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On the Harary Eccentric Euler Sombor Index of a Graph

Fateme Rezapour, Mohammad Habibi*

Department of Mathematics, Tafresh University, Tafresh, 39518-79611, Iran

rezapour@tafreshu.ac.ir, mhabibi@tafreshu.ac.ir

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Abstract

A novel distance-based topological index, the Harary-eccentric Euler Sombor index, $\mathcal{HE}_{EU}$, is introduced. It combines the reciprocal distance weighting of the Harary index, the eccentricity-based approach, and the geometric structure of the Euler Sombor index. Several basic properties and bounds for this index are established, including inequalities involving eccentric Euler Sombor index, diameter, radius, and Harary index. We prove that among trees and connected graphs of given order, the star and the complete graph are the unique minimizers. Exact values of $\mathcal{HE}_{EU}$ are computed for several familiar graph classes. We also introduce the Harary-eccentric Euler Sombor matrix and investigate its spectral properties.

1 Introduction

In this paper, G is a finite, simple and connected graph with vertex set $V(G)$, edge set $E(G)$, order $n = |V(G)|$ and size $m = |E(G)|$. The distance between two vertices u and v in G is denoted by $d(u, v)$. The eccentricity of a vertex v, denoted by $e(v)$, is defined as $e(v) = \max_{u \in V(G)} d(u, v)$.

*Corresponding author.

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The diameter and radius of G are denoted by $\mathrm{diam}(G)$ and $R(G)$, respectively. The maximum and minimum degrees are denoted by $\Delta(G)$ and $\delta(G)$. A graph is called self-centered if $R(G) = \mathrm{diam}(G)$. In a tree (a connected graph with no cycle) T, a *leaf* is a vertex of degree 1. Let P_n, S_n, $DS_{p,q}$, C_n, K_n, and $K_{p,q}$ $(p+q=n)$ denote the path, the star, the double star, the cycle, the complete, and the complete bipartite graph of order n, respectively. Recall that the *double star* $DS_{p,q}$ is a tree obtained from S_p and S_q by connecting the center of S_p with that of S_q. The join $G = G_1 + G_2$ of graphs G_1 and G_2 is a graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1) \text{ and } y \in V(G_2)\}$. In this work, we use the symbol W_n to denote the wheel graph $C_n + K_1$.

Topological indices are numerical invariants associated with the molecular structure of graphs, playing a significant role in mathematical chemistry, particularly in quantitative structure- property relationships (QSPR) and quantitative structure-activity relationships (QSAR). Among the most studied families of topological indices are the distance-based and degree-based indices.

The Harary index, introduced independently by Plavšić et al. [17] and by Ivashchenko [14], is defined as

$$H(G) = \sum_{\substack{\{u,v\} \subseteq V(G) \\ u \neq v}} \frac{1}{d(u,v)}.$$

Unlike the Wiener index, which sums the distances themselves, the Harary index emphasizes pairs of vertices that are close to each other, making it a valuable tool in the study of molecular connectivity and network analysis. In recent years, many extensions and weighted versions of the Harary index have been investigated, including the additively weighted Harary index [3] and the multiplicatively weighted Harary index [4]. More recently, the combination of the Harary index with Sombor-type indices has been initiated through the study of the Harary-Sombor index [9] and the Harary-Euler Sombor index [8].

On the other hand, the Sombor index, introduced in 2021 [7], initi-

ated a new direction in degree-based topological indices with a geometric interpretation. The Sombor index is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}\,.$$

There are numerous papers devoted to the study of the Sombor index of graphs. The interested reader can consult [1, 2, 10, 12, 18, 19] and the references cited therein for various extremal results, bounds, and chemical applications of this index. A natural generalization of the Sombor index, known as the Euler Sombor index, was recently introduced by replacing the Euclidean norm with Euler's approximation for the perimeter of an ellipse:

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}\,.$$

This modification, proposed by Tang, Li, and Deng [20], has been shown to possess interesting extremal properties and has been studied for various graph classes [5, 6].

