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Extremal Local Values of Vertex-Degree Function Indices

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Abstract

Let $n \geq 2$ and let f be a real-valued function on $\{1, \ldots, n-1\}$. For a graph G of order n without isolated vertices, the local value at u is

$$f_G(u) = (1 - d_G(u))f(d_G(u)) + \sum_{w \in N_G(u)} f(d_G(w)).$$

These values depend on the degrees of a vertex and its neighbors, and their sum equals the vertex-degree function index induced by f. For every monotone f, we determine the exact minimum and maximum over all pairs (G, u) and characterize all equality cases. One extremum reduces to an optimization over the degree of u; the other is attained at the center of a star. For the local zeroth-order general Randić index, we obtain both extrema for every nonzero exponent. When $n \geq 3$, the degree optimization reduces to comparing the values at the floor and ceiling of a unique critical point. We also correct an earlier upper bound for general graphs and prove that it remains valid for trees. More generally, we obtain sharp fixed-degree bounds on trees using concentration and balancing of neighbor degrees, and characterize equality. These bounds determine both global extrema on trees.

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1 Introduction

Throughout this paper, all graphs are finite, simple, and undirected. We study local values on graphs without isolated vertices. We write $V(G)$ and $E(G)$ for the vertex and edge sets of a graph G. The degree and neighborhood of a vertex u are denoted by $d_G(u)$ and $N_G(u)$; when the graph is clear, we write d_u and $N(u)$. We use standard terminology as in [2].

Let $n \geq 2$ and let $f : \{1, \ldots, n-1\} \to \mathbb{R}$. The corresponding vertex-degree function index is

$$f(G) = \sum_{u \in V(G)} f(d_G(u)).$$

Extremal problems for sums of functions of vertex degrees include the work of Linial and Rozenman [12] and have been studied under various monotonicity, convexity, and structural assumptions; see, for example, [3, 5, 9, 14, 15]. Related global extremal problems on trees have also been studied via branching operations [4].

An important example is the zeroth-order general Randić index, also known as the general first Zagreb index and considered by Li and Zheng [11]; see also the early extremal study of Hu et al. [10] and the survey in [1]:

$$R_\alpha(G) = \sum_{u \in V(G)} d_G(u)^\alpha, \qquad \alpha \in \mathbb{R}.$$

We suppress the customary superscript 0 from the notation. The cases $\alpha = 2$ and $\alpha = 3$ give the first Zagreb index [8] and the forgotten index [7], respectively.

The local value of a vertex-degree function index was introduced in [13]. It is defined by

$$f_G(u) = (1 - d_u) f(d_u) + \sum_{w \in N(u)} f(d_w),$$

or, equivalently,

$$f_G(u) = f(d_u) + \sum_{w \in N(u)} \big(f(d_w) - f(d_u)\big). \tag{1}$$

Each edge contributes opposite corrections at its two ends. Hence

$$\sum_{u \in V(G)} f_G(u) = f(G).$$

The local value therefore preserves the total index while taking neighbor degrees into account. It can distinguish vertices of the same degree, and in a regular graph it reduces to $f_G(u) = f(d_u)$.

Our first objective is to determine both global extrema of $f_G(u)$ for every monotone function f. At a fixed degree $d_G(u) = k$, the largest local value for nondecreasing f, and the smallest for nonincreasing f, is attained when all neighbors of u are universal. This gives

$$\Phi_{f,n}(k) = (1-k)f(k) + kf(n-1).$$

The corresponding global extremum is obtained by optimizing this expression over $1 \le k \le n-1$. The opposite extremum is

$$\Lambda_{f,n} = (2-n)f(n-1) + (n-1)f(1).$$

It is attained at the center of a star. We characterize all equality cases, including those arising when f has constant intervals. For strictly monotone functions and $n \ge 3$, only the star with its center distinguished attains $\Lambda_{f,n}$.

For $f(x) = x^\alpha$, these results determine both extrema of $(R_\alpha)_G(u)$ for every $\alpha \neq 0$. The bound given by $\Lambda_{f,n}$ was proved in [13, Theorems 4 and 5]. Das [6, Theorems 3 and 4] obtained the bounds corresponding to $\Phi_{f,n}$ for $0 < \alpha < 1$ and $-1 < \alpha < 0$. He also studied comparisons of local values under induced subgraphs and between two vertices. Here we obtain the degree bounds for arbitrary monotone functions, prove their attainability, and characterize all extremal pairs. For power functions, we

treat all nonzero exponents and show that the optimal integer degree is among the floor and ceiling of a unique critical point when $n \geq 3$.

We also correct the upper bound $(R_\alpha)_G(u) \leq (n-1)^\alpha$, stated for $\alpha \geq 1$ in [13, Theorem 6]. An infinite family of counterexamples shows that this bound fails for general graphs. Its conclusion does hold for trees, where the neighbor degrees satisfy

$$\sum_{w \in N_T(u)} d_T(w) \leq n - 1.$$

This constraint leads to sharp fixed-degree bounds through an elementary integer-optimization argument.

