

Degree-Transfer and Deficit Formulae for the Euler–Sombor Index of Trees

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Abstract

The Euler–Sombor index is a degree-based graph invariant obtained by summing the weights $\sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}$ over all edges uv . This paper studies exact degree-transfer formulae for trees and uses them to refine extremal results for fixed matching number, fixed pendent vertices, and fixed diameter. The problem is to identify which local changes force the index to increase, and how the extremal values vary when the parameters change. The method replaces global grafting arguments by verifiable identities for $\omega(x, y) = \sqrt{x^2 + y^2 + xy}$. We prove increment formulae, subdivision and leaf-transfer identities, and a strict branch-concentration theorem. These tools give closed expressions, equality cases, monotonicity laws, structure analysis, and stability gaps for starlike trees, brooms, matching trees, and one-hub diameter trees. The numerical tables correct representative computations and make each edge-degree count reproducible. Several open problems for chemical trees are stated.

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1 Introduction

Degree-based topological indices are among the most useful numerical descriptors in chemical graph theory and extremal graph theory. Their common principle is local, as each edge receives a weight determined by the degrees of its two end-vertices. This local definition makes such indices sensitive to branching, while still allowing exact proofs by transformations of the graph. Classical Zagreb-type and Randić-type descriptors already show that the degree distribution of a molecular graph may encode structural information of chemical interest. The Sombor index, introduced by Gutman in [13], added a geometric viewpoint by assigning to an edge uv the Euclidean length $\sqrt{d(u)^2 + d(v)^2}$. Since then, a large body of work has considered extremal values, chemical applicability, and relations with graph energy; see, for example, [1, 5–11, 19–21, 24, 30, 31].

The Euler–Sombor index is a newer member of this family. It was introduced in the context of geometric constructions of Sombor-type descriptors and is defined by

$$\text{EU}(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}.$$

The additional mixed term $d_G(u)d_G(v)$ makes the index more responsive to adjacent high-degree vertices than the original Sombor weight. Gutman discussed relations between Sombor and Euler-type quantities in [14], and Tang, Li and Deng studied the Euler–Sombor index as a graph invariant in [29]. Recent papers have already treated maximal outerplanar graphs [15], tricyclic graphs [2, 12, 18, 22, 23], unicyclic and bicyclic graphs [26–28], graph products [17], and energy-type questions [3]. For trees, Khanra and Das studied Euler–Sombor extremal problems in several standard classes [16], while Ren, Cao, Wang and Zhou established sharp upper bounds for trees with prescribed matching number, prescribed number of pendent vertices, and prescribed diameter [25].

The present paper is motivated by the following observation. The extremal results in [25] identify the final optimal tree under each constraint, but many computations inside the proofs are local and reusable. When a

branch is moved from one vertex to another, only the degrees in a small neighborhood change. When a pendent edge is subdivided, exactly one edge is replaced by two edges. When a broom is lengthened, a single edge of type (2,2) is added. These elementary changes are often the real engine of the proof. Recording them as independent formulae makes later work easier, especially when one studies new constraints where the global extremal tree is not known in advance.

We work throughout with finite, simple, undirected trees. For $x, y \geq 1$, let $\omega(x, y) = \sqrt{x^2 + y^2 + xy}$. The main structured trees are as follows. A starlike tree $S(\ell_1, \dots, \ell_k)$ has one center and k pendent paths of lengths $\ell_1, \dots, \ell_k$. The broom $B_{n,p}$ is the tree obtained from a star by replacing one pendent edge by a pendent path, so that the order is n and the number of pendent vertices is p . The matching tree $M_{n,a}$ is obtained from a star by subdividing $a - 1$ pendent edges once; it has order n and matching number a . The one-hub diameter tree $D_{n,d}^i$ is obtained from a diametrical path $u_0 u_1 \dots u_d$ by attaching all extra leaves to u_i . These families are not arbitrary examples. They are the compressed objects produced by degree-transfer operations.

In this paper, we give complete proofs for the edge-weight monotonicity and for transfer identities that are sometimes used only implicitly. In particular, we write out the algebra behind the increment $\omega(x, y) - \omega(x - 1, y)$ and prove a branch-concentration theorem by comparing every affected edge. For the structured families above we derive exact formulae, equality cases, strict parameter monotonicity, and explicit stability gaps. These statements describe, in a precise sense, the behavior of the extremal values as the parameters vary and they compare different extremal families with each other.