In parallel, eccentricity-based indices have proven to be powerful descriptors in chemical graph theory, as they capture the global position of vertices within the graph. The eccentric connectivity index, the eccentric Zagreb indices, and the eccentric Sombor index are among the most prominent examples in this category. The eccentric Sombor index, defined as

$$\mathcal{E}_{SO}(G) = \sum_{uv \in E(G)} \sqrt{e(u)^2 + e(v)^2},$$

was introduced by Kulli and has been studied for its predictive capabilities [15]. In a subsequent work, Habibi et al. [11] investigated the eccentric version of the Euler Sombor index, denoted by $\mathcal{E}_{EU}(G)$, where the vertex degrees are replaced by eccentricities. More precisely, the eccentric Euler Sombor index is defined as

$$\mathcal{E}_{EU}(G) = \sum_{uv \in E(G)} \sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}\,.$$

This index serves as a bridge between the eccentricity-based and geometric approaches to topological indices. Motivated by these developments, in this paper we introduce a novel topological index that combines the Harary distance weighting, the eccentricity-based approach, and the geometric structure of the Euler Sombor index. We define the *Harary-eccentric Euler Sombor index* of a graph G as

$$\mathcal{HE}_{EU}(G) = \sum_{\substack{\{u,v\}\subseteq V(G)\\ u\neq v}} \frac{\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}}{d(u,v)}.$$

This index integrates three fundamental concepts in graph theory: the reciprocal distance weighting from the Harary index, the eccentricity-based framework from eccentric Sombor-type indices, and the elliptic geometric term from the Euler Sombor index. In fact, our definition extends the eccentric Euler Sombor index from edges to all pairs of vertices, incorporating the Harary weighting scheme. Although $\mathcal{HE}_{EU}(G)$ is defined using an expression of algebraic form similar to degree-based indices, it is important to note that this is a distance-based topological index, since the summation runs over all pairs of vertices and incorporates the distance $d(u,v)$ in the denominator. Thus, $\mathcal{HE}_{EU}(G)$ does not belong to the family of vertex-degree-based topological indices such as the Sombor or Euler-Sombor indices.

We establish several basic properties and bounds for this index, including inequalities involving the eccentricity-based Euler Sombor index $\mathcal{E}_{EU}(G)$, the diameter, the radius, and the Harary index $H(G)$. We also obtain tight bounds for graphs with given order and size, and we characterize the extremal cases for self-centered graphs. Exact values of $\mathcal{HE}_{EU}(G)$ are computed for several well-known classes of graphs, including complete graphs, cycles, stars, double-stars, complete bipartite graphs, and wheel graphs. Furthermore, we introduce the Harary-eccentric Euler Sombor matrix $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$ and investigate its spectral properties. Using the Rayleigh–Ritz variational principle, we establish bounds on the largest and smallest eigenvalues of this matrix in terms of the $\mathcal{HE}_{EU}$ index, and we characterize the graphs for which equality holds. Finally, we prove ex-

tremal results for the $\mathcal{HE}_{EU}$ index. We show that among all trees of order n, the star S_n uniquely minimizes the index, and among all connected graphs of order n, the complete graph K_n is the unique minimizer. These results reveal an interesting inversion compared to the classical Harary index, where the roles of the star and the path are reversed.

2 Basic properties and bounds

We start by establishing some fundamental properties and bounds for the newly defined Harary-eccentric Euler Sombor index.

Lemma 1. *For any connected graph G, the following inequality holds:*

$$\mathcal{E}_{EU}(G) \leq \mathcal{HE}_{EU}(G) < \sum_{\substack{\{u,v\}\subseteq V(G)\\ u\neq v}} \frac{e(u)+e(v)}{d(u,v)}.$$

Moreover, the first inequality becomes an equality if and only if $G \cong K_n$.

Proof. The upper bound follows from the strict inequality

$$\sqrt{e(u)^2+e(v)^2+e(u)e(v)} < e(u)+e(v).$$

The lower bound is immediate since edge contributions in $\mathcal{HE}_{EU}(G)$ coincide with those of $\mathcal{E}_{EU}(G)$, and equality occurs precisely when there are no non-adjacent pairs, i.e., $G \cong K_n$. ■

Lemma 2. *For any connected graph G, the following inequalities hold:*

$$\frac{\sqrt{3}}{\operatorname{diam}(G)} \sum_{\substack{\{u,v\}\subseteq V(G)\\ u\neq v}} \frac{e(u)e(v)}{d(u,v)} \leq \mathcal{HE}_{EU}(G) \leq \frac{\sqrt{3}}{R(G)} \sum_{\substack{\{u,v\}\subseteq V(G)\\ u\neq v}} \frac{e(u)e(v)}{d(u,v)}.$$

Moreover, equality holds in both inequalities if and only if G is a self-centered graph.