Section 2 gives the general monotonicity results. Section 3 treats the local zeroth-order general Randić index, including the correction and explicit examples. Section 4 determines sharp bounds for trees. The final section discusses a parametric extension and a question about the changes in its optimal degree.

2 Exact extremal values for monotone functions

For $n \geq 2$, let $\mathcal{G}_n$ denote the family of all simple graphs of order n without isolated vertices. A vertex of degree $n-1$ is *universal*, and a vertex of degree 1 is *pendant*. We write $S_n = K_{1,n-1}$ for the star; when $n = 2$, either vertex may be regarded as its center. For vertex-disjoint graphs G_1, G_2, their join $G_1 \vee G_2$ is obtained by adding every edge between the two vertex sets. We write $\overline{K}_t$ for the edgeless graph of order t.

Fix $f : \{1, \ldots, n-1\} \to \mathbb{R}$. We seek the minimum and maximum of $f_G(u)$ over all $G \in \mathcal{G}_n$ and $u \in V(G)$, together with all pairs attaining them.

We begin by fixing the degree of the distinguished vertex. For $1 \leq k \leq n-1$, define

$$\Phi_{f,n}(k) = (1-k)f(k) + kf(n-1). \tag{2}$$

Theorem 1. *Let $G \in \mathcal{G}_n$ and $d_G(u) = k$.*

(i) *If f is nondecreasing, then $f_G(u) \le \Phi_{f,n}(k)$.*

(ii) *If f is nonincreasing, then $f_G(u) \ge \Phi_{f,n}(k)$.*

In either case, equality holds if and only if

$$f(d_G(w)) = f(n-1) \qquad \text{for every } w \in N_G(u).$$

If f is strictly monotone, this is equivalent to every neighbor of u being universal. The bound is attained for every k, by taking $G = K_k \vee \overline{K}_{n-k}$ and u in the independent set.

Proof. If f is nondecreasing, then $f(d_G(w)) \le f(n-1)$ for every neighbor w of u. Thus

$$f_G(u) = (1-k)f(k) + \sum_{w \in N_G(u)} f(d_G(w)) \le (1-k)f(k) + kf(n-1).$$

The inequality reverses when f is nonincreasing. Equality holds precisely when every term in the neighbor sum equals $f(n-1)$, which gives the stated conditions. In $K_k \vee \overline{K}_{n-k}$, a vertex in the independent set has degree k and all its neighbors are universal, so the bound is attained. ■

To describe the complementary extremum, define

$$\Lambda_{f,n} = (2-n)f(n-1) + (n-1)f(1). \tag{3}$$

Equivalently,

$$\Lambda_{f,n} = f(1) - (n-2)\big(f(n-1) - f(1)\big).$$

Theorem 2. *Let $G \in \mathcal{G}_n$ and $u \in V(G)$.*

(i) *If f is nondecreasing, then*

$$\Lambda_{f,n} \le f_G(u) \le \max_{1 \le k \le n-1} \Phi_{f,n}(k).$$

(ii) *If f is nonincreasing, then*

$$\min_{1\le k\le n-1} \Phi_{f,n}(k) \le f_G(u) \le \Lambda_{f,n}.$$

All four bounds are sharp. Equality at the bound involving $\Phi_{f,n}$ holds if and only if $k = d_G(u)$ attains its corresponding maximum or minimum and

$$f(d_G(w)) = f(n-1) \qquad \text{for every } w \in N_G(u).$$

If f is nonconstant, equality at $\Lambda_{f,n}$ holds if and only if

$$d_G(u) = n-1, \qquad f(d_G(w)) = f(1) \quad \text{for every } w \in N_G(u).$$

If f is constant, every pair attains both extrema. All extremal values remain unchanged when the optimization is restricted to connected graphs.

Proof. The bounds involving $\Phi_{f,n}$, their sharpness, and their equality conditions follow from Theorem 1.

Suppose that f is nondecreasing and set $d = d_G(u)$. Then

$$\begin{aligned} f_G(u) &\ge (1-d)f(d) + df(1) \\ &= f(1) - (d-1)\big(f(d) - f(1)\big) \\ &\ge f(1) - (n-2)\big(f(n-1) - f(1)\big) = \Lambda_{f,n}. \end{aligned}$$

The last step uses $0 \le d-1 \le n-2$ and $0 \le f(d)-f(1) \le f(n-1)-f(1)$. If f is nonconstant, then $n \ge 3$ and $f(n-1)-f(1) > 0$. For $d < n-1$, the product in the second line is strictly smaller than $(n-2)(f(n-1)-f(1))$. Hence equality requires $d = n-1$. Equality in the first step holds exactly when $f(d_G(w)) = f(1)$ for every neighbor w.