The results also clarify how the constraints differ. A star maximizes EU among all trees of fixed order, but it is not admissible if the diameter or the matching number is prescribed. A broom maximizes EU for a fixed number of pendent vertices, while a subdivided star maximizes EU for a fixed matching number. The same tree may arise at boundary values, but in general these constraints produce distinct extremal shapes. Exact formulae make this distinction visible: the maximum under a matching

constraint decreases as the matching number grows, whereas the broom value increases as the number of leaves grows at fixed order.

The paper is organized as: Section 2 fixes notation, recalls standard graph-theoretic facts, and states the known Euler–Sombor extremal results for trees. Section 3 proves the local degree calculus, including increment formulae and exact transfer identities. Section 4 treats starlike trees and shows that the Euler–Sombor value depends on long branches only through their number. Section 5 derives formulae, monotonicity laws, and gaps for brooms. Section 6 analyzes the subdivided-star family attached to matching number. Section 7 studies one-hub trees with fixed diameter and gives the endpoint-near stability gap. Section 8 summarizes the contributions and gives future directions.

2 Preliminaries

Let G be a finite simple graph. The vertex set and edge set are denoted by $V(G)$ and $E(G)$. The order and size of G are $|V(G)|$ and $|E(G)|$. The degree of a vertex v is $d_G(v)$, or simply $d(v)$ when the graph is clear. A pendent vertex, or leaf, is a vertex of degree 1. The diameter of G is $\text{diam}(G) = \max\{\text{dist}_G(u, v) : u, v \in V(G)\}$. A tree is a connected acyclic graph.

Definition 1. For $x, y \geq 1$, let $\omega(x, y) = \sqrt{x^2 + y^2 + xy}$. The Euler–Sombor index of a graph G is

$$\text{EU}(G) = \sum_{uv \in E(G)} \omega(d_G(u), d_G(v)).$$

We write P_n for the path of order n and S_n for the star of order n . A matching is a set of pairwise nonincident edges. The matching number is denoted by $\alpha'(G)$. A vertex cover is a set of vertices meeting every edge, and the minimum size of a vertex cover is $\beta(G)$. An independent set is a set of pairwise nonadjacent vertices, and its maximum size is $\alpha(G)$.

The following standard facts are used in equality and comparison arguments.

Lemma 1 (König theorem, [4]). *If G is bipartite, then $\alpha'(G) = \beta(G)$.*

Lemma 2 (Gallai identity, [4]). *If G has order n , then $\alpha(G) + \beta(G) = n$.*

The Euler–Sombor extremal results for trees that motivate the present paper are the next three theorems.

Theorem 1 ([25]). *If T is a tree of order $n \geq 3$, then*

$$2\sqrt{7} + 2(n-3)\sqrt{3} \leq \text{EU}(T) \leq (n-1)\sqrt{(n-1)^2 + n}.$$

The left equality holds if and only if $T \cong P_n$, and the right equality holds if and only if $T \cong S_n$.

Theorem 2 ([25]). *Let $n > 2a \geq 2$. Among all trees of order n with matching number a , the unique tree maximizing EU is the tree obtained from S_{n-a+1} by subdividing $a-1$ pendent edges once.*

Theorem 3 ([25]). *Let $n \geq p+1$ and $p \geq 2$. Among all trees of order n with exactly p pendent vertices, the unique tree maximizing EU is the broom obtained from S_{p+1} by subdividing one pendent edge $n-p-1$ times.*

Theorem 4 ([25]). *Let $2 \leq d \leq n-2$. Among all trees of order n and diameter d , the unique maximizers of EU are the two mirror-image trees obtained by attaching all extra leaves to a vertex adjacent to an endpoint of a diametrical path.*

Above theorems give sharp global extremal graphs, but they leave several useful computations scattered inside proofs. For example, they do not separately state the exact effect of subdividing an edge, the exact dependence of a starlike tree on its short branches, or the parameter monotonicity of the extremal values. In addition, numerical examples in this area are easy to miscalculate because one must count edge types carefully. The remainder of the paper fills these gaps by giving complete local proofs, closed formulae, strict inequalities, and numerical tables.

3 Validated local degree calculus

This section contains the local tools used later. Each statement is written so that the changed edge weights can be checked directly.

The following lemma gives the monotonicity of the one-unit increment. It is the analytic backbone of degree-transfer arguments.