Proof. Using the definition of $\mathcal{HE}_{EU}(G)$ and the identity

$$\sqrt{e(u)^2+e(v)^2+e(u)e(v)} = e(u)e(v)\sqrt{\frac{1}{e(u)^2}+\frac{1}{e(v)^2}+\frac{1}{e(u)e(v)}},$$

we obtain

$$\mathcal{HE}_{EU}(G) = \sum_{\substack{\{u,v\}\subseteq V(G)\\ u\neq v}} \frac{e(u)e(v)}{d(u,v)}\sqrt{\frac{1}{e(u)^2}+\frac{1}{e(v)^2}+\frac{1}{e(u)e(v)}}.$$

Since $R(G) \leq e(u), e(v) \leq \operatorname{diam}(G)$, we have

$$\frac{\sqrt{3}}{\operatorname{diam}(G)} \leq \sqrt{\frac{1}{e(u)^2}+\frac{1}{e(v)^2}+\frac{1}{e(u)e(v)}} \leq \frac{\sqrt{3}}{R(G)}.$$

Multiplying by $\frac{e(u)e(v)}{d(u,v)}$ and summing over all unordered pairs yields the result. Equality holds in both inequalities if and only if

$$e(u) = e(v) = R(G) = \operatorname{diam}(G)$$

for all pairs, which is equivalent to G being self-centered. ■

Theorem 1. *Let G be a connected graph of order n and size m. Then*

$$\frac{\sqrt{3}R(G)}{\operatorname{diam}(G)}[\binom{n}{2} - m] \leq \mathcal{HE}_{EU}(G) - \mathcal{E}_{EU}(G) \leq \frac{\sqrt{3}}{2}\operatorname{diam}(G)[\binom{n}{2} - m].$$

In the particular case when G is a self-centered graph, we have

$$\mathcal{E}_{EU}(G) + \sqrt{3}[\binom{n}{2} - m] \leq \mathcal{HE}_{EU}(G) \leq \mathcal{E}_{EU}(G) + \frac{\sqrt{3}}{2}\operatorname{diam}(G)[\binom{n}{2} - m]$$

and equality holds in both inequalities if and only if either $G \cong K_n$ or G is a self-centered graph with $\operatorname{diam}(G) = 2$.

Proof. We split the sum defining $\mathcal{HE}_{EU}(G)$ into two parts: edges and

non-edges:

$$\sum_{uv\in E(G)} \sqrt{e(u)^2+e(v)^2+e(u)e(v)} + \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} \frac{\sqrt{e(u)^2+e(v)^2+e(u)e(v)}}{d(u,v)}.$$

The first sum is exactly $\mathcal{E}_{EU}(G)$. For the second sum, since $R(G) \leq e(u), e(v) \leq \mathrm{diam}(G)$ and $2 \leq d(u,v) \leq \mathrm{diam}(G)$ for every non-edge $\{u,v\}$, we have

$$\frac{\sqrt{3}R(G)}{\mathrm{diam}(G)} \leq \frac{\sqrt{e(u)^2+e(v)^2+e(u)e(v)}}{d(u,v)} \leq \frac{\sqrt{3}\mathrm{diam}(G)}{2}.$$

Summing over all $\binom{n}{2} - m$ non-edges and adding $\mathcal{E}_{EU}(G)$ yields the result. In the self-centered case, $R(G) = \mathrm{diam}(G)$, and equality holds in both inequalities if and only if either $G \cong K_n$ or G is a non-complete self-centered graph with $\mathrm{diam}(G) = 2$. ■

Lemma 3. *Let G be a connected graph of order n. Then*

$$\sqrt{3}R(G)H(G) \leq \mathcal{HE}_{EU}(G) \leq \sqrt{3}\,\mathrm{diam}(G)H(G).$$

In the particular case when G is a self-centered graph, we have

$$\mathcal{HE}_{EU}(G) = \sqrt{3}R(G)H(G).$$

Proof. The proof is straightforward by considering $R(G) \leq e(u), e(v) \leq \mathrm{diam}(G)$. ■

Remark 1. *Using the well-known bound* $\mathrm{diam}(G) \leq n - \delta(G)$*, the above theorem yields the weaker inequality*

$$\mathcal{HE}_{EU}(G) \leq \sqrt{3}(n - \delta(G))H(G).$$

Also, in Proposition 1(ii) of [11], the authors stated

$$\sqrt{3}m(n - \Delta(G)) \leq \mathcal{E}_{EU}(G) \leq \sqrt{3}m(n - \delta(G)).$$

We note that the lower bound of this inequality is incorrect, since $n -$

$\Delta(G) \leq R(G)$ *does not hold in general. The correct form is only the upper bound:*

$$\mathcal{E}_{EU}(G) \leq \sqrt{3}m(n - \delta(G)).$$

3 Exact values for special graph classes

In this section, we compute the $\mathcal{HE}_{EU}$ index for some well-known classes of graphs.

