The model achieves $R^2 = 0.998$, confirming that these four variables jointly explain virtually all variance in $DRSO$.

We further examined the SO–$DRSO$ relationship separately for each cyclomatic class. The regression slope decreases monotonically with c (Ta-

Table 2. Multivariate regression results for the model in Eq. (2). Dependent variable: $DRSO$; $N = 15\,673$.

Variable	Coefficient	Std. error	t	p
Constant	0.9319	0.016	59.8	< 0.001
SO	0.0151	0.001	19.3	< 0.001
c	-0.9681	0.004	-236.0	< 0.001
n	-0.8576	0.002	-460.1	< 0.001
SDD	1.1345	0.001	1032.7	< 0.001

$R^2 = 0.9983$, Adj. $R^2 = 0.9983$

$F(4, 15668) = 2.31 \times 10^6$, $p < 0.001$

ble 3), from 0.921 for trees to 0.592 for $c = 5$. This trend indicates that as the graph becomes more cyclically dense, a change in SO induces a smaller change in $DRSO$. The R^2 values also decline (from 0.874 to 0.848), suggesting that the linear SO–$DRSO$ relationship weakens with increasing c.

Table 3. Simple linear regression $DRSO = \alpha_c + \beta_c SO$ for each cyclomatic class c, aggregated across $n \in \{8, 9, 10\}$.

c	Graphs	β_c (slope)	α_c (intercept)	R^2
0	128	0.921	-6.55	0.874
1	733	0.836	-8.39	0.889
2	2547	0.769	-10.21	0.893
3	6479	0.712	-11.86	0.887
4	2591	0.652	-12.64	0.870
5	3195	0.592	-13.18	0.848

While c governs the large-scale layering, the within-cluster fine structure is controlled by the (n_3, n_4) degree distribution. Eq. (1) implies that graphs with different (n_3, n_4) combinations can share the same M_1 value, provided that $n_3 + 3n_4$ is constant. Each such combination yields a distinct range of SDD values, producing resolvable sub-lines within a cluster.

For molecular trees, Gutman et al. [4] observed the simplest instance of this phenomenon: clusters with two (n_3, n_4) combinations exhibited two distinct sub-lines. In cycle-containing graphs, the additional degree of freedom from c enables richer splitting behavior, including three-way splits.

Figure 3 illustrates the progression from double to triple splitting as n and c increase. For $n = 9$, $c = 3$, the $M_1 = 60$ cluster contains three distinct (n_3, n_4) combinations—$(0, 2)$, $(3, 1)$, and $(6, 0)$—each forming a separate linear sub-structure.

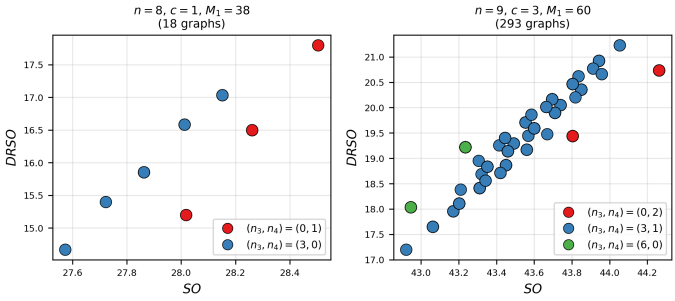

Figure 3. Selected examples of within-cluster splitting. Left: $M_1 = 38$ cluster of $n = 8$, $c = 1$ with two (n_3, n_4) combinations (double split). Right: $M_1 = 60$ cluster of $n = 9$, $c = 3$ with three combinations (triple split). Colors indicate (n_3, n_4) pairs as shown in the legends.

The complete inventory of mixed (n_3, n_4) clusters is summarized in Table 4. A cluster is termed *k-mixed* if it contains k distinct (n_3, n_4) pairs.