Lemma 3. *For $x \geq 2$ and $y \geq 1$, let $\Delta_y(x) = \omega(x, y) - \omega(x-1, y)$. Then $\Delta_y(x)$ is strictly increasing in x and strictly decreasing in y .*

Proof. As $\frac{\partial}{\partial x} \omega(x, y) = \frac{2x+y}{2\omega(x, y)}$, so we have

$$\frac{\partial}{\partial x} \Delta_y(x) = \frac{2x+y}{2\omega(x, y)} - \frac{2x+y-2}{2\omega(x-1, y)}.$$

The last expression is positive exactly when

$$(2x+y)^2((x-1)^2 + y^2 + (x-1)y) > (2x+y-2)^2(x^2 + y^2 + xy).$$

Which is equivalent to $3y^2(2x+y-1) > 0$. Thus, $\Delta_y(x)$ is strictly increasing in x .

Similarly,

$$\frac{\partial}{\partial y} \Delta_y(x) = \frac{2y+x}{2\omega(x, y)} - \frac{2y+x-1}{2\omega(x-1, y)}.$$

The sign is negative, as the corresponding squared comparison gives

$$\begin{aligned} (2y+x)^2((x-1)^2 + y^2 + (x-1)y) - (2y+x-1)^2(x^2 + y^2 + xy) \\ = -3y(x^2 + 2xy - x - y) < 0. \end{aligned}$$

Here $x^2 + 2xy - x - y = x(x-1) + y(2x-1) > 0$. Hence $\Delta_y(x)$ is strictly decreasing in y . ■

The monotonicity is used in [25]. The contribution here is the algebraic verification, written in a form that can be reused without appealing to an omitted calculation.

The next proposition records the exact effect of subdividing an edge.

Proposition 5. *Let G be a graph and let $uv \in E(G)$ with $d_G(u) = x$ and $d_G(v) = y$. Let G' be obtained from G by subdividing uv once. Then*

$$\text{EU}(G') - \text{EU}(G) = \omega(x, 2) + \omega(2, y) - \omega(x, y).$$

In particular, if $y = 1$, then

$$\text{EU}(G') - \text{EU}(G) = \omega(x, 2) + \sqrt{7} - \omega(x, 1) > 0.$$

Proof. Let w be the vertex inserted on the edge uv . Then the edge uv is deleted and the two edges uw and wv are added. The vertex u loses the neighbour v but gains the neighbour w , and similarly v loses u but gains w . So, $d_{G'}(u) = d_G(u) = x$, and $d_{G'}(v) = d_G(v) = y$, whereas the new vertex has degree $d_{G'}(w) = 2$. For every edge $ab \in E(G) \setminus \{uv\}$, the edge ab is still present in G' and the degrees of its endpoints are the same as before. Thus all these edges contribute equally to $\text{EU}(G)$ and to $\text{EU}(G')$. The only part of the sum that changes is therefore the contribution of the edge uv . In G this contribution is $\omega(x, y)$, while in G' it is replaced by $\omega(x, 2) + \omega(2, y)$. It follows at once that $\text{EU}(G') - \text{EU}(G) = \omega(x, 2) + \omega(2, y) - \omega(x, y)$. Now, suppose that $y = 1$. As $\omega(2, 1) = \sqrt{2^2 + 1^2 + 2 \cdot 1} = \sqrt{7}$, the preceding identity gives $\text{EU}(G') - \text{EU}(G) = \omega(x, 2) + \sqrt{7} - \omega(x, 1)$. Also, $\omega(x, 2)^2 - \omega(x, 1)^2 = (x^2 + 4 + 2x) - (x^2 + 1 + x) = x + 3 > 0$ as $x \geq 1$. So we obtain $\omega(x, 2) > \omega(x, 1)$. Therefore $\text{EU}(G') - \text{EU}(G) = \omega(x, 2) + \sqrt{7} - \omega(x, 1) > 0$. ■

Subdivision is implicit in the extremal constructions of [5, 25]. The exact Euler–Sombor increment is isolated here as a standalone identity.

The next formula is a useful validation tool for adding a new pendent vertex.

Proposition 6. *Let T be a tree and let $v \in V(T)$ have degree x . Let T^+ be obtained by attaching a new leaf to v . If $N_T(v) = \{v_1, \dots, v_x\}$, then*

$$\text{EU}(T^+) - \text{EU}(T) = \omega(x + 1, 1) + \sum_{i=1}^x (\omega(x + 1, d_T(v_i)) - \omega(x, d_T(v_i))).$$

In particular, $\text{EU}(T^+) > \text{EU}(T)$.