Example 1. (i) $\mathcal{HE}_{EU}(K_n) = \mathcal{E}_{EU}(K_n) = \frac{\sqrt{3}}{2}n(n-1)$.
(ii) The cycle graph with n vertices C_n $(n \geq 3)$ is self-centered with $R(C_n) = \lfloor \frac{n}{2} \rfloor$. By Lemma 3, we have

$$\mathcal{HE}_{EU}(C_n) = \sqrt{3}\left\lfloor \frac{n}{2} \right\rfloor H(C_n) = \begin{cases} \frac{\sqrt{3}}{2}n(n-1)H_{\frac{n-1}{2}} & n \text{ is odd} \\ \frac{\sqrt{3}}{2}n(nH_{\frac{n}{2}} - 1) & n \text{ is even} \end{cases},$$

where $H_t = \sum_{k=1}^{t} \frac{1}{k}$ is the t-th harmonic number.
(iii) For the star graph S_n with n vertices, we have $e(c) = 1$ for the center vertex c and $e(v) = 2$ for every leaf v. Thus,

$$\mathcal{HE}_{EU}(S_n) = \sqrt{7}(n-1) + \sqrt{3}\binom{n-1}{2} = \sqrt{7}(n-1) + \frac{\sqrt{3}}{2}(n-1)(n-2).$$

(iv) For the double-star graph $DS_{p,q}$ with $p + q = n$, let u and v be the two adjacent central vertices. Then $e(u) = e(v) = 2$, and every leaf has eccentricity 3. Suppose $p, q \geq 2$, then a direct computation gives

$$\begin{aligned} \mathcal{HE}_{EU}(DS_{p,q}) &= (p+q-2)\sqrt{19} + 2\sqrt{3} + \frac{3\sqrt{3}}{2}[\binom{p-1}{2} + \binom{q-1}{2}] \\ &\quad + (p+q-2)\frac{\sqrt{19}}{2} + (p-1)(q-1)\sqrt{3} \\ &= \frac{3n-6}{2}\sqrt{19} + 2\sqrt{3} + \frac{3\sqrt{3}}{4}(p^2+q^2-3n+4) \\ &\quad + (pq-n+1)\sqrt{3}. \\ &= \frac{3n-6}{2}\sqrt{19} + \frac{3\sqrt{3}}{4}(p^2+q^2+\frac{4}{3}pq) - \frac{13\sqrt{3}}{4}n + 6\sqrt{3} \end{aligned}$$

If $p > q = 2$, then

$$\begin{aligned}\mathcal{HE}_{EU}(DS_{p,2}) &= (p-1+1)\sqrt{19} + 2\sqrt{3} + \frac{3\sqrt{3}}{2}\binom{p-1}{2} + (p-1+1)\frac{\sqrt{19}}{2} \\ &\quad + (p-1)\sqrt{3} \\ &= \frac{3n-6}{2}\sqrt{19} + 2\sqrt{3} + \frac{3\sqrt{3}}{4}(n^2-7n+12) + (n-3)\sqrt{3}. \\ &= \frac{3n-6}{2}\sqrt{19} + \frac{3\sqrt{3}}{4}(n^2 - \frac{17}{3}n) + 8\sqrt{3}.\end{aligned}$$

If $p = q = 2$, then

$$\mathcal{HE}_{EU}(DS_{2,2}) = \mathcal{HE}_{EU}(P_4) = 3\sqrt{19} + 3\sqrt{3}.$$

(v) For the complete bipartite graph $K_{p,q}$ of order n with $p, q \geq 2$, we have $e(u) = 2$ for every vertex $u \in V(K_{p,q})$. Thus by Theorem 1 we have:

$$\mathcal{HE}_{EU}(K_{p,q}) = \mathcal{E}_{EU}(K_{p,q}) + \frac{2\sqrt{3}}{2}[\binom{p}{2} + \binom{q}{2}] = 2\sqrt{3}pq + \frac{\sqrt{3}}{2}(p^2+q^2-n).$$

(vi) For the wheel graph W_n with $n \geq 4$, let c be the central vertex and let $v_1, \ldots, v_{n-1}$ be the vertices of the cycle. Then $e(c) = 1$ and $e(v_i) = 2$ for all i. Hence,

$$\begin{aligned}\mathcal{HE}_{EU}(W_n) =& (n-1)\sqrt{7} + 2\sqrt{3}(n-1) + [\binom{n-1}{2} - (n-1)]\frac{2\sqrt{3}}{2} \\ =& (n-1)\sqrt{7} + \frac{n(n-1)}{2}\sqrt{3}.\end{aligned}$$

