

Invertible Matrix Method for Power Law Systems: New Deficiency Results for Power Law Reactant-Determined Kinetic Systems

Jaysie Mher G. Tiongson^{a,*}

^a *University of the Philippines Los Baños, Laguna, Philippines*

jgtiongson@up.edu.ph

(Received July 19, 2026)

Abstract

A Power Law Reactant-Determined Kinetics is $\hat{T}$ –rank maximal (to type PL-TIK) if the associated $\hat{T}$ –matrix has a maximal column rank, that is, Power Law $\hat{T}$ –independent Kinetic (PL-TIK) systems have $\hat{T}$ –matrices with linearly independent column vectors per linkage class. The concept of PL-TIK systems, introduced by Talabis et al. in 2018, is important to characterize complex balanced dynamical systems with power law kinetics. This study proposes an invertible matrix-based approach to construct these systems instead of relying to the linear independence of column vectors. Moreover, this approach allows us to reformulate known results associated to PL-TIK systems, such as the Zero Kinetic Reactant Deficiency Theorem (ZKRDT) and the Weak Reversibility Theorem for PL-RDK systems based on PL-TIK Decompositions. Lastly, we apply this new construct to qualitatively describe the long term behavior of some known complex dynamical systems, such as the Glucose and Ketone Body Metabolism with Medium-Chain Triglycerides model studied by Espina et al. (2022), and the Human Dihydrofolate Reductase (DHFR) Catalysis model studied by Craciun et al. (2006).

*Corresponding author.



1 Introduction

Recently, Chemical Reaction Network Theory (CRNT) becomes an indispensable tool for qualitatively describing complex dynamical systems that induce a system of ordinary differential equations (ODEs). This emerging field uses the concept of network deficiency to determine whether the systems of ODEs can admit (positive vector) solutions. The first deficiency-led theorems, the so-called Deficiency Zero Theorem (DZT) and Deficiency One Theorem (DOT), were due to Horn, Feinberg, and Jackson [6]. These results are mainly applied to Deficiency Zero and Deficiency One Mass Action Kinetic (MAK) systems via network deficiency and weak reversibility.

After several decades, Müller and Regensburger [18] established the first result beyond MAK systems, that is, characterizing the existence of complex balanced General Mass Action Kinetic (GMAK) systems through the concept of kinetic deficiency. This is true only for cycle terminal GMAK systems, whatever the network deficiency the systems have.

Talabis et al. established another kinetics-based characterization by introducing the concept of kinetic reactant deficiency. They proved the Zero Kinetic Reactant Deficiency Theorem (ZKRDT) [21] for Power Law $\hat{T}$ -independent Kinetic (PL-TIK) systems. Regardless of the systems' network deficiencies, this theorem applies also to non-cycle terminal systems, as opposed to GMAK systems. However, it led to a problem with how to describe the dynamics of non-PL-TIK systems.

Feinberg [6] developed the Network Decomposition Theory mainly to characterize large complex systems via the properties of their subnetworks. He proposed that, using independent decomposition, the set of positive equilibria for the whole system is equal to the intersection of the sets of positive equilibria of its subnetworks. Also, Farinas et al. [5] used the notion of Incidence-independent decomposition to show that the set of complex balanced equilibria for the whole system is equal to the intersection of the sets of complex balanced equilibria of its subnetworks. Farinas et al. also established the Zero Deficiency Decomposition (ZDD) for any chemical kinetic system; that is, each subnetwork has zero deficiency [5].

Recently, Tiongson et al. in [24] presented kinetics-based decomposi-

tions such as the Zero Kinetic Deficiency decomposition (ZKDD) and the PL-TIK Decomposition, wherein the latter was used to characterize the existence of positive equilibria of weakly reversible, Power Law Reactant-Determined Kinetic (PL-RDK) systems.

This study focuses on constructing PL-TIK systems, the subclass of PL-RDK systems, by using invertible matrices. This approach allows us to reformulate the results concerning PL-TIK systems and PL-TIK decompositions. Lastly, we apply this new construct to some known complex dynamical systems, such as the Glucose and Ketone Body Metabolism with Medium-Chain Triglycerides model studied by Espina et al. in [4], and the Human Dihydrofolate Reductase (DHFR) Catalysis model studied by Craciun et al. in [3].

2 Basic notions of kinetic systems

In this section, we discuss some fundamental concepts on chemical kinetic systems (CKS) and known results on the existence of positive equilibria. In addition, we also discuss some known network decomposition classes and their properties.

2.1 Fundamental concepts of a chemical reaction network

In a chemical reaction network (CRN), three non-empty, finite sets can be defined. These are the species set, the complex set, and the reaction set.

The **species set**, denoted by $\mathcal{S}$, consists of all species or variables in the given CRN. Meanwhile, a **complex** is a nonnegative, integral linear combination of the species. Thus, the **complex set**, denoted by $\mathcal{C}$, is the set of all complexes in the given CRN. Note that the numerical coefficients of the species in a complex are called **stoichiometric coefficients**. Lastly, the **reaction set**, denoted by $\mathcal{R}$, is the set of all reactions in the given CRN.

A **reaction** is an ordered pair (X_i, X_j) , denoted by $X_i \rightarrow X_j$, such that $X_i, X_j \in \mathcal{C}$, $i \neq j$. For a given reaction, say $X_1 \rightarrow X_2$, the **reactant** (or

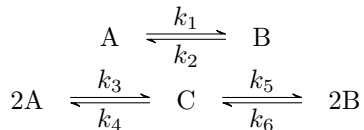
source) complex is X_1 and the **product complex** is X_2 . If a reaction is given by $0 \rightarrow X_i$, we call this as an **inflow reaction**, where 0 is the **zero complex** and an outside reactor. If a reaction is given by $X_i \rightarrow 0$, we call this as an **outflow reaction**. We formally define below what a CRN is.