Proof. The new pendent edge contributes $\omega(x+1, 1)$ to T^+ . Every old edge not incident with v keeps the same endpoint degrees. Each old edge vv_i changes from weight $\omega(x, d_T(v_i))$ to $\omega(x+1, d_T(v_i))$. Adding the contribution of the new edge and the changes along all edges incident with v , we obtain

$$\text{EU}(T^+) - \text{EU}(T) = \omega(x+1, 1) + \sum_{i=1}^x (\omega(x+1, d_T(v_i)) - \omega(x, d_T(v_i))).$$

Moreover, for each i , we have $d_T(v_i) \geq 1$, and

$$\begin{aligned} \omega(x+1, d_T(v_i))^2 - \omega(x, d_T(v_i))^2 &= ((x+1)^2 + d_T(v_i)^2 + (x+1)d_T(v_i)) \\ &\quad - (x^2 + d_T(v_i)^2 + xd_T(v_i)) \\ &= 2x + 1 + d_T(v_i) > 0. \end{aligned}$$

Thus, we obtain $\omega(x+1, d_T(v_i)) > \omega(x, d_T(v_i))$, and hence $\text{EU}(T^+) > \text{EU}(T)$. ■

This is a local counterpart of the usual order-increase comparisons. It separates the new leaf contribution from the secondary gains caused by the degree increase at the attachment vertex.

The following exact leaf-transfer identity is often useful when one compares two candidate trees with the same order.

Proposition 7. *Let T be a tree containing a leaf z adjacent to v . Let u be a vertex not adjacent to v and not equal to z . Put $a = d_T(u)$ and $b = d_T(v)$. Let T' be obtained from T by deleting vz and adding uz . Then*

$$\begin{aligned} \text{EU}(T') - \text{EU}(T) &= \omega(a+1, 1) - \omega(b, 1) \\ &\quad + \sum_{r \in N_T(u)} (\omega(a+1, d_T(r)) - \omega(a, d_T(r))) \\ &\quad + \sum_{s \in N_T(v) \setminus \{z\}} (\omega(b-1, d_T(s)) - \omega(b, d_T(s))). \end{aligned}$$

Proof. Let $E(T') = (E(T) \setminus \{vz\}) \cup \{uz\}$. Since z is a leaf of T , its neighbour changes from v to u . Thus z remains a leaf in T' , while $d_{T'}(u) = a+1$, and $d_{T'}(v) = b-1$, and all other vertices keep their degrees. Only edges incident with u or v , together with the deleted edge vz and the added edge uz , can change their Euler–Sombor contributions. The edge vz contributes $\omega(b, 1)$ in T , while the new edge uz contributes $\omega(a+1, 1)$ in T' . For each $r \in N_T(u)$, the degree of r is unchanged, so the contribution of ur changes by $\omega(a+1, d_T(r)) - \omega(a, d_T(r))$. Similarly, for each $s \in N_T(v) \setminus \{z\}$, the contribution of vs changes by $\omega(b-1, d_T(s)) - \omega(b, d_T(s))$. All remaining edges possess identical endpoint degrees in T and T' , hence their contributions negate each other. Thus, we have the result

$$\begin{aligned} \text{EU}(T') - \text{EU}(T) &= \omega(a+1, 1) - \omega(b, 1) \\ &\quad + \sum_{r \in N_T(u)} (\omega(a+1, d_T(r)) - \omega(a, d_T(r))) \\ &\quad + \sum_{s \in N_T(v) \setminus \{z\}} (\omega(b-1, d_T(s)) - \omega(b, d_T(s))). \quad \blacksquare \end{aligned}$$

Global branch-moving transformations appear in [25]. Proposition 7 is a precise one-leaf version, as it is intentionally stated as an identity rather than as a broad monotonicity claim, since the sign depends on the local neighborhood.

The final local theorem proves the strict branch-concentration operation used repeatedly in extremal tree arguments.