For $n = 3$, $W_3 \cong K_4$, and so $\mathcal{HE}_{EU}(W_3) = 6\sqrt{3}$.
(vii) The Petersen graph P is a well-known 3-regular graph on 10 vertices with diameter 2. It is self-centered, so $e(v) = 2$ for every vertex $v \in V(P)$. Since P has $m = 15$ edges, by Theorem 1 we have:

$$\mathcal{HE}_{EU}(P) = \mathcal{E}_{EU}(P) + \sqrt{3}\left[\binom{10}{2} - 15\right] = 30\sqrt{3} + \sqrt{3}(45-15) = 60\sqrt{3}.$$

Note that the first equality holds because P is self-centered with diameter

2, and the second equality follows from $\mathcal{E}_{EU}(P) = 30\sqrt{3}$ since every edge connects two vertices of eccentricity 2.

4 Spectral properties of the Harary-eccentric Euler Sombor matrix

Now, we introduce the Harary-eccentric Euler Sombor matrix of a graph G, and establish some spectral bounds for its eigenvalues. Throughout this section, let G be a connected graph of order n with vertex set $V(G) = \{v_1, \ldots, v_n\}$.

Definition 1. The *Harary-eccentric Euler Sombor matrix* of a graph G, denoted by $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$, is an $n \times n$ matrix whose (i, j)-entry is given by

$$m_{ij} = \begin{cases} \dfrac{\sqrt{e(v_i)^2 + e(v_j)^2 + e(v_i)e(v_j)}}{d(v_i, v_j)} & i \neq j \\ 0 & i = j \end{cases}.$$

We denote the eigenvalues of $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$ by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

The following classical result will be used in the proof of the next theorem.

Theorem 2 (Rayleigh-Ritz variational principle)**.** *[13,16] Let $\mathcal{M}$ be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then for any non-zero real vector $X \in \mathbb{R}^n$,*

$$\lambda_n \leq \frac{X^T \mathcal{M} X}{X^T X} \leq \lambda_1.$$

Moreover, equality on the left (resp. right) holds if and only if X is an eigenvector corresponding to λ_n (resp. λ_1).

Theorem 3. *Let G be a connected graph of order $n \geq 2$. Then:*

(i) $\mu_1 \geq \dfrac{2}{n(n-1)}\mathcal{HE}_{EU}(G)$. *Moreover, equality holds if and only if* $G \cong K_2$. *In this case,* $\mu_1 = \mathcal{HE}_{EU}(K_2) = \sqrt{3}$.

(ii) $\mu_n \leq -\dfrac{2}{n(n-1)}\mathcal{HE}_{EU}(G)$. *Moreover, equality holds if and only if* $G \cong K_n$. *In this case,* $\mu_n = \sqrt{3}(1-n)$ *and* $\mathcal{HE}_{EU}(K_n) = \frac{\sqrt{3}}{2}n(n-1)$.

Proof. Let $1 \leq r < s \leq n$ be arbitrary distinct indices. Define the vector $X_{r,s} = (x_1, \ldots, x_n)^T$ by

$$x_i = \begin{cases} 1 & i \in \{r, s\} \\ 0 & \text{otherwise} \end{cases}.$$

By the Rayleigh-Ritz principle applied to $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$, we have

$$\mu_1 \geq \frac{X_{r,s}^T \mathcal{M}_{\mathcal{HE}_{EU}}(G) X_{r,s}}{X_{r,s}^T X_{r,s}}.$$

A direct computation gives $X_{r,s}^T \mathcal{M}_{\mathcal{HE}_{EU}}(G) X_{r,s} = 2m_{rs}$ and $X_{r,s}^T X_{r,s} = 2$. Hence,

$$\mu_1 \geq h_{rs}.$$

Since this holds for every pair $r \neq s$, summing over all unordered pairs yields

$$\binom{n}{2}\mu_1 \geq \sum_{1 \leq r < s \leq n} m_{rs} = \mathcal{HE}_{EU}(G),$$

which proves the inequality in (i). For the equality case, suppose $\mu_1 = \frac{2}{n(n-1)}\mathcal{HE}_{EU}(G)$. Then equality must hold in every inequality above, so $X_{r,s}$ is an eigenvector of $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$ corresponding to μ_1 for every $r < s$. By the Perron-Frobenius theorem, the entries of the eigenvector corresponding to the largest eigenvalue must be strictly positive. Thus, $X_{r,s}$ must have all positive entries, which forces $n = 2$. Therefore G must be a connected graph on two vertices, i.e., $G \cong K_2$. Conversely, if $G \cong K_2$, a direct computation gives $\mu_1 = \mathcal{HE}_{EU}(K_2) = \sqrt{3}$.