Definition 1. [1] A chemical reaction network (CRN) is an ordered triple $(\mathcal{S}, \mathcal{C}, \mathcal{R})$, where $\mathcal{S}$ is the finite set of all species or variables in the CRN, $\mathcal{C}$ is the set of all complexes in the CRN, and $\mathcal{R}$ is the set of all reactions involved in the CRN such that

- a. $(y, y) \notin \mathcal{R}$ for any $y \in \mathcal{C}$;
- b. for each $y \in \mathcal{C}$, there exists $y' \in \mathcal{C}$ such that $(y, y') \in \mathcal{R}$ or $(y', y) \in \mathcal{R}$.

In other words, a CRN has no self-loops and no isolated complex/es. We denote the cardinality of $\mathcal{S}$ by m , the cardinality of $\mathcal{C}$ by n , and the cardinality of $\mathcal{R}$ by r .

Example 1. Consider the toy CRN below.



In Example 1, $\mathcal{S} = \{A, B, C\}$, $m = 3$, $\mathcal{C} = \{A, B, 2A, 2B, C\}$, $n = 5$, and $\mathcal{R} = \{A \rightarrow B, B \rightarrow A, 2A \rightarrow C, C \rightarrow 2A, C \rightarrow 2B, 2B \rightarrow C\}$, and $r = 6$.

On the other hand, a CRN can also be described using matrices, such as the molecularity matrix Y , the incidence matrix I_a , and the stoichiometric matrix N .

The **molecularity matrix** of a CRN, denoted by Y , is the $m \times n$ matrix whose entries are the stoichiometric coefficients of the complexes. The **incidence matrix** of a CRN, denoted by I_a , is the $n \times r$ matrix whose entries are -1 (if the complex is a reactant complex), 1 (if the complex is a product complex), and 0 if a complex is neither a reactant nor a product complex in the i^{th} reaction. Lastly, the **stoichiometric matrix**, denoted by N , is the $m \times r$ matrix whose entries are the stoichiometric coefficients of

the associated reaction vector of the i^{th} reaction. The associated **reaction vector** of the i^{th} reaction, say $y_i \rightarrow y'_i$, is the difference between the reactant complex and the product complex, denoted by $y'_i - y_i$.

In Example 1, the molecularity matrix of the toy CRN is

$$Y = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Also, the incidence matrix of the toy CRN is

$$I_a = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Moreover, the stoichiometric matrix of the toy CRN is

$$N = \begin{bmatrix} -1 & 1 & -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Now, we discuss some graph theoretic notions of a CRN. A **linkage class** is a connected subgraph of the CRN, where the number of linkage classes is denoted by l . A connected subgraph implies that there is an undirected path between every pair of complexes. On the other hand, a **strong linkage class** is a strong connected subgraph of the given CRN, where the number of strong linkage classes is denoted by sl . A strong connected subgraph implies that there is a directed path between every pair of complexes. A **terminal strong linkage class** is a strong linkage class $\mathcal{L}$, such that for any reaction $y \rightarrow y'$, $y \in \mathcal{L}$ implies $y' \in \mathcal{L}$ [1]. This means that no complex in $\mathcal{L}$ reacts to a complex outside $\mathcal{L}$. Denote the number of terminal strong linkage classes by t .

In Example 1, $l = sl = t = 2$. Note that the linkage classes, strong linkage classes, and terminal strong linkage classes of the toy CRN coincide. We next define the concepts of weak reversibility and t-minimality.

Definition 2. [1] A CRN is called weakly reversible if $sl = l$ and t-minimal if $t = l$, respectively.

Based on Definition 2, Example 1 is a weakly reversible network since $l = sl = 2$. Moreover, it is also t -minimal since $t = l = 2$.

Arceo et al. [2] showed that there are two types of terminal strong linkage classes, namely, cycles (not necessarily simple) and singletons (or terminal points). Denote the number of terminal points by t_p and cycles by t_c . A CRN is called **cycle terminal** if $t_p = 0$ and **point terminal** if $t_c = 0$. If both t_p and t_c are positive, then the given CRN is a **point-and-cycle terminal** network. Observe that $t_p = 0$ and hence, Example 1 is a cycle terminal network.

Now, the **stoichiometric subspace**, denoted by S , is the vector space spanned by the reaction vectors. Denote the cardinality of S given by s . In Example 1, $s = 2$ and $S = \text{span}\{C - 2A, C - 2B\}$.

Lastly, we introduce the concept of network deficiency, an important network parameter that can be used to describe long term behaviors of a complex dynamical system. The deficiency of the CRN, denoted by δ , is a good criterion to determine the linear independence of its reactions. It is defined as follows:

Definition 3. [1] The deficiency of a CRN is the nonnegative integer $\delta = n - l - s$.

In Example 1, note that $\delta = 5 - 2 - 2 = 1$. We are now ready to include kinetics to our CRNs; thus, forming a chemical kinetic system (CKS).

2.2 Chemical kinetic systems

A Chemical Kinetic System (CKS) is the ordered pair $(\mathcal{N}, K)$, where $\mathcal{N}$ is a CRN and K its given kinetics. We shall focus on power law kinetics, specifically Power Law Reactant-Determined Kinetics (PL-RDK), and study its equilibria properties using the new construct that we will propose in this study.