Theorem 8. *Let T be a tree with an edge uv such that $d_T(u) = x \geq 2$ and $d_T(v) = y \geq 2$. Let $A = N_T(v) \setminus \{u\}$, and let T' be obtained from T by deleting all edges vw with $w \in A$ and adding all edges uw with $w \in A$. Then T' is a tree and $\text{EU}(T') > \text{EU}(T)$.*

Proof. The operation moves the components attached at v through vertices in A from v to u . As each $w \in A$ lies in a component of $T - \{v\}$ not containing u , and these components are pairwise distinct. It does not create a cycle and does not disconnect the graph, so T' is a tree. In T' , the degree of u is $x + y - 1$, the degree of v is 1, and all other degrees are

unchanged. Let $B = N_T(u) \setminus \{v\}$. Then the affected part of the index is

$$\begin{aligned} \text{EU}(T') - \text{EU}(T) &= (\omega(x + y - 1, 1) - \omega(x, y)) \\ &\quad + \sum_{w \in A} (\omega(x + y - 1, d_T(w)) - \omega(y, d_T(w))) \\ &\quad + \sum_{r \in B} (\omega(x + y - 1, d_T(r)) - \omega(x, d_T(r))). \end{aligned}$$

Since $x + y - 1 > y$ and $x + y - 1 > x$, every summand in the two sums is positive. It remains to check the edge uv . By squaring, we have $\omega(x + y - 1, 1)^2 - \omega(x, y)^2 = (x - 1)(y - 1) > 0$. Thus the whole difference is positive, and $\text{EU}(T') - \text{EU}(T) > 0$. ■

The same transformation is a main ingredient in [25]. The proof above fills the gap by showing that every affected edge contributes positively, including the central edge uv .

Example 1. The transformation from P_4 to S_4 is the smallest nontrivial concentration step. Clearly, $\text{EU}(P_4) = 2\sqrt{7} + 2\sqrt{3} \approx 8.7557$ and $\text{EU}(S_4) = 3\sqrt{13} \approx 10.8167$. The gain is $3\sqrt{13} - 2\sqrt{7} - 2\sqrt{3} \approx 2.0610$.

4 Starlike trees

A starlike tree is the simplest setting in which one can see the difference between immediate leaves and long pendent paths.

Definition 2. For integers $k \geq 2$ and $\ell_1, \dots, \ell_k \geq 1$, the starlike tree $S(\ell_1, \dots, \ell_k)$ is obtained by identifying one end of k paths of lengths $\ell_1, \dots, \ell_k$ at a common center. Its order is $1 + \ell_1 + \dots + \ell_k$, and it has k pendent vertices. See $S(2, 3, 4, 5)$ in Figure 1.

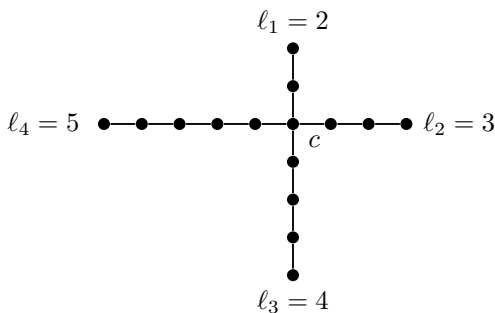


Figure 1. The starlike tree $S(2, 3, 4, 5)$.

The following formula is the main counting result for starlike trees.

Theorem 9. Let $T = S(\ell_1, \dots, \ell_k)$, where $k \geq 3$, $n = |V(T)|$, and let $s = |\{i : \ell_i = 1\}|$. Then

$$\text{EU}(T) = s\sqrt{k^2 + k + 1} + (k - s)(\sqrt{k^2 + 2k + 4} + \sqrt{7}) + 2\sqrt{3}(n - 1 - 2k + s).$$

Proof. The center has degree k . A branch of length one contributes one edge of type $(k, 1)$, hence weight $\sqrt{k^2 + k + 1}$. A branch of length at least two contributes one edge of type $(k, 2)$, one terminal edge of type $(2, 1)$, and $\ell_i - 2$ internal edges of type $(2, 2)$. These weights are $\sqrt{k^2 + 2k + 4}$, $\sqrt{7}$, and $2\sqrt{3}$. Summing over all branches, we have

$$\text{EU}(T) = s\sqrt{k^2 + k + 1} + (k - s)(\sqrt{k^2 + 2k + 4} + \sqrt{7}) + 2\sqrt{3} \sum_{\ell_i \geq 2} (\ell_i - 2).$$

Since $\sum_{\ell_i \geq 2} (\ell_i - 2) = n - 1 - 2k + s$, the formula follows. ■

Starlike trees occur naturally in Sombor-type extremal problems [5, 10, 25]. Theorem 9 records the complete Euler–Sombor edge count in closed form.