Table 4. Distribution of (n_3, n_4)-mixed clusters across the dataset. Only M_1 values with more than one (n_3, n_4) combination are listed.

n	c	M_1	(n_3, n_4) pairs
8	2	50	$(0, 2)$, $(3, 1)$
8	2	48	$(2, 1)$, $(5, 0)$
8	3	56	$(0, 2)$, $(3, 1)$, $(6, 0)$
9	3	60	$(0, 2)$, $(3, 1)$, $(6, 0)$
9	3	62	$(1, 2)$, $(4, 1)$
9	4	72	$(0, 3)$, $(3, 2)$, $(6, 1)$
10	2	58	$(0, 2)$, $(3, 1)$, $(6, 0)$
10	3	64	$(0, 2)$, $(3, 1)$, $(6, 0)$
10	3	66	$(1, 2)$, $(4, 1)$, $(7, 0)$

These splitting patterns are fully consistent with Eq. (1): every M_1 value that admits multiple (n_3, n_4) solutions—subject to $n_2 \geq 0$—produces a resolvable multi-line structure in the *DRSO–SO* scatter plot.

The results above demonstrate that the cluster-and-line framework established for molecular trees in [4] extends naturally to cycle-containing molecular graphs. The cyclomatic number c plays a predictable role: it shifts the M_1 baseline by $6(c-1)$ (Eq. (1)), thereby translating the entire cluster configuration upward and producing distinct c-layers. The within-cluster structure is governed by the same (n_3, n_4) pairing mechanism, with c broadening the range of admissible degree distributions. The regression analysis quantifies the c-effect as approximately -0.97 per cycle, with the *SO–DRSO* slope decreasing monotonically.

3 Extremal theory

3.1 Preliminaries

Define

$$h(x) = \sqrt{x^2 + x^{-2}}, \qquad x \geq 1.$$

Then the following lemma holds.

Lemma 1. *For $x \geq 1$,*

(i) *$h(1) = \sqrt{2}$, and $h(x) > \sqrt{2}$ for $x > 1$;*

(ii) *$x < h(x) < x + \frac{1}{2x^3}$ for $x \geq 1$;*

(iii) *h is strictly convex;*

(iv) *$h(x) > \frac{\sqrt{2}}{2}(x+1)$ for $x \geq 5/2$.*

Proof. (i) and (ii) are trivial. (iii) $h''(x) = 2(3x^4+1)/[x^3(x^4+1)^{3/2}] > 0$. (iv) Noting that $2h^2(x) - (x+1)^2 = x^2 - 2x - 1 + \frac{2}{x^2} > x^2 - 2x - 1$ and $x \geq 5/2$, we have that $h(x) > \frac{\sqrt{2}}{2}(x+1)$. ∎

For each $uv \in E(G)$, denote

$$r_{uv} := \frac{\max\{d_G(u), d_G(v)\}}{\min\{d_G(u), d_G(v)\}}.$$

Then the DRSO-contribution of uv is $h(r_{uv})$. Hence

$$DRSO(G) = \sum_{e \in E(G)} h(r_e).$$

A useful auxiliary quantity is the *degree-ratio sum*

$$R(G) = \sum_{e \in E(G)} r_e.$$

The following bound on $R(G)$ will be used in the proofs of Theorems 1 and 2.

Lemma 2. *For any connected n-vertex graph G,*

$$R(G) + m \leq n\Delta,$$

where $\Delta = \Delta(G)$. Equality holds if and only if every vertex whose degree is strictly less than Δ is adjacent exclusively to vertices of degree Δ.

Proof. For a vertex v, let $N_>(v) := \{u \in N(v) : d(u) > d(v)\}$, and let

$$\tau(v) := \sum_{u \in N_>(v)} \frac{d(u) - d(v)}{d(v)}.$$

Then

$$\begin{aligned} \tau(v) &\leq \sum_{u \in N_>(v)} \frac{\Delta - d(v)}{d(v)} = |N_>(v)| \cdot \frac{\Delta - d(v)}{d(v)} \\ &\leq d(v) \cdot \frac{\Delta - d(v)}{d(v)} = \Delta - d(v). \end{aligned}$$