To prove (ii), define for each $1 \leq r < s \leq n$ the vector $Y_{r,s} = (y_1, \ldots, y_n)^T$

by

$$y_i = \begin{cases} -1 & i = r \\ 1 & i = s \\ 0 & \text{otherwise} \end{cases}.$$

Applying Rayleigh-Ritz again gives

$$\mu_n \leq \frac{Y_{r,s}^T \mathcal{M}_{\mathcal{HE}_{EU}}(G) Y_{r,s}}{Y_{r,s}^T Y_{r,s}}.$$

Since $Y_{r,s}^T \mathcal{M}_{\mathcal{HE}_{EU}}(G) Y_{r,s} = -2m_{rs}$ and $Y_{r,s}^T Y_{r,s} = 2$, we obtain

$$\mu_n \leq -m_{rs}.$$

Summing over all unordered pairs yields

$$\binom{n}{2} \mu_n \leq -\mathcal{HE}_{EU}(G),$$

which is equivalent to (ii).
For equality in (ii), suppose $\mu_n = -\frac{2}{n(n-1)} \mathcal{HE}_{EU}(G)$. Then equality holds in each step, so every $Y_{r,s}$ is an eigenvector of $\mathcal{M}_{\mathcal{HE}_{EU}}(G)$ corresponding to μ_n. Hence, for every $r < s$,

$$\mathcal{M}_{\mathcal{HE}_{EU}}(G) Y_{r,s} = \mu_n Y_{r,s}.$$

This implies that $m_{rs} = -\mu_n$ for all $r < s$. In particular, for any vertex u, let v_ℓ and v_t be two neighbors of u. Then

$$\sqrt{e(u)^2 + e(v_\ell)^2 + e(u)e(v_\ell)} = \sqrt{e(u)^2 + e(v_t)^2 + e(u)e(v_t)}.$$

Squaring and simplifying gives

$$e(v_\ell)^2 + e(u)e(v_\ell) = e(v_t)^2 + e(u)e(v_t).$$

If $e(v_\ell) \neq e(v_t)$, then $e(v_\ell) + e(v_t) = -e(u)$, a contradiction since all eccentricities are positive. Hence $e(v_\ell) = e(v_t)$. Thus, all neighbors of

any vertex have the same eccentricity. By the connectedness of G, this implies that all vertices of G have the same eccentricity, i.e., G is self-centered. Furthermore, since $m_{rs} = -\mu_n$ is constant for all $r \neq s$, we have $d(v_r, v_s) = 1$ for every pair. Therefore G is a complete graph. Hence $G \cong K_n$. Conversely, if $G \cong K_n$, then $m_{ij} = \sqrt{3}$ for all $i \neq j$. The matrix $\mathcal{M}_{\mathcal{HE}_{EU}}(K_n)$ has eigenvalues $\sqrt{3}(n-1)$ (with multiplicity 1) and $-\sqrt{3}$ (with multiplicity $n-1$). Thus $\mu_n = -\sqrt{3} = -\frac{2}{n(n-1)}\mathcal{HE}_{EU}(K_n)$, since $\mathcal{HE}_{EU}(K_n) = \frac{\sqrt{3}}{2}n(n-1)$ and the proof is complete. ■

5 Extremal results

In this section, we prove that the star uniquely minimizes the $\mathcal{HE}_{EU}$ index among trees, and the complete graph uniquely minimizes it among all connected graphs.

Theorem 4. (i) *If T is a tree of order n, then $\mathcal{HE}_{EU}(T) \geq \mathcal{HE}_{EU}(S_n)$, with equality if and only if $T \cong S_n$.*
(ii) *If G is an arbitrary connected graph of order n, then $\mathcal{HE}_{EU}(G) \geq \mathcal{HE}_{EU}(K_n)$, with equality if and only if $G \cong K_n$.*