Definition 4. [1] A kinetics K for a reaction network $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ associates to each reaction $j \in \mathcal{R}$ a mapping $K_j : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}_{\geq 0}$ such that $K_j(c) > 0$ iff $\text{supp } y$ is contained in $\text{supp } c$ (positivity condition). The ordered pair $(\mathcal{N}, K)$ is called a chemical kinetic system (CKS).

Now, the dynamical system of a CKS is defined below:

Definition 5. [1] The species formation rate function (SFRF) $f : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^n$ of a CKS is defined as

$$f(x) = N \cdot K(x) = \sum_{y \rightarrow y'} K_{y \rightarrow y'}(x)(y \rightarrow y'), \forall x \in \mathbb{R}_{\geq 0}^m.$$

The ODE system of a CKS is defined by $dx/dt = f(x)$. Also, a zero of f is called an equilibrium point or a steady state of the kinetic system.

We then define the sets of positive equilibria and complex balanced equilibria below:

Definition 6. [19] The set of positive equilibria, denoted as $E_+(\mathcal{N}, K)$, is defined as

$$E_+(\mathcal{N}, K) = \{x \in \mathbb{R}_+^m | f(x) = 0\}.$$

Definition 7. [21] A positive vector c in $\mathbb{R}^m$ is called complex balanced (CB) if $K(c)$ is contained in $\ker I_a$. The set of complex balanced equilibria, denoted as $Z_+(\mathcal{N}, K)$, is defined as

$$Z_+(\mathcal{N}, K) = \{x \in \mathbb{R}_+^m | I_a \cdot K(x) = 0\} \subseteq E_+(\mathcal{N}, K).$$

Below is a proposition relating the set of complex balanced equilibria and the weak reversibility structure of the underlying chemical reaction network.

Proposition 1. [15] *If $Z_+(\mathcal{N}, K) \neq \emptyset$, then $\mathcal{N}$ is weakly reversible.*

We are now ready to define a power law kinetics on a dynamical system.

Definition 8. [19] A kinetics K is a Power Law Kinetics (PLK) if

$$K_i(x) = k_i \prod_{j=1}^r x^{F_{ij}} \quad \forall i \in \overline{1, r}$$

with $k_i, F_{ij} \in \mathbb{R}_+$.

A **Power Law Kinetics** (PLK) is defined by an $r \times m$ matrix $F = [F_{ij}]$, called the **kinetic order matrix**, and a vector $k \in \mathbb{R}^r$, called the

(kinetic) rate vector. A well-known example of PLK systems is the Mass Action Kinetic (MAK) systems, where the kinetic orders are simply the stoichiometric coefficients of the reactant complexes. As the focus of this study, a PL-RDK system will now be introduced.

Definition 9. [19] A Power Law Kinetic system has Reactant-Determined Kinetics (of type PL-RDK) if for any two reactions i, j with identical reactant complexes, the corresponding rows of kinetic orders in F are identical, i.e., $f_{ik} = f_{jk}$ for $k = 1, \dots, m$. Otherwise, the given PLK system has Non-Reactant-Determined Kinetics (of type PL-NDK).

In Example 1, the associated kinetic order matrix is

$$F = \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & 0 \\ f_{31} & 0 & 0 \\ 0 & 0 & f_{43} \\ 0 & 0 & f_{53} \\ 0 & f_{62} & 0 \end{bmatrix}.$$

The CKS is a PL-RDK system if and only if $f_{43} = f_{53}$.

Moreover, the corresponding SFRF of Example 1 is

$$f(x) = N \cdot K(x) = \begin{bmatrix} -1 & 1 & -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 A^{f_{11}} \\ k_2 B^{f_{22}} \\ k_3 A^{f_{31}} \\ k_4 C^{f_{43}} \\ k_5 C^{f_{53}} \\ k_6 B^{f_{62}} \end{bmatrix}.$$

Definition 10. [19] The $m \times n_r$ T -matrix is the truncated $\tilde{Y}$ where the non-reactant columns are deleted. The $\tilde{Y}$ -matrix is defined below.

$$(\tilde{Y})_{ij} = \begin{cases} (F)_{ki}, & \text{if } j \text{ is a reactant complex of reaction } k \\ 0, & \text{otherwise.} \end{cases}$$

In Example 1, the associated $\tilde{Y}$ -matrix is

$$\tilde{Y} = \begin{bmatrix} f_{11} & 0 & f_{31} & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 & 0 & f_{62} \\ 0 & 0 & 0 & f_{43} & f_{53} & 0 \end{bmatrix}.$$

Since every complex is a reactant complex and assuming PL-RDK, it follows that

$$T = \begin{bmatrix} f_{11} & 0 & f_{31} & 0 & 0 \\ 0 & f_{22} & 0 & 0 & f_{62} \\ 0 & 0 & 0 & f_{43} & 0 \end{bmatrix}.$$

Talabis et al. [19] defined a block matrix $\hat{T} \in \mathbb{R}^{(m+l) \times n_r}$ by $\hat{T} = \begin{bmatrix} T \\ L^T \end{bmatrix}$, where $L = [e^1, \dots, e^l]$ and $e^1, \dots, e^l \in \{0, 1\}^n$. The $\hat{T}$ from Example 1 is given by

$$\hat{T} = \begin{bmatrix} f_{11} & 0 & f_{31} & 0 & 0 \\ 0 & f_{22} & 0 & 0 & f_{62} \\ 0 & 0 & 0 & f_{43} & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Definition 11. [19] A Power Law Reactant-Determined Kinetics is $\hat{T}$ -rank maximal (of type PL-TIK) if its column-rank is maximal.