On the other hand,

$$\begin{aligned} R(G) + m &= \sum_{e \in E(G)} (r_e + 1) \\ &= 2m + \sum_{e \in E(G)} (r_e - 1) = 2m + \sum_{v \in V(G)} \tau(v) \\ &\leq 2m + n\Delta - \sum_{v \in V(G)} d(v) = n\Delta. \end{aligned}$$

Note that the equality in $R(G) + m = n\Delta$ holds if and only if $\tau(v) = \Delta - d(v)$ for every vertex v. Thus equality forces $|N_>(v)| \in \{0, d(v)\}$ for every v, and every neighbor of a vertex with degree $< \Delta$ must have degree Δ. ■

3.2 The global maximum

We now determine the maximum value of DRSO among connected graphs. The results in Subsections 3.2–3.3 apply to general connected graphs (no $\Delta \leq 4$ restriction), unless otherwise noted.

Theorem 1. *For any connected n-vertex graph G with $n \geq 4$,*

$$DRSO(G) \leq DRSO(K_{1,n-1}) = (n-1)\sqrt{(n-1)^2 + \frac{1}{(n-1)^2}},$$

with equality if and only if $G \cong K_{1,n-1}$.

Proof. If $\Delta \leq 2$, then G is a path or a cycle. For $n \geq 4$, $DRSO(G) \leq n \cdot h(2) < (n-1)h(n-1) = DRSO(K_{1,n-1})$, so the theorem holds. Hence we assume $\Delta \geq 3$ for the remainder of the proof.

Since h is convex, for any $x \in [1, \Delta]$, we have the chord bound as follows:

$$h(x) \leq h(1) + \frac{h(\Delta) - h(1)}{\Delta - 1}(x - 1) = \sqrt{2} + \frac{h(\Delta) - \sqrt{2}}{\Delta - 1}(x - 1). \qquad (3)$$

Summing over all edges,

$$\begin{aligned} DRSO(G) &= \sum_{e\in E(G)} h(r_e) \\ &\le m\sqrt{2} + \frac{h(\Delta)-\sqrt{2}}{\Delta-1} \sum_{e\in E(G)} (r_e - 1) \\ &= m\sqrt{2} + \frac{h(\Delta)-\sqrt{2}}{\Delta-1}\big(R(G) - m\big). \end{aligned}$$

By Lemma 2, $R(G) - m \le n\Delta - 2m$. Hence

$$DRSO(G) \le m\sqrt{2} + \frac{h(\Delta)-\sqrt{2}}{\Delta-1}(n\Delta - 2m). \tag{4}$$

Define $k_\Delta := (h(\Delta) - \sqrt{2})/(\Delta - 1)$, and $f(m) := m\sqrt{2} + k_\Delta(n\Delta - 2m)$ which is the right-hand side of (4). By (iv) of Lemma 1, we have that $k_\Delta > \sqrt{2}/2$. Thus $f'(m) = \sqrt{2} - 2k_\Delta < 0$, hence $f(m)$ is decreasing in m.

Since the vertex of degree Δ contributes at least Δ edges, we have $m \ge \Delta$. Therefore, $f(m) \le f(\Delta)$, and hence

$$\begin{aligned} DRSO(G) &\le f(\Delta) \\ &= \Delta\sqrt{2} + k_\Delta(n\Delta - 2\Delta) \\ &= \Delta\sqrt{2} + \Delta(n-2)\, k_\Delta. \end{aligned}$$

Now, $\Delta \le n - 1$ for any n-vertex graph. Since k_Δ is increasing in Δ (convexity of h implies the secant slope increases with Δ), the right-hand side is maximized when $\Delta = n - 1$:

$$\begin{aligned} DRSO(G) &\le (n-1)\sqrt{2} + (n-1)(n-2)\,\frac{h(n-1)-\sqrt{2}}{n-2} \\ &= (n-1)\sqrt{2} + (n-1)(h(n-1) - \sqrt{2}) \\ &= (n-1)h(n-1) \\ &= DRSO(K_{1,n-1}). \end{aligned}$$

Equality requires $\Delta = n - 1$ and $m = \Delta$, which together force $G \cong K_{1,n-1}$. ■

3.3 Graphs containing a cycle

Theorem 1 identifies the star $K_{1,n-1}$, a tree, as the global DRSO maximizer. We now determine the maximum among all connected n-vertex graphs that contain at least one cycle (equivalently, $m \geq n$).