The formula immediately gives an invariance principle.

Corollary 1. For fixed n , k , and s , all starlike trees $S(\ell_1, \dots, \ell_k)$ with exactly s branches of length one have the same Euler–Sombor index.

Proof. The right side of Theorem 9 depends only on n , k , and s , not on the individual lengths of the long branches. ■

This invariance is usually hidden by drawing a particular representative. It shows that balancing long starlike branches has no effect on EU once the number of short branches is fixed.

The next theorem gives a strict ordering inside the starlike class.

Theorem 10. *For fixed n and $k \geq 3$, among all starlike trees $S(\ell_1, \dots, \ell_k)$ of order n , the Euler–Sombor index is strictly increasing in the number s of branches of length one.*

Proof. By Theorem 9, increasing s by one, changes EU by

$$\eta_k = \sqrt{k^2 + k + 1} - \sqrt{k^2 + 2k + 4} + 2\sqrt{3} - \sqrt{7}.$$

It remains to show $\eta_k > 0$. Let $h(k) = \sqrt{k^2 + 2k + 4} - \sqrt{k^2 + k + 1}$. For real $k \geq 3$, we have

$$h'(k) = \frac{k+1}{\sqrt{k^2 + 2k + 4}} - \frac{2k+1}{2\sqrt{k^2 + k + 1}} < 0,$$

since, after squaring positive quantities, the required inequality is equivalent to

$$(2k+1)^2(k^2 + 2k + 4) - 4(k+1)^2(k^2 + k + 1) = 3k(3k+2) > 0.$$

Thus $h(k) \leq h(3) = \sqrt{19} - \sqrt{13}$. A direct square comparison gives $\sqrt{19} - \sqrt{13} < 2\sqrt{3} - \sqrt{7}$. Hence $\eta_k > 0$ for every $k \geq 3$. ■

The fixed-pendent-vertex theorem in [25] identifies a broom globally. Theorem 10 explains the internal starlike mechanism, as every additional immediate leaf at the center strictly increases the index.

The following corollary gives the exact starlike maximum and minimum.

Corollary 2. *For fixed n , in the class of starlike trees of order n with $k \geq 3$ leaves, the maximum is attained by the broom $B_{n,k}$. The minimum is attained exactly by the starlike trees with $s = \max\{0, 2k - n + 1\}$.*

Proof. Theorem 10 shows that the maximum is obtained by maximizing s , the number of branches of length one. If $n = k + 1$, the tree is a star. Otherwise the largest possible value is $s = k - 1$, which is exactly the broom structure. The smallest value of s is obtained by making as many branches as possible have length at least two. Since the total branch length is $n - 1$, this minimum is $\max\{0, 2k - n + 1\}$. ■

The minimum statement is not part of the fixed-leaf maximum theorem. It gives all equality cases at the opposite end of the starlike class.

The next proposition records the exact effect of lengthening a long branch.

Proposition 11. *Let $T = S(\ell_1, \dots, \ell_k)$ with $k \geq 3$. If some branch has length at least two and is lengthened by one edge, then EU increases by exactly $2\sqrt{3}$.*

Proof. Lengthening a branch of length at least two adds one internal edge of type (2, 2) and does not change the center degree or the types of the previous edges. The added contribution is $\omega(2, 2) = 2\sqrt{3}$. ■

This proposition is a local version of the order-increment law for brooms. It is useful in asymptotic comparisons with fixed leaf number.

Example 2. Let $n = 10$ and $k = 4$. The trees $S(1, 1, 1, 6)$ (see Figure 2), $S(1, 1, 2, 5)$, and $S(1, 2, 3, 3)$ have $s = 3$, $s = 2$, and $s = 1$. Their Euler–Sombor indices are strictly ordered, in agreement with Theorem 10, see Table 1.

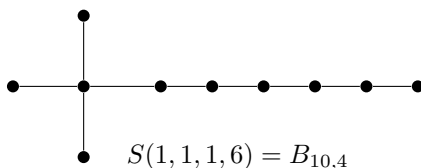


Figure 2. The starlike maximizer for $n = 10$ and $k = 4$.

Table 1. Validated starlike comparison for $n = 10$ and $k = 4$.