Remark. [19] PL-TIK systems are PL-RDK systems such that, for each linkage class, reactions with the same reactant complexes have the same kinetic order vectors (or kinetic interactions) which are linearly independent.

Note that the columns of the associated $\hat{T}$ -matrix of the toy CRN are linearly independent. Hence, it is column-rank maximal, i.e., the PL-RDK system is a PL-TIK system.

The following definition establishes a kinetic analogue of a network deficiency.

Definition 12. [21]. Let $\mathcal{N}$ be a network with n_r reactant complexes and K a Power Law Reactant-Determined Kinetics with T -matrix T . If $\hat{q} = \text{rank}(\hat{T})$, then the kinetic reactant deficiency $\hat{\delta}$ is defined as $\hat{\delta} = n_r - \hat{q}$.

In Example 1, $\hat{q} = 5$ since it is column-rank maximal. Thus, $\hat{\delta} = n_r - \hat{q} = 5 - 5 = 0$. The next proposition shows a new way to define the kinetic reactant deficiency of a PL-RDK system.

Proposition 2. [21] Let $(\mathcal{N}, K)$ be a PL-RDK system with T -matrix T and kinetic reactant deficiency $\hat{\delta}$. Then, $\hat{\delta} = \dim \ker(\hat{T})$.

As proposed by Talabis et al., the next proposition argues that zero kinetic reactant deficiency systems are precisely PL-TIK systems.

Proposition 3. [21] $(\mathcal{N}, K)$ is a PL-TIK system if and only if $\hat{\delta} = 0$.

The next result shows that a PL-TIK system has a decomposition (a partitioning of $\mathcal{R}$) in which every subnetwork is also PL-TIK.

Proposition 4. [20]. Let $(\mathcal{N}, K)$ be a PL-TIK system. Then,

1. $\dim(\ker(\hat{T})) = 0$; and
2. $\text{Im } \hat{T} = \text{Im } \hat{T}^1 \oplus \text{Im } \hat{T}^2 \oplus \cdots \oplus \text{Im } \hat{T}^l$.

Talabis et al. proposed and proved the Zero Kinetic Reactant Deficiency Theorem, which is a kinetic analogue of the Deficiency Zero Theorem (DZT).

Theorem 5. (Zero Kinetic Reactant Deficiency Theorem (ZKRDT)) [21]. Let $(\mathcal{N}, K)$ be a PL-RDK system with T -matrix T on $\mathcal{N}$ and $\hat{\delta} = 0$. Then $\mathcal{N}$ is weakly reversible if and only if $Z_+(\mathcal{N}, K) \neq \emptyset$.

2.3 Network decomposition theory

In this subsection, we discuss some well-known network decomposition classes. Also, we list down some important theorems applicable to this study.

A **decomposition** is simply a partitioning of the reaction set of a given CRN. We call each subgraph as a **subnetwork**. The simplest example of a network decomposition is the so-called **linkage class decomposition**, where each linkage class of the CRN constitutes a subnetwork of the said decomposition.

The next decomposition class is common when studying Network Decomposition Theory. This is obtained when we partition the reaction set based on the induced stoichiometric subspaces.

Definition 13. [5] A decomposition is independent if and only if S is the direct sum of the subnetworks' stoichiometric subspaces S_i . Equivalently, $s = s_1 + \cdots + s_k$.

A result that relates the set of positive equilibria and independent decomposition is given below.

Theorem 6. (*Feinberg Decomposition Theorem (1987)*) [5] Let $P(\mathcal{R}) = \mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k$ be a partition of $\mathcal{N}$ and let K be the kinetics. If $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_k$ is a network decomposition of $P(\mathcal{R})$ and

$$E_+(\mathcal{N}_i, K_i) = \{x \in \mathbb{R}_+^m | \mathcal{N}_i K_i(x) = 0\},$$

then

$$E(\mathcal{N}_1, K_1) \cap \dots \cap E_+(\mathcal{N}_k, K_k) \subseteq E_+(\mathcal{N}, K).$$

If the network decomposition is independent, then equality holds.

The next decomposition class relates the incidence map of the parent network and its subnetworks.

Definition 14. [5]. A decomposition $\mathcal{N} = \mathcal{N}_1 \cup \dots \cup \mathcal{N}_k$ is incidence-independent if and only if the image of the incidence map of $\mathcal{N}$ is the direct sum of the images of the incidence maps of $\mathcal{N}_i$'s. Equivalently, $n - l = \sum (n_i - l_i)$.

The following result is analogous to Theorem 6.

Theorem 7. [5]. Let $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ be a CRN and $\mathcal{N}_i = (\mathcal{S}_i, \mathcal{C}_i, \mathcal{R}_i)$, $i = 1, \dots, k$ be the subnetworks of a decomposition. Let K be any kinetics and $Z_+(\mathcal{N}, K)$, $Z_+(\mathcal{N}_i, K_i)$ be as defined in Theorem 6 [5]. Then,

$$i. \quad \cap Z_+(\mathcal{N}_i, K_i) \subset Z_+(\mathcal{N}, K).$$

If the decomposition is incidence-independent, then

$$ii. \quad Z_+(\mathcal{N}, K) = \cap Z_+(\mathcal{N}_i, K_i); \text{ and}$$

$$iii. \quad \text{if } Z_+(\mathcal{N}, K) \neq \emptyset, \text{ then } Z_+(\mathcal{N}_i, K_i) \neq \emptyset \text{ for each } i.$$

The next decomposition class partitions not only the reaction set $\mathcal{R}$ but also the complex set $\mathcal{C}$, which was introduced by Farinas et al. in [5].