For nonincreasing f, apply the same argument to $-f$, noting that $(-f)_G(u) = -f_G(u)$, $\Phi_{-f,n} = -\Phi_{f,n}$, and $\Lambda_{-f,n} = -\Lambda_{f,n}$. If f is constant, then $f_G(u) = f(1)$ for every pair.

The value $\Lambda_{f,n}$ is attained at the center of S_n. Both S_n and the graphs $K_k \vee \overline{K}_{n-k}$ are connected, so restricting to connected graphs leaves the extrema unchanged. ∎

For strictly monotone functions, all extremal pairs admit a simple structural description.

Corollary 1. *Assume that f is strictly monotone.*

A pair (G, u) attains the maximum in Theorem 2 when f is strictly increasing, or the minimum when f is strictly decreasing, if and only if

$$G \cong K_k \vee H,$$

where $k = d_G(u)$ attains the corresponding extremum of $\Phi_{f,n}$, and H is a graph of order $n - k$ in which u is isolated.

A pair (G, u) attains the complementary extremum $\Lambda_{f,n}$ if and only if

$$G \cong S_n$$

and u is the center of the star.

Proof. The case $n = 2$ is immediate, so assume $n \geq 3$. Suppose first that (G, u) attains the extremum determined by $\Phi_{f,n}$, and set

$$A = N_G(u), \qquad B = V(G) \setminus A.$$

By Theorem 2, every vertex of A is universal. Consequently, $G[A] = K_k$, every vertex of A is adjacent to every vertex of B, and

$$G \cong K_k \vee G[B].$$

Since $N_G(u) = A$, the vertex u is isolated in $G[B]$.

Conversely, suppose that

$$G \cong K_k \vee H,$$

where u is isolated in H and k attains the corresponding extremum of $\Phi_{f,n}$. Then $d_G(u) = k$ and every neighbor of u is universal. The conclusion follows from Theorem 2.

Finally, strict monotonicity gives

$$f\big(d_G(w)\big) = f(1) \quad \Longleftrightarrow \quad d_G(w) = 1.$$

Hence equality at the complementary endpoint requires u to be universal and every other vertex to be pendant. This is equivalent to $G = S_n$ with u as its center. The converse is immediate. ■

3 The local zeroth-order general Randić index

For $\alpha \in \mathbb{R}$, let $R_\alpha(x) = x^\alpha$. Then

$$(R_\alpha)_G(u) = \big(1 - d_G(u)\big)d_G(u)^\alpha + \sum_{w \in N_G(u)} d_G(w)^\alpha.$$

When $\alpha = 0$, every local value equals 1. We therefore assume $\alpha \neq 0$ throughout this section.

The following example shows that the upper bound stated in Theorem 6 of [13] does not hold for general graphs.

Example 1. Let $n \geq 4$ and let

$$G = K_2 \vee \overline{K}_{n-2}.$$

If u belongs to the independent set, then $d_G(u) = 2$ and its two neighbors are universal. Hence

$$(R_\alpha)_G(u) = (1-2)2^\alpha + 2(n-1)^\alpha = -2^\alpha + 2(n-1)^\alpha.$$

Consequently,

$$(R_\alpha)_G(u) - (n-1)^\alpha = (n-1)^\alpha - 2^\alpha.$$

Since $n-1>2$, it follows that

$$(R_\alpha)_G(u) > (n-1)^\alpha \qquad \text{for every } \alpha > 0,$$

whereas

$$(R_\alpha)_G(u) < (n-1)^\alpha \qquad \text{for every } \alpha < 0.$$

Remark. Example 1 gives an infinite family of counterexamples to Theorem 6 of [13]. There are two problems in the proof given there. First, the inequality

$$\sum_{w\in N_G(u)} d_G(w) \le n-1$$

is false for general graphs. Second, for $\alpha > 1$, under a fixed-sum constraint, $\sum_i x_i^\alpha$ is maximized by concentrating the variables and minimized by balancing them. The original proof uses the opposite conclusion. Consequently, the upper bounds in Corollaries 1 and 2 of that paper also fail for general graphs.

The same family also contradicts the general-graph extremal claims in [13, Example 6]: take $n=4$ and $\alpha = 1/2$ or $\alpha = -1/2$. Finally, the original equality characterization is not valid for general graphs either, since every vertex of K_n has local value $(n-1)^\alpha$.

We now give both exact bounds. Set $N = n-1$ and define

$$H_{\alpha,n}(k) = (1-k)k^\alpha + kN^\alpha, \qquad 1 \le k \le N, \tag{4}$$

and

$$L_{\alpha,n} = (1-N)N^\alpha + N.$$

Theorem 3. *Let $G \in \mathcal{G}_n$ and $u \in V(G)$.*

(i) *If $\alpha > 0$, then*

$$L_{\alpha,n} \le (R_\alpha)_G(u) \le \max_{1\le k\le n-1} H_{\alpha,n}(k).$$

(ii) *If $\alpha < 0$, then*

$$\min_{1\le k\le n-1} H_{\alpha,n}(k) \le (R_\alpha)_G(u) \le L_{\alpha,n}.$$

All bounds are sharp. If $n \geq 3$, equality at $L_{\alpha,n}$ holds if and only if $G \cong S_n$ and u is its center. Equality at the other bound holds if and only if

$$G \cong K_k \vee F,$$

where $k = d_G(u)$ attains the corresponding extremum of $H_{\alpha,n}$, and F is a graph of order $n-k$ in which u is isolated. If $n = 2$, both vertices of K_2 attain both bounds, which equal 1.