For $n = 4$ and $n = 5$, the claim below follows by exhaustive enumeration: the only cycle-containing connected graphs on four vertices are K_4, $K_4 - e$, C_4, and the paw, whose DRSO values are all bounded by $DRSO(K_4) = 6\sqrt{2}$; similarly K_5 yields $10\sqrt{2}$ for $n = 5$. We therefore assume $n \geq 6$ throughout the proof and set

$$D := n - 1 \geq 5.$$

The benchmark graph is $J_n = K_2 \vee \overline{K}_{n-2}$, which has two universal vertices (degree D) and $D - 1$ vertices of degree 2. Its DRSO value is

$$DRSO(J_n) = 2(D-1)\,h\left(\frac{D}{2}\right) + \sqrt{2} = \frac{D-1}{D}\,\sqrt{D^4 + 16} + \sqrt{2}.$$

Theorem 2. *Let $\mathcal{C}_n$ be the set of all connected n-vertex graphs that contain at least one cycle. Then*

$$\max_{G \in \mathcal{C}_n} DRSO(G) = \begin{cases} 6\sqrt{2}, & n = 4, \\ 10\sqrt{2}, & n = 5, \\ 2(n-2)\,h\Big(\dfrac{n-1}{2}\Big) + \sqrt{2}, & n \geq 6. \end{cases}$$

The unique maximizer is K_4 for $n = 4$, K_5 for $n = 5$, and $J_n = K_2 \vee \overline{K}_{n-2}$ for $n \geq 6$.

The proof proceeds in three steps: (i) every maximizer has a universal vertex (Lemma 3); (ii) among graphs with at least two universal vertices, J_n is uniquely maximal (Lemma 4); (iii) the single-universal-vertex case yields strictly smaller DRSO than J_n (Lemmas 5 and 6).

Lemma 3. *For $n \geq 6$, every graph that maximizes DRSO over $\mathcal{C}_n$ satisfies $\Delta(G) = n - 1$.*

Proof. Define

$$\phi(t) = \frac{(t-2)h(t) + \sqrt{2}}{t-1}, \qquad t \geq 3.$$

Using the chord bound (3) and Lemma 2, for any $G \in \mathcal{C}_n$ (so $m \geq n$) we obtain

$$\begin{aligned} DRSO(G) &\leq m\sqrt{2} + k_\Delta\big(R(G) - m\big) \\ &\leq m\sqrt{2} + k_\Delta(n\Delta - 2m) \end{aligned}$$

where $k_\Delta = (h(\Delta) - \sqrt{2})/(\Delta - 1)$. If $\Delta \leq 2$, then G is a path or a cycle; for $n \geq 6$, $DRSO(G) \leq n\,h(2) < DRSO(J_n)$, so G cannot be a maximizer. Hence $\Delta \geq 3$ for any potential maximizer, and by (iv) of Lemma 1 the slope $\sqrt{2} - 2k_\Delta < 0$. Thus the right-hand side is maximized at the minimal possible m. For a connected graph with given Δ and containing a cycle, the minimum number of edges is $m = n$. Hence

$$DRSO(G) \leq n\sqrt{2} + k_\Delta(n\Delta - 2n) = n\,\phi(\Delta). \tag{5}$$

The function ϕ is strictly increasing for $t \geq 3$; this follows from $\phi'(t) > 0$. Now compute a lower bound for $DRSO(J_n)$. Since $h(x) > x$ for $x > 1$,

$$DRSO(J_n) = 2(D-1)h(D/2) + \sqrt{2} > D(D-1) + \sqrt{2} = D^2 - D + \sqrt{2}, \tag{6}$$

where $D = n - 1$. On the other hand, suppose that $\Delta \leq D - 1$, then by (5) and the monotonicity of ϕ,

$$DRSO(G) \leq n\,\phi(D-1) = (D+1)\,\phi(D-1).$$

We now prove $(D+1)\phi(D-1) < D^2 - D + \sqrt{2}$ for $D \geq 5$. By Lemma 1(ii), $h(D-1) < (D-1) + \frac{1}{2(D-1)^3}$, hence

$$\phi(D-1) < \frac{(D-3)\big(D - 1 + \frac{1}{2(D-1)^3}\big) + \sqrt{2}}{D-2}.$$