Definition 15. [5]. A decomposition $\mathcal{N} = \mathcal{N}_1 \cup \dots \cup \mathcal{N}_k$ with $\mathcal{N}_i = (\mathcal{S}_i, \mathcal{C}_i, \mathcal{R}_i)$ is a $\mathcal{C}$ -decomposition if and only if $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$.

Based on Definition 15, any linkage class decomposition is also $\mathcal{C}$ -decomposition. Moreover, Farinas et al. [5] established the relationship between $\mathcal{C}$ -decomposition and Incidence-independent decomposition.

Corollary. [5]. *Any $\mathcal{C}$ -decomposition is Incidence-independent.*

In their study, Farinas et al. [5] proved a vital formulation to characterize the set of complex balanced equilibria of any weakly reversible CKS via $\mathcal{C}$ -decomposition.

Theorem 8. [5]. *Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_k$ be a weakly reversible $\mathcal{C}$ -decomposition of a chemical kinetic system $(\mathcal{N}, K)$. Then, if $Z_+(\mathcal{N}_i, K_i) \neq \emptyset$ for each subnetwork, $Z_+(\mathcal{N}, K) \neq \emptyset$.*

An important deficiency-based decomposition class was introduced by Farinas et al. in [5].

Definition 16. [5]. A decomposition is a zero deficiency decomposition (ZDD) if and only if each subnetwork has zero deficiency.

2.4 Several results involving PL-TIK decompositions

This subsection highlights some results on the application of a kinetics-based decomposition class to MAK and PL-RDK systems. The objective of this paper is to present an alternative result by applying the aforementioned decomposition, utilizing invertible matrices as the main tool.

Kinetics-based decomposition classes introduced in [24] are fundamental to the results of this study. One of these decomposition classes was used in [22] to explore Deficiency Zero, PL-NDK systems. The said decomposition class is formally defined below.

Definition 17. [24] A decomposition $\mathcal{N} = \mathcal{N}_1 \cup \dots \cup \mathcal{N}_k$ of a PLK system is called a PL-TIK decomposition if and only if each subnetwork $(\mathcal{N}_i, K_i)$ has an associated column-rank maximal $\hat{T}_i$ -matrix. Equivalently, $\hat{\delta}_i = 0$.

Tiongson et al. in [24] showed that for weakly reversible MAK systems, ZDD and PL-TIK decomposition coincide since the entries of $\hat{T}$ are the stoichiometric coefficients of the complexes, and hence $\hat{T}_i = \hat{Y}_i$, that is,

$\hat{\delta}_i = \dim \ker(\hat{T}_i) = \dim \ker(\hat{Y}_i) = \delta_i$. The following results for MAK systems can be found in [23]:

Proposition 9. *Let $(\mathcal{N}, K)$ be a MA-TIK system. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_k$ be any decomposition.*

1. *If $\delta = 0$, then $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if $(\mathcal{N}, K)$ has a weakly reversible decomposition.*
2. *If $\delta > 0$, then $Z_+(\mathcal{N}, K) = \emptyset$.*

Theorem 10. *Let $(\mathcal{N}, K)$ be a non-zero deficiency MA-NTIK system. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_k$ be a zero deficiency, $\mathcal{C}$ -decomposition. Then, $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if the decomposition is weakly reversible.*

Remark. Proposition 9 showed that only Deficiency Zero MA-TIK systems with weakly reversible decomposition are complex balanced. Also, Theorem 10 showed that Non-Zero Deficiency MA-NTIK systems with weakly reversible, zero deficiency, $\mathcal{C}$ -decomposition (and hence weakly reversible, PL-TIK, $\mathcal{C}$ -decomposition) are complex balanced.

The next result, established in [24], is an extension of Theorem 10 applied to a weakly reversible, PL-RDK system.

Theorem 11. *Let $(\mathcal{N}, K)$ be a PL-RDK system with PL-TIK, $\mathcal{C}$ -decomposition. Then $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if the decomposition is weakly reversible.*

Remark. Theorem 11 asserts that any PL-RDK system (whether PL-TIK or not) with a weakly reversible, PL-TIK, $\mathcal{C}$ -decomposition is complex balanced.

3 Results and discussions

This section presents the main results of the study, particularly the results related to PL-TIK systems and PL-TIK decompositions. Moreover, the new construct that we propose in this paper allows an alternative characterization of the set of complex balanced equilibria via the concept of

invertible matrices. Lastly, we apply the aforementioned construct to the biological models of Glucose and Ketone Body Metabolism with Medium-Chain Triglycerides studied by Espina et al. in [4], and the Human Dihydrofolate Reductase (DHFR) Catalysis model studied by Craciun et al. in [3].

3.1 Constructing PL-TIK systems using invertible matrices

The concept of an invertible matrix is commonly known to mathematics students and researchers specializing in Linear Algebra. However, in this study, its usefulness extends to Chemical Reaction Network Theory, particularly to characterize the equilibria of complex dynamical systems with Power Law Reactant-Determined Kinetics formulation. We recall a crucial result which can be found in [14].

Proposition 12. *For any reaction network, if $Y^T \cdot Y$ is invertible, then $\delta = 0$.*

Remark. Note that Y^T is the transpose of the molecularity matrix Y . Also, Hernandez et al. [14] showed in Proposition D.2 that Y has a left generalized inverse if and only if $\text{rank } Y = n$ and $n \leq m$. This means that the invertibility of the matrix $Y^T \cdot Y$ depends on the number of complexes, which should be at most the number of the species in the given CRN.