Proof. Apply Theorem 2 and Corollary 1 to $f(x) = x^\alpha$, which is strictly increasing for $\alpha > 0$ and strictly decreasing for $\alpha < 0$. In this case, $\Phi_{f,n} = H_{\alpha,n}$ and $\Lambda_{f,n} = L_{\alpha,n}$. The bounds given by $L_{\alpha,n}$ are those of [13, Theorems 4 and 5]. ■

Theorem 3 reduces the graph-theoretic problem to the discrete optimization of $H_{\alpha,n}$. We study it through the continuous function

$$H_{\alpha,n}(x) = (1-x)x^\alpha + xN^\alpha, \qquad 1 \leq x \leq N.$$

Theorem 4. *Let $n \geq 3$ and $\alpha \neq 0$. The equation*

$$(n-1)^\alpha = y^{\alpha-1}\big((\alpha+1)y - \alpha\big) \tag{5}$$

has a unique solution $y_{\alpha,n} \in (1, n-1)$.

(i) *If $\alpha > 0$, then*

$$\max_{1\leq k\leq n-1} H_{\alpha,n}(k) = \max\left\{H_{\alpha,n}\big(\lfloor y_{\alpha,n}\rfloor\big), H_{\alpha,n}\big(\lceil y_{\alpha,n}\rceil\big)\right\}.$$

(ii) *If $\alpha < 0$, then*

$$\min_{1\leq k\leq n-1} H_{\alpha,n}(k) = \min\left\{H_{\alpha,n}\big(\lfloor y_{\alpha,n}\rfloor\big), H_{\alpha,n}\big(\lceil y_{\alpha,n}\rceil\big)\right\}.$$

Proof. Set $N = n-1$ and define

$$F_\alpha(x) = x^{\alpha-1}\big((\alpha+1)x - \alpha\big).$$

Then

$$H'_{\alpha,n}(x) = N^{\alpha} - F_{\alpha}(x)$$

and

$$F'_{\alpha}(x) = \alpha x^{\alpha-2}\big((\alpha+1)x - (\alpha-1)\big).$$

Moreover,

$$H'_{\alpha,n}(1) = N^{\alpha} - 1, \qquad H'_{\alpha,n}(N) = \alpha N^{\alpha-1}(1-N).$$

If $\alpha > 0$, then $F'_{\alpha}(x) > 0$ for $x \geq 1$. Since $H'_{\alpha,n}(1) > 0$ and $H'_{\alpha,n}(N) < 0$, the derivative has exactly one zero in $(1, N)$.

Now suppose that $\alpha < 0$. Then $H'_{\alpha,n}(1) < 0$ and $H'_{\alpha,n}(N) > 0$, so the derivative has at least one zero. Any such zero satisfies $F_{\alpha}(x) = N^{\alpha} > 0$. On the interval where F_{α} is positive, both

$$(\alpha+1)x - \alpha > 0 \quad \text{and} \quad (\alpha+1)x - (\alpha-1) > 0.$$

The second expression equals the first plus 1. The positivity set of F_{α} on $[1, \infty)$ is an interval containing 1, because $x^{\alpha-1} > 0$ and $(\alpha+1)x - \alpha$ is linear and equals 1 at $x = 1$. Moreover, $F'_{\alpha}(x) < 0$ throughout this interval. Therefore, the equation $F_{\alpha}(x) = N^{\alpha}$ has at most one solution, and the zero of $H'_{\alpha,n}$ is unique.

Denote this zero by $y_{\alpha,n}$. The equation $H'_{\alpha,n}(y_{\alpha,n}) = 0$ is precisely (5). If $\alpha > 0$, the derivative is positive before $y_{\alpha,n}$ and negative after it; if $\alpha < 0$, the signs are reversed. Thus the continuous function has a unique maximum in the first case and a unique minimum in the second. The extremum over the integers is therefore attained among $\lfloor y_{\alpha,n} \rfloor$ and $\lceil y_{\alpha,n} \rceil$. If these are distinct, either one or both can be optimal; their function values must be compared. ■

We conclude the section with several cases in which the optimal degree and the corresponding extremal value can be described explicitly.

Example 2. For $\alpha = 1$,

$$H_{1,n}(k) = (1-k)k + k(n-1) = k(n-k).$$

Therefore,

$$\max_{\substack{G\in\mathcal{G}_n\\ u\in V(G)}} (R_1)_G(u) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

The optimal degrees are

$$k \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right\}.$$

For either such value of k, the maximum is attained at every vertex of the independent set in $K_k \vee \overline{K}_{n-k}$.