Proof. (i) For $n = 2$, the only tree is $P_2 \cong S_2$ and the result is trivial. Also, for $n = 3$, the only tree is $P_3 \cong S_3$ and so the result is trivial. Now, assume $n \geq 4$ and let T be a tree of order n. We split $\mathcal{HE}_{EU}(T)$ into two parts:

$$\sum_{uv \in E(T)} \sqrt{e(u)^2 + e(v)^2 + e(u)e(v)} + \sum_{\substack{\{u,v\} \subseteq V(T) \\ uv \notin E(T)}} \frac{\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}}{d(u,v)}.$$

For every edge $uv \in E(T)$, since T is triangle free, we have $(e(u), e(v)) \neq (1,1)$. Hence $e(u) + e(v) \geq 3$ and so

$$\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)} \geq \sqrt{7}.$$

Thus,

$$\sum_{uv \in E(T)} \sqrt{e(u)^2 + e(v)^2 + e(u)e(v)} \geq (n-1)\sqrt{7}.$$

For every non-adjacent pair $\{u, v\}$, since $e(u), e(v) \geq d(u, v)$, we have

$$\frac{\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}}{d(u, v)} \geq \sqrt{3}.$$

There are exactly $\binom{n}{2} - (n-1) = \binom{n-1}{2}$ non-adjacent pairs in any tree, so

$$\sum_{\substack{\{u,v\} \subseteq V(T) \\ uv \notin E(T)}} \frac{\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}}{d(u, v)} \geq \sqrt{3}\binom{n-1}{2}.$$

Combining the two bounds gives

$$\mathcal{HE}_{EU}(T) \geq (n-1)\sqrt{7} + \frac{\sqrt{3}}{2}(n-1)(n-2) = \mathcal{HE}_{EU}(S_n).$$

Equality holds in the first inequality if and only if every edge has eccentricity pair $(1, 2)$. It concludes that T has a unique vertex with eccentricity 1 and hence $T \cong S_n$.

(ii) Let G be any connected graph of order n. For every pair of distinct vertices u, v, we have $e(u), e(v) \geq d(u, v)$. Hence,

$$\frac{\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}}{d(u, v)} \geq \sqrt{3}.$$

Summing over all $\binom{n}{2}$ pairs gives

$$\mathcal{HE}_{EU}(G) \geq \sqrt{3}\binom{n}{2} = \mathcal{HE}_{EU}(K_n).$$

Equality holds if and only if $e(u) = e(v) = 1$ for every pair, which is equivalent to $G \cong K_n$. ■

6 Chemical motivation and potential applications

6.1 Why a new index?

The rapid development of topological indices in mathematical chemistry is driven by the fact that no single index can fully capture the structural complexity of molecules. Different indices emphasize different aspects of molecular structure:

- **Degree-based indices** (e.g., Sombor index, Zagreb indices) focus on local connectivity and bonding patterns.
- **Distance-based indices** (e.g., Wiener index, Harary index) emphasize the global separation between atoms.
- **Eccentricity-based indices** capture the positional importance of atoms within the molecule, distinguishing between central and peripheral atoms.

Unlike degree-based indices, which only consider local connectivity, the Harary-eccentric Euler Sombor index $\mathcal{HE}_{EU}(G)$ is designed to integrate distance-based and eccentricity-based approaches by simultaneously incorporating:

1. **Reciprocal distance weighting**: Pairs of atoms that are close to each other contribute more to the index. This is chemically meaningful because nearby atoms interact more strongly through space and across bonds, influencing properties such as boiling point, solubility, and molecular reactivity.
2. **Eccentricity information**: Atoms with high eccentricity are located at the periphery of the molecule, while those with low eccentricity are centrally positioned. This captures molecular shape and compactness, which are critical for understanding steric effects, molecular volume, and surface area.

3. **The elliptic geometric term**: The expression

$$\sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}$$

provides a smooth, non-linear combination of eccentricities. Recent studies have shown that such geometric terms correlate well with molecular stability and electronic properties [6, 20].

6.2 Comparison with related indices

To understand the novelty of $\mathcal{HE}_{EU}(G)$, it is useful to compare it with related indices. Table 1 summarizes the key differences.

Table 1. Comparison of $\mathcal{HE}_{EU}(G)$ with related indices

Index	Definition	Type
$H(G)$	$\sum_{\{u,v\}} \frac{1}{d(u,v)}$	Distance-based, all pairs
$\mathcal{E}_{SO}(G)$	$\sum_{uv\in E} \sqrt{e(u)^2 + e(v)^2}$	Eccentricity-based, edge-only
$\mathcal{E}_{EU}(G)$	$\sum_{uv\in E} \sqrt{e(u)^2 + e(v)^2 + e(u)e(v)}$	Eccentricity-based, edge-only
$\mathcal{HSO}(G)$	$\sum_{\{u,v\}} \frac{\sqrt{d(u)^2+d(v)^2}}{d(u,v)}$	Degree-based, all pairs
$\mathcal{HE}_{EU}(G)$	$\sum_{\{u,v\}} \frac{\sqrt{e(u)^2+e(v)^2+e(u)e(v)}}{d(u,v)}$	Eccentricity-based, all pairs

As Table 1 shows, $\mathcal{HE}_{EU}(G)$ is the only index that simultaneously considers all vertex pairs, uses eccentricity information, and incorporates the elliptic geometric term.