Our first main result is an analogous result to Proposition 12, where the proof follows the same argument as in the aforementioned proposition. To prove it, we recall from [24] an alternative formulation of $\hat{\delta}$.

Proposition 13. [24] *For a PL-RDK system $(\mathcal{N}, K)$ with T -matrix T , $\ker \hat{T} = \ker T \cap \text{Im } I_a$.*

We are now ready to present our alternative approach to construct a PL-TIK system based on a PL-RDK system's T -matrix.

Proposition 14. *For any PL-RDK system $(\mathcal{N}, K)$ with T -matrix T , if $T^T \cdot T$ is invertible, then $\hat{\delta} = 0$. Equivalently, $(\mathcal{N}, K)$ is a PL-TIK system.*

Proof. Suppose $(\mathcal{N}, K)$ is a PL-RDK system with T-matrix T . Suppose further that $T^T \cdot T$ is invertible. It follows that $\ker(T^T \cdot T) = \{0\}$. Since $\ker T \subseteq \ker(T^T \cdot T)$, it follows that $\ker T = \{0\}$. Hence, $\ker T \cap \text{Im } I_a = \{0\}$. Therefore, by Proposition 2 and Proposition 13,

$$\begin{aligned}\hat{\delta} &= \dim \ker \hat{T} \\ &= \dim (\ker T \cap \text{Im } I_a) \\ &= 0. \blacksquare\end{aligned}$$

Remark. Analogous to Remark 3.1, T has a left generalized inverse if and only if $\text{rank } T = n_r$ and $n_r \leq m$. This means that the invertibility of the matrix $T^T \cdot T$ depends on the number of reactant complexes, which should be at most the number of the species in the given CRN.

3.2 Modified results on PL-TIK systems

This subsection is divided into two parts: (1) Modified Results involving PL-RDK systems, and (2) Modified Results involving MAK systems (since any MAK system is a PL-RDK system).

3.2.1 Modified results on PL-RDK systems

From Proposition 14, we can reformulate known results on PL-TIK systems such as the ZKRDT proposed by Talabis et al. [21], and the Weak Reversibility Theorem for PL-RDK systems established by Tiongson et al. [24].

Proposition 15. *Let $(\mathcal{N}, K)$ be a PL-RDK system with T-matrix T on $\mathcal{N}$ such that $T^T \cdot T$ is invertible. Then $\mathcal{N}$ is weakly reversible if and only if $Z_+(\mathcal{N}, K) \neq \emptyset$.*

Theorem 16. *Let $(\mathcal{N}, K)$ be a PL-RDK system with T-matrix T . Suppose that the decomposition of $\mathcal{N}$ is C-decomposition and for each $(\mathcal{N}_i, K_i)$, $T_i^T \cdot T_i$ is invertible. Then $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if each $\mathcal{N}_i$ is weakly reversible.*

Proof. Suppose $Z_+(\mathcal{N}, K) \neq \emptyset$. Since any $\mathcal{C}$ -decomposition is Incidence-independent from Corollary 2.3, it follows from Theorem 7 that

$Z_+(\mathcal{N}_i, K_i) \neq \emptyset$. Since for each $(\mathcal{N}_i, K_i)$, $T_i^T \cdot T_i$ is invertible, it follows from Proposition 15 that each N_i is weakly reversible.

On the other hand, suppose that each N_i is weakly reversible. Since for each $(\mathcal{N}_i, K_i)$, $T_i^T \cdot T_i$ is invertible, it follows from Proposition 15 that $Z_+(\mathcal{N}_i, K_i) \neq \emptyset$. Since the decomposition is $\mathcal{C}$ -decomposition, it follows from Theorem 8 that $Z_+(\mathcal{N}, K) \neq \emptyset$. ■

Remark. The novelty of the above results lies with the fact that we do not assume the concept of a PL-TIK system; thus, allowing us a PL-TIK decomposition-free analysis of complex balanced equilibria of PL-RDK systems. The only limitation is that, based on Remark 3.1, this result is applicable only when $\text{rank } T = n_r$ and $n_r \leq m$.

3.2.2 Modified results on MAK systems

Since any MAK system is a PL-RDK system, we can adapt the new formulation from Proposition 14 to a generate analogous results from the results in [23]. Moreover, for weakly reversible MAK systems, $T = Y$. Hence, we have the following results.

Proposition 17. *Consider a weakly reversible MAK system $(\mathcal{N}, K)$. If $Y^T \cdot Y$ is invertible, then $\delta = 0 = \hat{\delta}$. Equivalently, $(\mathcal{N}, K)$ is a MA-TIK system.*

Proof. Since any MAK system is PL-RDK and $T = Y$ for a weakly reversible MAK system, it follows from Proposition 14 that $\delta = 0 = \hat{\delta}$, or $(\mathcal{N}, K)$ is a Mass Action, $\hat{T}$ -independent Kinetic (MA-TIK) system. ■

Proposition 18. *Let $(\mathcal{N}, K)$ be a MAK system such that $Y^T \cdot Y$ is invertible. Then $\mathcal{N}$ is weakly reversible if and only if $Z_+(\mathcal{N}, K) \neq \emptyset$.*

Proof. Note that any MAK system is a PL-RDK system. Moreover, for a weakly reversible MAK system, $T = Y$. Hence, this follows directly from Proposition 15. ■

Theorem 19. *Let $(\mathcal{N}, K)$ be a MAK system. Suppose that the decomposition of $\mathcal{N}$ is $\mathcal{C}$ -decomposition and for each $(\mathcal{N}_i, K_i)$, $Y_i^T \cdot Y_i$ is invertible. Then, $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if each N_i is weakly reversible.*

Proof. Suppose $Z_+(\mathcal{N}, K) \neq \emptyset$. Since any $\mathcal{C}$ -decomposition is Incidence-independent from Corollary 2.3, it follows from Theorem 7 that $Z_+(\mathcal{N}_i, K_i) \neq \emptyset$. It follows from Proposition 1 that each N_i is weakly reversible.