Example 3 (The first Zagreb index)**.** For $\alpha = 2$ and $n \geq 3$,

$$H_{2,n}(k) = (1-k)k^2 + k(n-1)^2.$$

The unique continuous maximizer is

$$y_{2,n} = \frac{1+\sqrt{1+3(n-1)^2}}{3}.$$

Thus the optimal integer is one of $\lfloor y_{2,n} \rfloor$ and $\lceil y_{2,n} \rceil$.

For $n = 4$, the optimal degree is $k = 2$, and

$$\max_{\substack{G\in\mathcal{G}_4\\ u\in V(G)}} (R_2)_G(u) = H_{2,4}(2) = -4 + 2 \cdot 9 = 14.$$

This value is attained in

$$K_2 \vee \overline{K}_2 = K_4 - e$$

at either vertex of degree 2. In particular, $14 > 3^2 = 9$, so this is a counterexample of the smallest possible order to the claimed upper bound discussed in Example 1; for $n \leq 3$, every admissible graph is a star or a complete graph.

Example 4 (The forgotten index)**.** For $\alpha = 3$ and $n = 4$, the optimal

degree is again $k = 2$. Hence

$$\max_{\substack{G\in\mathcal{G}_4\\ u\in V(G)}} (R_3)_G(u) = H_{3,4}(2) = -8 + 2\cdot 27 = 46.$$

The maximum is attained at either degree-2 vertex of $K_4 - e$, while a pendant vertex of S_4 has local value 27.

Example 5 (The case $\alpha = -1$)**.** For $\alpha = -1$ and $n \geq 3$,

$$H_{-1,n}(k) = \frac{1-k}{k} + \frac{k}{n-1} = \frac{1}{k} - 1 + \frac{k}{n-1}.$$

Equation (5) gives

$$y_{-1,n} = \sqrt{n-1}.$$

Therefore,

$$\min_{\substack{G\in\mathcal{G}_n\\ u\in V(G)}} (R_{-1})_G(u) = \min\left\{H_{-1,n}\left(\lfloor\sqrt{n-1}\rfloor\right), H_{-1,n}\left(\lceil\sqrt{n-1}\rceil\right)\right\}.$$

If $n - 1 = m^2$, then $k = m$ is the unique optimal degree, and

$$\min_{\substack{G\in\mathcal{G}_n\\ u\in V(G)}} (R_{-1})_G(u) = \frac{2}{m} - 1.$$

This value is attained at every vertex of the independent set in $K_m \vee \overline{K}_{n-m}$.

Example 6 (The case $\alpha = -2$)**.** Let $n \geq 3$ and set $N = n-1$. Equation (5) becomes

$$N^{-2} = y_{-2,n}^{-3}(2 - y_{-2,n}),$$

so $1 < y_{-2,n} < 2$. Hence the only possible optimal degrees are 1 and 2. Since

$$H_{-2,n}(1) = \frac{1}{N^2} \qquad \text{and} \qquad H_{-2,n}(2) = -\frac{1}{4} + \frac{2}{N^2},$$

both degrees are optimal when $n = 3$, whereas $k = 2$ is the unique optimal

degree when $n \geq 4$. Consequently, for $n \geq 4$,

$$\min_{\substack{G \in \mathcal{G}_n \\ u \in V(G)}} (R_{-2})_G(u) = -\frac{1}{4} + \frac{2}{(n-1)^2},$$

and the extremal pairs are precisely those described in Theorem 3 with $k = 2$.

4 Sharp bounds for trees

The conclusion of the incorrect upper bound from [13] remains valid for trees. We prove a stronger fixed-degree result.

Let T be a tree of order n, let $u \in V(T)$ have degree k, and write

$$N_T(u) = \{w_1, \ldots, w_k\}, \qquad x_i = d_T(w_i).$$

The sets of vertices at distance two reached through different neighbors are disjoint. Indeed, for each i, the vertex w_i itself and its $x_i - 1$ neighbors other than u form a set of x_i vertices at distance one or two from u. These sets are pairwise disjoint, since an intersection would create a cycle in T. Hence

$$\sum_{i=1}^{k} x_i = k + \sum_{i=1}^{k} (x_i - 1) = \big|\{v \in V(T) : 1 \leq d_T(u, v) \leq 2\}\big|,$$

and consequently,

$$\sum_{i=1}^{k} x_i \leq n - 1. \tag{6}$$

Equality holds if and only if every vertex of T is at distance at most two from u.

We use the following elementary integer-optimization lemma.