Clearing the positive denominator $2(D-2)(D-1)^3$ reduces the inequality to checking that a single-variable polynomial in D is positive—a task

readily discharged by symbolic computation (e.g., SymPy or Mathematica). Thus $(D+1)\phi(D-1) < D^2 - D + \sqrt{2}$. Comparing with (6),

$$DRSO(G) < D^2 - D + \sqrt{2} < DRSO(J_n),$$

so no graph with $\Delta \leq D-1$ can be extremal. Hence $\Delta = D$. ■

Lemma 4. *Among all graphs in $\mathcal{C}_n$ with at least two universal vertices, J_n is the unique DRSO maximizer for $n \geq 6$.*

Proof. If G has at least two universal vertices, then every vertex has degree at least 2, hence the degree ratio of every edge is at most $D/2$. Apply the chord bound (3) on the interval $[1, D/2]$:

$$h(x) \leq \sqrt{2} + k_{D/2}(x-1)$$

for $1 \leq x \leq D/2$, where $k_{D/2} = (h(D/2) - \sqrt{2})/(D/2 - 1)$. Summing over all edges,

$$DRSO(G) \leq m\sqrt{2} + k_{D/2}\big(R(G) - m\big).$$

Using Lemma 2, $R(G) - m \leq nD - 2m$, so

$$DRSO(G) \leq m\sqrt{2} + k_{D/2}(nD - 2m) = m(\sqrt{2} - 2k_{D/2}) + nD\,k_{D/2}.$$

By property (iv) of Lemma 1, $h(D/2) > \frac{\sqrt{2}}{2}(D/2+1) = \frac{\sqrt{2}}{4}(D+2)$, which is equivalent to $2k_{D/2} > \sqrt{2}$, i.e., $\sqrt{2} - 2k_{D/2} < 0$. Thus the bound decreases with m.

Two universal vertices contribute $2D-1$ edges. Any additional edges only increase m and therefore strengthen the inequality, so the tightest bound—and hence the upper bound for the whole class—is attained at the minimal m, namely $m = 2D-1$. At $m = 2D-1$, the bound equals $DRSO(J_n)$. Equality in the chord bound additionally requires every degree ratio to be either 1 or $D/2$; in J_n, the edge between the two universal vertices has ratio 1, and the edges from universal vertices to degree-2 vertices have ratio $D/2$, so equality is attained uniquely by J_n. ■

Lemma 5. *Let*

$$\ell(t) := h(D) - h\Big(\frac{D}{t}\Big), \qquad t \geq 1.$$

Then for integers $2 \le r \le s \le D-1$,

$$h\Big(\frac{s}{r}\Big) \le \frac{\ell(r)}{r-1} + \frac{\ell(s)}{s-1} - \frac{2\ell(2)-\sqrt{2}}{r+s-3}. \tag{7}$$

Equality holds only for $(r,s) = (2,2)$.

Proof. If $(r,s) = (2,2)$, the right-hand side is $2\ell(2) - (2\ell(2) - \sqrt{2}) = \sqrt{2} = h(1)$, so equality holds. Now assume $(r,s) \neq (2,2)$. From the elementary bounds

$$x \le h(x) \le x + \frac{1}{2x^3} \qquad (x \ge 1),$$

we obtain

$$\begin{aligned} \ell(t) &\ge \frac{D(t-1)}{t} - \frac{t^3}{2D^3}, \\ 2\ell(2) - \sqrt{2} &\le D + D^{-3} - \sqrt{2}, \\ h(s/r) &\le \frac{s}{r} + \frac{r^3}{2s^3}. \end{aligned}$$