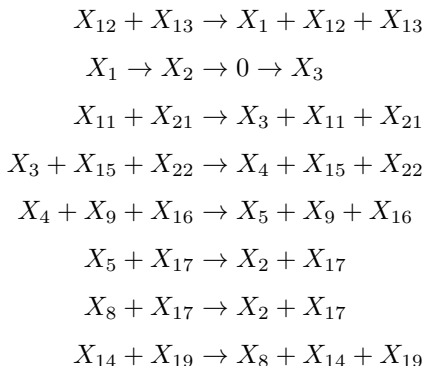
Now, suppose that each N_i is weakly reversible. Hence, $T_i = Y_i$. Since for each $(\mathcal{N}_i, K_i)$, $Y_i^T \cdot Y_i$ is invertible and $T_i = Y_i$, it follows from Proposition 15 that $Z_+(\mathcal{N}_i, K_i) \neq \emptyset$. Thus, since the decomposition is $\mathcal{C}$ -decomposition, it follows from Theorem 8 that $Z_+(\mathcal{N}, K) \neq \emptyset$. ■

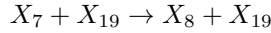
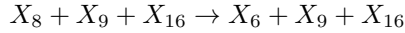
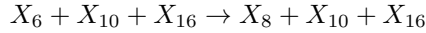
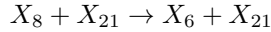
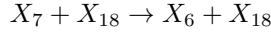
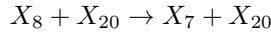
3.3 Application to complex dynamical systems

To conclude this section, we applied our main results to two known dynamical systems. First, to apply either Proposition 15 or Theorem 16, we consider the Glucose and Ketone Body Metabolism with Medium-Chain Triglycerides studied by Espina et al. in [4]. On the other hand, to apply either Proposition 18 or Theorem 19, we consider the Human Dihydrofolate Reductase (DHFR) Catalysis model studied by Craciun et al. in [3].

3.3.1 Glucose and ketone body metabolism with medium-chain triglycerides model

Here, we consider the model explains the importance of Medium-Chain Triglycerides (MCTs) to aid for the body's glucose and ketone body metabolism, which was studied by Espina et al. [4]. Ferreras et al. [11] provided the total CRN representation of the model given below.





Note that both the reactions $X_5 + X_{17} \rightarrow X_2 + X_{17}$ and $X_8 + X_{17} \rightarrow X_2 + X_{17}$ constitute a single linkage class. Thus, we only have $l = 13$. Also, observe that $n_r = 15 < 22 = m$. Note that we removed the zero complex (originally, $n_r = 16$) since when formulating the T -matrix, the kinetic orders of the zero complex are just zero. We then present the T-matrix T , where its associated $\hat{T}$ -matrix can be found in [11], to verify Remark 3.1 that the induced T-matrix T has a left generalized inverse.

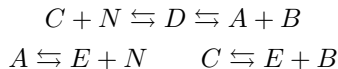
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.1 & 0 & 1.85 & 0 & 1.85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.75 & 0 & 0 \\ 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.005 & 0.005 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By computing the rank of the above matrix, we have $\text{rank } T = 15 = n_r$. This means that the T-matrix T has, indeed, a left generalized inverse.

Using Python in Google Colab to compute the product of large matrices, one can easily verify that the matrix $T^T \cdot T$ is invertible. Thus, it follows from Proposition 14 that the model is a PL-TIK system. However, observe that the total CRN representation of the model is not weakly reversible. Therefore, it follows from Proposition 15 that $Z_+(\mathcal{N}, K) = \emptyset$, that is, the model does not admit complex balanced equilibria.

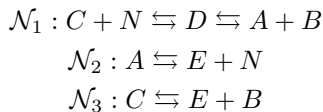
3.3.2 Human dihydrofolate reductase (DHFR) catalysis model

Now, we consider the model that explains the function of the enzyme Dihydrofolate Reductase (DHFR) studied by Craciun et al. in [3]. Feinberg [9] provided the CRN representation of the three groups of reactions, where Tiongson et al. [23] chose the second group of reactions and simplify the corresponding CRN representation below.



where E is DHFR, N is NADP^+ , A is the product of E and N , B is tetrahydrofolate, C is the product of E and B , and D is the product of E , N , and B .

Observe that $n = 7 > 6 = m$. Thus, we cannot apply Proposition 18. Now, we consider the weakly reversible, $\mathcal{C}$ -decomposition of the system.



Computing each $Y_i^T \cdot Y_i$, we have

$$Y_1^T \cdot Y_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$Y_2^T \cdot Y_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$Y_3^T \cdot Y_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that from Linear Algebra, a diagonal matrix with non-zero diagonal entries are invertible [17]. It follows that each $Y_i^T \cdot Y_i$ is invertible. Therefore, since the decomposition is a weakly reversible, $\mathcal{C}$ -decomposition, it follows from Theorem 19 that the subnetwork (second group of reactions) of the Human DHFR Catalysis model is complex balanced.