Lemma 1. *Let $1 \leq k \leq N$ be integers, and let $x_1, \ldots, x_k$ be positive integers with $\sum_{i=1}^{k} x_i \leq N$. Write*

$$N = qk + r, \qquad 0 \leq r < k.$$

(i) *If* $\alpha > 1$, *then*

$$\sum_{i=1}^{k} x_i^{\alpha} \leq (N-k+1)^{\alpha} + k - 1.$$

(ii) *If* $0 < \alpha < 1$, *then*

$$\sum_{i=1}^{k} x_i^{\alpha} \leq r(q+1)^{\alpha} + (k-r)q^{\alpha}.$$

(iii) *If* $\alpha < 0$, *then*

$$\sum_{i=1}^{k} x_i^{\alpha} \geq r(q+1)^{\alpha} + (k-r)q^{\alpha}.$$

In part (i), *equality holds if and only if, up to a permutation,*

$$(x_1, \ldots, x_k) = (N-k+1, 1, \ldots, 1).$$

In parts (ii) *and* (iii), *equality holds if and only if* $\sum_i x_i = N$ *and the entries differ by at most one; equivalently,* r *entries are* $q+1$ *and the remaining* $k-r$ *entries are* q.

Proof. Let

$$F(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i^{\alpha}.$$

If $\alpha > 0$, then F is strictly increasing in each coordinate, whereas if $\alpha < 0$, it is strictly decreasing in each coordinate. Therefore, when maximizing F for $\alpha > 0$, or minimizing it for $\alpha < 0$, the constraint must be attained with equality:

$$\sum_{i=1}^{k} x_i = N.$$

Suppose first that $\alpha > 1$. If $a \geq b \geq 2$, the strict convexity of x^{α} gives

$$(a+1)^{\alpha} - a^{\alpha} > b^{\alpha} - (b-1)^{\alpha},$$

and hence

$$(a+1)^\alpha + (b-1)^\alpha > a^\alpha + b^\alpha.$$

Thus, transferring one unit from an entry $b \geq 2$ to another entry $a \geq b$ strictly increases F. Repeating this operation, all the excess above 1 is concentrated in a single coordinate, and we obtain, up to a permutation,

$$(N-k+1, 1, \ldots, 1).$$

Consequently,

$$\sum_{i=1}^{k} x_i^\alpha \leq (N-k+1)^\alpha + k - 1.$$

Since every nontrivial transfer is strict, equality holds if and only if the original vector is a permutation of $(N-k+1, 1, \ldots, 1)$.

Now suppose that $0 < \alpha < 1$. If $a \geq b+2$, the strict concavity of x^α gives

$$(a-1)^\alpha + (b+1)^\alpha > a^\alpha + b^\alpha.$$

Therefore, balancing two entries whose difference is at least two strictly increases F. Repeating this operation yields a vector whose entries differ by at most one. Since $N = qk + r$, with $0 \leq r < k$, this vector has, up to a permutation, r entries equal to $q+1$ and $k-r$ entries equal to q. This proves part (ii), including its equality condition.

Finally, if $\alpha < 0$, the function x^α is strictly convex, so the same balancing operation satisfies

$$(a-1)^\alpha + (b+1)^\alpha < a^\alpha + b^\alpha.$$

Hence it strictly decreases F, and the minimum is attained precisely at the same balanced vectors. This proves part (iii) and its equality condition. ■

For each $1 \leq k \leq n-1$, write

$$n - 1 = qk + r, \qquad 0 \leq r < k,$$

and define

$$B_{\alpha,n}(k) = (1-k)k^{\alpha} + r(q+1)^{\alpha} + (k-r)q^{\alpha}.$$

Theorem 5. *Let T be a tree of order $n \geq 2$, and let $u \in V(T)$ have degree k.*

(i) *If $\alpha \geq 1$, then*

$$(R_{\alpha})_T(u) \leq (1-k)k^{\alpha} + (n-k)^{\alpha} + k - 1. \tag{7}$$

For $\alpha > 1$, equality holds if and only if one neighbor of u has degree $n-k$ and the other $k-1$ neighbors have degree 1. For $\alpha = 1$, equality holds if and only if

$$\sum_{w \in N_T(u)} d_T(w) = n - 1.$$

(ii) *If $0 < \alpha < 1$, then*

$$(R_{\alpha})_T(u) \leq B_{\alpha,n}(k).$$

Equality holds if and only if $\sum_{w \in N_T(u)} d_T(w) = n-1$ and the neighbor degrees differ by at most one.

(iii) *If $\alpha < 0$, then*

$$(R_{\alpha})_T(u) \geq B_{\alpha,n}(k).$$

Equality holds if and only if $\sum_{w \in N_T(u)} d_T(w) = n-1$ and the neighbor degrees differ by at most one.

All bounds are sharp for every admissible n and k.