Then

$$\begin{aligned} &\text{RHS of (7)} - \text{LHS of (7)} \\ \ge &\frac{D}{r} + \frac{D}{s} - \frac{D + D^{-3} - \sqrt{2}}{r+s-3} - \frac{r^3}{2D^3(r-1)} - \frac{s^3}{2D^3(s-1)} - \frac{s}{r} - \frac{r^3}{2s^3}. \end{aligned}$$

Denote the right hand side of above inequality by $B(D,r,s)$. It suffices to prove $B(D,r,s) \ge 0$. Differentiating in D,

$$\frac{\partial B}{\partial D} = \frac{1}{r} + \frac{1}{s} - \frac{1}{r+s-3} + \frac{3}{D^4}\Big[\frac{1}{r+s-3} + \frac{r^3}{2(r-1)} + \frac{s^3}{2(s-1)}\Big].$$

The second group is clearly positive. Moreover,

$$\frac{1}{r} + \frac{1}{s} - \frac{1}{r+s-3} = \frac{r^2+s^2+rs-3r-3s}{rs(r+s-3)} \ge 0,$$

with strict inequality for $(r,s) \neq (2,2)$. Hence $\partial B/\partial D > 0$, so $B(D,r,s)$ is strictly increasing in D, and it suffices to verify $B(s+1,r,s) \ge 0$ since $D \ge s+1$. At $D = s+1$, using $\sqrt{2} > 5/4$, we obtain the lower bound

$B(s+1,r,s) > \Psi(r,s)$, where

$$\Psi(r,s) = 1 + \frac{1}{r} + \frac{1}{s} - \frac{s - \frac{1}{4} + (s+1)^{-3}}{r+s-3} - \frac{r^3}{2(s+1)^3(r-1)} - \frac{s^3}{2(s+1)^3(s-1)} - \frac{r^3}{2s^3}.$$

It remains to verify $\Psi(r,s) > 0$ for all $2 \le r \le s$, $(r,s) \ne (2,2)$. As a rational inequality over the rational numbers, this can be verified by straightforward hand calculation or with symbolic computation tools such as SymPy or Mathematica. For completeness, we briefly outline the computational proof. Clearing the positive denominators reduces the problem to showing that the resulting numerator polynomial in r and s is positive.

- For $r = 2$ (hence $s \ge 3$ since $(r,s) \ne (2,2)$), the numerator factors as $2(s-1)P(s)$ with $P(s) = 2s^7+3s^6-s^5-45s^4-33s^3-4s^2+32s+16$. Discarding the positive factor $2(s-1)$, substituting $s = u+3$ $(u \ge 0)$ yields a polynomial in u whose coefficients are all strictly positive; hence $P(s) > 0$.

- For $r \ge 3$, write $s = r + k$ with $k \ge 0$. The numerator expands as $\sum_{j=0}^{8} A_j(r)\, k^j$. Substituting $r = w + 3$ $(w \ge 0)$, each $A_j(w+3)$ is a polynomial in w with strictly positive coefficients; hence every $A_j(r) > 0$ and the sum is positive.

Thus $\Psi(r,s) > 0$, which implies $B(s+1,r,s) > 0$, and hence $B(D,r,s) > 0$. This establishes (7) for all $(r,s) \ne (2,2)$. Equality holds only for $(r,s) = (2,2)$. ■

Lemma 6. *For $n \ge 6$, if $G \in \mathcal{C}_n$ has the unique universal vertex u, then $DRSO(G) < DRSO(J_n)$.*

Proof. Since $n \ge 6$, we have that $D = n - 1 \ge 5$. Let $H = G - u$. Then $G = K_1 \vee H$, $|V(H)| = D$ and $\Delta(H) \le D - 2$. Since G contains a cycle, H has at least one edge. Thus the DRSO index of G is

$$DRSO(G) = \sum_{v \in V(H)} h\left(\frac{D}{d_G(v)}\right) + \sum_{uv \in E(H)} h(r_{uv}). \tag{8}$$