6.3 Theoretical illustrations for small molecules

To provide concrete intuition, we compute $\mathcal{HE}_{EU}(G)$ for several simple molecular graphs. These examples demonstrate how the index reflects structural differences such as branching and cyclicity.

Key observations from Table 2:

1. **Branching effect**: Isobutane (S_4) has a significantly lower $\mathcal{HE}_{EU}$ value than n-butane (P_4). This is consistent with the known fact

Table 2. $\mathcal{HE}_{EU}(G)$ values for simple molecular graphs

Molecule	**Graph**	$\mathcal{HE}_{EU}(G)$	**Interpretation**
Ethane	K_2	$\sqrt{3}$	Two equiv. C atoms
Propane	P_3	$2\sqrt{7}+\sqrt{3}$	Linear chain, 3 atoms
n-Butane	P_4	$3\sqrt{19}+3\sqrt{3}$	Linear chain, 4 atoms
Isobutane	S_4	$3\sqrt{7}+3\sqrt{3}$	Branched (star)
Cyclohexane	C_6	$30\sqrt{3}$	Cyclic structure

that branched alkanes have different physical properties (e.g., lower boiling points, higher octane numbers) than their linear isomers.

2. **Chain length effect**: As the carbon chain length increases, the index increases, reflecting the growing molecular size.

3. **Cyclic vs. acyclic**: Cyclohexane (C_6) and its acyclic counterpart have different $\mathcal{HE}_{EU}$ values, reflecting the structural distinction between ring and chain compounds.

6.4 Potential for QSAR/QSPR studies

Based on the properties discussed above, $\mathcal{HE}_{EU}(G)$ has potential utility in Quantitative Structure-Property Relationship (QSPR) and Quantitative Structure-Activity Relationship (QSAR) studies for several reasons:

- **Boiling points of alkanes**: The index's sensitivity to branching suggests it could model boiling point trends, where branched isomers have lower values than linear ones.

- **Molecular shape and surface area**: The eccentricity-based nature of the index makes it sensitive to molecular compactness, which is relevant for properties such as density and refractive index.

- **Position of substituents**: In substituted aromatic compounds, the eccentricity of atoms changes depending on substitution patterns, making the index potentially useful for modeling biological activity where the position of functional groups matters.

- **All-pairs weighting**: By considering all atom pairs, the index accounts for both short-range and long-range interactions, which is important for properties like polarizability and van der Waals forces.

6.5 Limitations and future work

We acknowledge the following limitations:

1. The index is defined for simple graphs and does not directly account for bond orders (e.g., double bonds in aromatic compounds) or heteroatoms. Extensions to weighted graphs would be needed for full chemical applicability.

2. A comprehensive QSAR/QSPR study on a large dataset of organic compounds is necessary to validate the predictive power of $\mathcal{HE}_{EU}(G)$. Such a study is beyond the scope of this theoretical paper.

3. The index, like all topological indices, captures structural information but cannot replace quantum-chemical calculations for properties that depend on electronic distribution.

Despite these limitations, the theoretical properties established in this paper, combined with the structural sensitivity demonstrated in the examples above, suggest that $\mathcal{HE}_{EU}(G)$ is a promising candidate for future applications in mathematical chemistry and chemoinformatics.

7 Conclusion

In this paper, we introduced the Harary-eccentric Euler Sombor index $\mathcal{HE}_{EU}(G)$ and established its basic properties, bounds, and extremal values. We computed exact formulas for several well-known graph classes and introduced the associated matrix with spectral bounds. Among the main results, we proved that the star uniquely minimizes the index among trees, and the complete graph uniquely minimizes it among all connected graphs. The chemical motivation and potential applications of the index were discussed, highlighting its sensitivity to structural features such as

branching and molecular shape. Future work may focus on characterizing graphs with maximum $\mathcal{HE}_{EU}$ index, studying the index for additional graph classes, performing comprehensive QSAR/QSPR analyses, and exploring its chemical applications.

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