4 Conclusion and outlook

The results of this study, as well as some research problems for further research, are summarized as follows:

1. We established the applicability of the concept of an invertible matrix to construct a PL-TIK system without using the known $\hat{T}$ -matrix. The resulting formulation provides an alternative algebraic perspective for analyzing complex balanced equilibria in complex dynamical systems. We plan to extend this new construct to GMAK systems, in the sense of Müller and Regensburger [18].
2. We modified and proved known results concerning PL-TIK systems and PL-TIK decompositions using invertible matrices. We also plan to modify the results to Poly-PL Kinetic Systems.
3. We applied these results to the Human DHFR catalysis model and Glucose and Ketone Body Metabolism with MCT model. We recommend the application of the above results to other complex systems with Power Law Reactant-Determined Kinetics.

References

- [1] C. Arceo, E. Jose, A. Marin-Sanguino, E. Mendoza, Chemical reaction network approaches to biochemical systems theory, *Math. Biosci.* **269** (2015) 135–152.
- [2] C. Arceo, E. Jose, A. Lao, E. Mendoza, Reactant subspaces and kinetics of chemical reaction networks, *J. Math. Chem.* **56** (2017) 395–422.
- [3] G. Craciun, Y. Tang, M. Feinberg, Understanding bistability in complex enzyme-driven reaction networks, *Proceed. Nat. Acad. Sci.* **103** (2006) 8697–8702.
- [4] A. Espina, E. Mendoza, A. Lao, Modelling the Effects of Medium-Chain Triglycerides on Cerebral Ketone Body Metabolism, *Frontiers Sys. Biol.* **2** (2022) 907–957, **doi:** 10.3389/fsysb.2022.907957.
- [5] H. Farinas, E. Mendoza, A. Lao, Chemical Reaction Network decompositions and realizations of S -systems, *Philippine Sci. Lett.* **14** (2021) 147–157.
- [6] M. Feinberg, Chemical reaction network structure and the stability of complex isothermal reactors-I. The deficiency zero and deficiency one theorems, *Chem. Engin. Sci.* **42** (1987) 2229–2268.
- [7] M. Feinberg, Complex balancing in general kinetic systems, *Arch. Ration. Mech. Anal.* **49** (1972) 187–194.
- [8] M. Feinberg, The existence and uniqueness of steady states for a class of chemical reaction networks, *Arch. Rational Mech. Anal.* **132** (1995) 311–370, **doi:** 10.1007/BF00375614.
- [9] M. Feinberg, *Foundations of Chemical Reaction Network Theory*, Springer, 2019.
- [10] M. Feinberg, F. Horn, Dynamics of open chemical systems and the algebraic structure of the underlying reaction network, *Chemical Engineering Science* **29** (1974) 775–787.
- [11] M. Ferreras, F. Facundo, J. Tiongson, Reaction network analysis of glucose and ketone body metabolism with medium-chain triglycerides model. Research Square (preprint). (2026), **doi:** 10.21203/rs.3.rs-9846605/v1.
- [12] N. Fortun, A. Lao, L. Razon, E. Mendoza, A deficiency zero theorem for a class of power law kinetic systems with non-reactant-determined interactions, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 621–638.

-
- [13] B. Hernandez, D. Amistas, R. De la Cruz, L. Fontanil, A. de los Reyes V, E. Mendoza, Independent, incidence independent and weakly reversible decompositions of chemical reaction networks, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 367–396.
- [14] B. Hernandez, J. Santos, P. Lubenia, E. Mendoza, The finest decompositions’ architecture of a reaction network, *arXiv*, **doi:** 10.48550/arXiv.2412.15225.
- [15] F. Horn, Necessary and sufficient conditions for complex balancing in chemical kinetics, *Arch. Ration. Mech. Anal.* **49** (1972) 172–186.
- [16] F. Horn, R. Jackson, General mass action kinetics, *Arch. Ration. Mech. Anal.* **47** (1972) 81–116.
- [17] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 2013.
- [18] S. Müller, G. Regensburger, Generalized mass action systems and positive solutions of polynomial equations with real and symbolic exponents, in: V. P. Gerdt, W. Koepf, W. M. Seiler, E. V. Vorozhtsov (Eds.), *Computer Algebra in Scientific Computing*, Springer, Cham, 2014, 302–323, **doi:** 10.1007/978-3-319-10515-4_22.
- [19] D. Talabis, C. Arceo, E. Mendoza, Positive equilibria of a class of power law kinetics, *J. Math. Chem.* **56** (2018) 358–394.
- [20] D. Talabis, E. Jose, E. Mendoza, Positive equilibria of weakly reversible power law kinetics systems with linear independent interactions, *J. Math. Chem.* **56** (2018) 2643–2673.
- [21] D. Talabis, E. Jose, E. Mendoza, Complex balanced equilibria of weakly reversible power law kinetic systems, *MATCH Commun. Math. Comput. Chem.* **82** (2019) 601–624.
- [22] J. Tiongson, A new deficiency zero theorem for power law non-reactant-determined kinetic systems via pl-tik decomposition, *MATCH Commun. Math. Comput. Chem.* **97** (2027) 105–119.
- [23] J. Tiongson, D. Talabis, L. Fontanil, Absolutely complex balanced mass action kinetic systems via zero deficiency decomposition, *MATCH Commun. Math. Comput. Chem.* **97** (2027) 79–103.
- [24] J. Tiongson, D. Talabis, L. Fontanil, Positive equilibria of power law kinetic systems with kinetics-based decompositions, *J. Math. Chem.* **63** (2025) 14–38.