Noting that (computing by means of a simple contribution method)

$$\sum_{uv\in E(H)}\left(\frac{\ell(d_G(u))}{d_H(u)}+\frac{\ell(d_G(v))}{d_H(v)}\right)=\sum_{v\in V(H)}\ell(d_G(v)),$$

and $d_H(u)+d_H(v)-1\le|E(H)|$ for every $uv\in E(H)$, and applying Lemma 5, we have that

$$\begin{aligned}&\sum_{uv\in E(H)}h(r_{uv})\\&\le\sum_{uv\in E(H)}\left(\frac{\ell(d_G(u))}{d_H(u)}+\frac{\ell(d_G(v))}{d_H(v)}\right)-\sum_{uv\in E(H)}\frac{2\ell(2)-\sqrt{2}}{d_H(u)+d_H(v)-1}\\&\le\left(\sum_{v\in V(H)}\ell(d_G(v))\right)-(2\ell(2)-\sqrt{2}).\end{aligned}$$

Substituting into (8),

$$\begin{aligned}DRSO(G)&=\sum_{v\in V(H)}h\Big(\frac{D}{d_G(v)}\Big)+\sum_{xy\in E(H)}h(r_{xy})\\&\le\left(\sum_{v\in V(H)}h\Big(\frac{D}{d_G(v)}\Big)\right)+\left(\sum_{v\in V(H)}\ell(d_G(v))\right)-2\ell(2)+\sqrt{2}\\&=|V(H)|\,h(D)-2\ell(2)+\sqrt{2}\\&=D\,h(D)-2\left[h(D)-h\Big(\frac{D}{2}\Big)\right]+\sqrt{2}\\&=(D-2)h(D)+2h\left(\frac{D}{2}\right)+\sqrt{2}.\end{aligned}$$

On the other hand, noting that $h(x)<2h(x/2)$, we have that

$$\begin{aligned}DRSO(J_n)&=\sqrt{2}+2(D-1)\,h\left(\frac{D}{2}\right)\\&>(D-2)h(D)+2h\left(\frac{D}{2}\right)+\sqrt{2}.\end{aligned}$$

Hence, $DRSO(G) < DRSO(J_n)$. ■

Conclusion. Lemma 3 shows every maximizer has a universal vertex. Lemma 4 identifies J_n as the unique maximizer among graphs with at least two universal vertices. Lemma 6 proves that any graph G with exactly one universal vertex satisfies $DRSO(G) < DRSO(J_n)$ and therefore cannot be extremal. Hence $J_n = K_2 \vee \overline{K}_{n-2}$ is the unique DRSO maximizer in $\mathcal{C}_n$ for $n \geq 6$. Together with the small cases $n = 4, 5$, this completes the proof of Theorem 2.

4 Discussion and concluding remarks

We have investigated the degree-ratio Sombor index of cycle-containing graphs from two complementary perspectives. On the one hand, computational experiments reveal a rich structural pattern: DRSO and SO exhibit a near-linear correlation, organized into clusters by M_1 and further layered by cyclomatic number c, with the symmetric division deg index governing the vertical dispersion within each cluster (Section 2). These findings extend the cluster-and-line framework of Gutman et al. [4] to cycle-containing molecular graphs.

On the other hand, analytical methods determine the maximum DRSO value in two natural settings: globally (Theorem 1) and for all cycle-containing graphs (Theorem 2). The global maximizer is the star $K_{1,n-1}$, confirming that DRSO indeed rewards extreme degree disparity. For all cycle-containing graphs, $J_n = K_2 \vee \overline{K}_{n-2}$ is the unique maximizer for $n \geq 6$, with K_4 and K_5 as the small exceptions.

Several directions merit further investigation.

1. The relationship between $DRSO(G)$ and other spectral or structural parameters remains unexplored.

2. For molecular graphs ($\Delta \leq 4$), the maximum DRSO may be attained by a different family, since the star exceeds the degree bound when $n > 5$. This restricted problem is of potential chemical interest.

3. The precise relationship between the experimental observations of Section 2 (cluster-and-line structure) and the extremal graphs iden-

tified in Section 3 deserves further theoretical exploration. In particular, the observation that the *SO–DRSO* slope decreases with cyclomatic number may be connected to the structural properties of the extremal graphs.

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