Proof. Write

$$N_T(u) = \{w_1, \ldots, w_k\} \qquad \text{and} \qquad x_i = d_T(w_i).$$

Since T is a tree, each set $N_T(w_i) \setminus \{u\}$ consists of vertices at distance two

from u, and these sets are pairwise disjoint. Consequently,

$$\sum_{i=1}^{k} x_i \leq n-1.$$

Moreover, by the definition of the local value,

$$(R_\alpha)_T(u) = (1-k)k^\alpha + \sum_{i=1}^{k} x_i^\alpha.$$

For $\alpha > 1$, Lemma 1, applied with $N = n-1$, gives

$$\sum_{i=1}^{k} x_i^\alpha \leq (n-k)^\alpha + k - 1.$$

Equality holds if and only if, up to a permutation,

$$(x_1, \ldots, x_k) = (n-k, 1, \ldots, 1).$$

This proves part (i) when $\alpha > 1$. If $\alpha = 1$, then

$$\sum_{i=1}^{k} x_i \leq n-1,$$

with equality if and only if

$$\sum_{w \in N_T(u)} d_T(w) = n-1.$$

Adding the fixed term $(1-k)k^\alpha$ proves the remaining assertion in part (i).

For parts (ii) and (iii), write

$$n-1 = qk + r, \qquad 0 \leq r < k.$$

Lemma 1 gives the corresponding bounds. In both cases, equality holds if and only if

$$\sum_{i=1}^{k} x_i = n-1$$

and, up to a permutation,

$$(x_1, \ldots, x_k) = (\underbrace{q+1, \ldots, q+1}_{r}, \underbrace{q, \ldots, q}_{k-r}).$$

Equivalently, the neighbor degrees differ by at most one.

It remains to prove sharpness. For part (i), let u be adjacent to $k-1$ leaves and to one additional vertex v, and attach $n-k-1$ leaves to v. Then

$$d_T(v) = n - k,$$

while the other $k-1$ neighbors of u have degree 1. The resulting tree has order n and attains equality. This construction also proves sharpness when $\alpha = 1$.

For parts (ii) and (iii), note that $q \geq 1$ because $k \leq n-1$. Let u have neighbors $w_1, \ldots, w_k$. Attach q leaves to r of these neighbors and $q-1$ leaves to each of the remaining $k-r$ neighbors. Their degrees are therefore $q+1$ and q, respectively, and the order of the resulting tree is

$$1 + k + rq + (k-r)(q-1) = qk + r + 1 = n.$$

Thus equality is attained for every admissible pair n and k. ■

Corollary 2. *Let T be a tree of order $n \geq 2$, let $u \in V(T)$, and let $\alpha \geq 1$. Then*

$$(R_\alpha)_T(u) \leq (n-1)^\alpha.$$

Equality holds if and only if $T \cong S_n$ and u is a pendant vertex.

Proof. Let $k = d_T(u)$. By Theorem 5,

$$\begin{aligned}(R_\alpha)_T(u) &\leq (1-k)k^\alpha + (n-k)^\alpha + k - 1 \\ &= (n-k)^\alpha - (k-1)(k^\alpha - 1) \\ &\leq (n-k)^\alpha \\ &\leq (n-1)^\alpha.\end{aligned}$$

If $k > 1$, then $n - k < n - 1$, so the last inequality is strict. Therefore,

equality requires $k = 1$.

Let v be the unique neighbor of u. Equality in Theorem 5 then requires

$$d_T(v) = n - 1$$

both when $\alpha > 1$ and when $\alpha = 1$. Thus v is adjacent to every other vertex of T, and hence $T \cong S_n$, with u as a pendant vertex. The converse follows immediately. ■

The fixed-degree result also gives the exact global extrema in the remaining ranges.

Corollary 3. *Let $n \geq 2$.*

(i) *If $0 < \alpha < 1$, then*

$$\max_{\substack{T \text{ tree of order } n \\ u \in V(T)}} (R_\alpha)_T(u) = \max_{1 \leq k \leq n-1} B_{\alpha,n}(k).$$

(ii) *If $\alpha < 0$, then*

$$\min_{\substack{T \text{ tree of order } n \\ u \in V(T)}} (R_\alpha)_T(u) = \min_{1 \leq k \leq n-1} B_{\alpha,n}(k).$$

In either case, a pair (T, u) is extremal if and only if $k = d_T(u)$ optimizes $B_{\alpha,n}$ in the indicated direction, every vertex is at distance at most two from u, and the neighbor degrees differ by at most one.

Proof. Let (T, u) be any such pair and set $k = d_T(u)$. If $0 < \alpha < 1$, Theorem 5 gives

$$(R_\alpha)_T(u) \leq B_{\alpha,n}(k) \leq \max_{1 \leq j \leq n-1} B_{\alpha,n}(j).$$

Choose k_0 attaining the maximum on the right. By the sharpness construction in Theorem 5, there exists a pair (T_0, u_0) such that

$$d_{T_0}(u_0) = k_0 \qquad \text{and} \qquad (R_\alpha)_{T_0}(u_0) = B_{\alpha,n}(k_0).$$

This proves part (i).

Similarly, if $\alpha < 0$, then

$$(R_\alpha)_T(u) \geq B_{\alpha,n}(k) \geq \min_{1 \leq j \leq n-1} B_{\alpha,n}(j).$$

Taking a value k_0 that minimizes $B_{\alpha,n}$ and applying the sharpness construction from Theorem 5 proves part (ii).

In either case, equality in the global bound holds if and only if $k = d_T(u)$ optimizes $B_{\alpha,n}$ in the corresponding direction and equality holds in the fixed-degree bound of Theorem 5. The latter occurs if and only if

$$\sum_{w \in N_T(u)} d_T(w) = n - 1$$

and the neighbor degrees differ by at most one. For a tree, the first condition is equivalent to every vertex being at distance at most two from u. This proves the characterization of the extremal pairs. ■

For completeness, the complementary extrema over trees are as follows. For every $n \geq 2$,

$$\min_{\substack{T \text{ tree of order } n \\ u \in V(T)}} (R_\alpha)_T(u) = L_{\alpha,n} \qquad (\alpha > 0),$$

and

$$\max_{\substack{T \text{ tree of order } n \\ u \in V(T)}} (R_\alpha)_T(u) = L_{\alpha,n} \qquad (\alpha < 0).$$

Indeed, Theorem 3 applies to every tree, and taking $T \cong S_n$ with u as its center shows that these bounds are attained. By the equality conditions in that theorem, if $n \geq 3$, equality holds if and only if $T \cong S_n$ and u is its center. When $n = 2$, the unique tree is $S_2 \cong K_2$, and both vertices have local value 1.

Together with Corollaries 2 and 3, this determines both extrema over trees for every $\alpha \neq 0$.

5 Concluding remarks and a parametric extension

For every monotone vertex-degree function, we have determined both global extrema of its local value and all equality cases. The bound $\Lambda_{f,n}$ is explicit, while the opposite bound is a finite optimization of $\Phi_{f,n}(k)$. For power functions, this optimization requires comparison at no more than two candidate degrees. These results extend the bounds in [6, Theorems 3 and 4] to all nonzero exponents and provide their complete equality characterization.

The equality cases in general graphs are governed by universal neighbors. In trees, the restriction on the sum of neighbor degrees instead leads to concentrated or balanced degree distributions. The sharp fixed-degree bounds determine the remaining global extrema on trees, while the opposite bounds are attained at the star center. In particular, Corollary 2 recovers, for trees, the conclusion of [13, Theorem 6].

The correction term in (1) suggests a natural one-parameter extension of the local value. For $\lambda \in \mathbb{R}$, define

$$f_{G,\lambda}(u) = f(d_u) + \lambda \sum_{w \in N(u)} \big(f(d_w) - f(d_u)\big). \tag{8}$$

The parameter λ controls the influence of the neighborhood: $\lambda = 0$ gives the original vertex contribution, while $\lambda = 1$ gives the local value studied here. The edge corrections still cancel in pairs, and therefore

$$\sum_{u \in V(G)} f_{G,\lambda}(u) = f(G)$$

for every λ. Thus (8) changes the local decomposition without changing the global index.

For $f(x) = x^\alpha$, write $(R_{\alpha,\lambda})_G(u)$ for this value. For $\lambda > 0$, the proof of Theorem 1 gives the sharp fixed-degree bound

$$H^{(\lambda)}_{\alpha,n}(k) = k^\alpha + \lambda k\big((n-1)^\alpha - k^\alpha\big), \qquad 1 \le k \le n-1.$$

Consequently,

$$\max_{\substack{G\in\mathcal{G}_n\\ u\in V(G)}} (R_{\alpha,\lambda})_G(u) = \max_{1\le k\le n-1} H^{(\lambda)}_{\alpha,n}(k) \quad (\alpha > 0),$$

and

$$\min_{\substack{G\in\mathcal{G}_n\\ u\in V(G)}} (R_{\alpha,\lambda})_G(u) = \min_{1\le k\le n-1} H^{(\lambda)}_{\alpha,n}(k) \quad (\alpha < 0).$$

Equality holds precisely when $G \cong K_k \vee F$, where k is an optimal degree, F has order $n-k$, and u is isolated in F. The identities also hold for $\lambda = 0$, when the extremal value is $(n-1)^\alpha$ and equality requires only that u be universal.

Each function $H^{(\lambda)}_{\alpha,n}(k)$ is affine in λ. Thus the optimal value is a piecewise affine function of λ, and changes in the set of optimal degrees can occur only at intersections of these functions. The finite optimization and the graph equality cases therefore follow directly from our method. The following question concerns a more explicit description.

Problem. For fixed $n \ge 3$ and $\alpha \neq 0$, describe the degrees that maximize $H^{(\lambda)}_{\alpha,n}$ for $\alpha > 0$, or minimize it for $\alpha < 0$, as λ ranges over $[0,\infty)$. Determine the transition values at which the set of optimal degrees changes. Study the asymptotic behavior of these degrees and extremal values as $n \to \infty$, allowing $\lambda = \lambda_n$ to vary with n.

For $\lambda = 1$, Theorem 4 reduces the degree optimization to the floor and ceiling of a unique critical point. The question is to describe how this optimization changes with λ.

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