Tree	s	Exact expression	Decimal value
$S(1, 1, 1, 6)$	3	$3\sqrt{21} + \sqrt{28} + \sqrt{7} + 8\sqrt{3}$	35.5414
$S(1, 1, 2, 5)$	2	$2\sqrt{21} + 2(\sqrt{28} + \sqrt{7}) + 6\sqrt{3}$	35.4320
$S(1, 2, 3, 3)$	1	$\sqrt{21} + 3(\sqrt{28} + \sqrt{7}) + 4\sqrt{3}$	35.3225

5 Brooms with prescribed pendent vertices

The broom is the global maximizer for a fixed number of pendent vertices. Here we record additional formulae and parameter laws.

Definition 3. For $n \geq p + 1$ and $p \geq 2$, let $B_{n,p}$ be the tree obtained from S_{p+1} by subdividing one pendent edge $n - p - 1$ times. Equivalently, for $n > p + 1$ it is $S(1, \dots, 1, n - p)$ with $p - 1$ branches of length one. For $B_{10,4}$, see Figure 2.

The exact value of $\text{EU}(B_{n,p})$ is as follows.

Theorem 12. For $n \geq p + 2$ and $p \geq 2$,

$$\text{EU}(B_{n,p}) = (p - 1)\sqrt{p^2 + p + 1} + \sqrt{p^2 + 2p + 4} + 2(n - p - 2)\sqrt{3} + \sqrt{7}.$$

For the boundary case $n = p + 1$, $B_{n,p} = S_n$ and $\text{EU}(B_{p+1,p}) = p\sqrt{p^2 + p + 1}$.

Proof. If $n \geq p + 2$, the broom is the starlike tree with $k = p$ and $s = p - 1$. The formula is therefore Theorem 9. If $n = p + 1$, the broom is the star S_{p+1} , and all p edges have type $(p, 1)$. ■

The first formula is the fixed-leaf extremal value of [25]. The boundary case is written separately to avoid the common error of substituting $n = p + 1$ into the handle formula.

The next proposition gives the exact order increment.

Proposition 13. For $n \geq p + 2$ and $p \geq 2$,

$$\text{EU}(B_{n+1,p}) - \text{EU}(B_{n,p}) = 2\sqrt{3}.$$

Proof. Increasing n by one lengthens the handle by one internal edge of type $(2, 2)$. Thus the added contribution is $\omega(2, 2) = 2\sqrt{3}$. ■

This isolates the linear part of the fixed-leaf extremal value; the non-linear dependence is entirely in p .

The following theorem gives the strict leaf monotonicity at fixed order.

Theorem 14. *If $n \geq 4$, then*

$$\text{EU}(B_{n,2}) < \text{EU}(B_{n,3}) < \cdots < \text{EU}(B_{n,n-1}).$$

Proof. First suppose $2 \leq p \leq n-3$, so both $B_{n,p}$ and $B_{n,p+1}$ have a genuine handle. Let $A_p = \sqrt{p^2 + p + 1}$ and $C_p = \sqrt{p^2 + 2p + 4}$. By Theorem 12, we have

$$\text{EU}(B_{n,p+1}) - \text{EU}(B_{n,p}) = pA_{p+1} - (p-1)A_p + C_{p+1} - C_p - 2\sqrt{3}.$$

Since $A_{p+1} > A_p$ and $C_{p+1} > C_p$, the right side is greater than $A_{p+1} - 2\sqrt{3}$. For $p \geq 2$, $A_{p+1} \geq \sqrt{13} > 2\sqrt{3}$, so the difference is positive. It remains to compare $B_{n,n-2}$ with $B_{n,n-1} = S_n$. The tree $B_{n,n-2}$ is not the star, and Theorem 1 gives $\text{EU}(B_{n,n-2}) < \text{EU}(S_n)$. ■

The fixed-leaf theorem says which tree is best for each p . Theorem 14 compares those best values as p varies.

The next corollary gives the gap between a broom and the path.

Corollary 3. *For $n \geq p+2$ and $p \geq 2$,*

$$\text{EU}(B_{n,p}) - \text{EU}(P_n) = (p-1)\sqrt{p^2 + p + 1} + \sqrt{p^2 + 2p + 4} - \sqrt{7} - 2(p-1)\sqrt{3}.$$

This gap is nonnegative, and equality holds if and only if $p = 2$.

Proof. By Theorem 1, $\text{EU}(P_n) = 2\sqrt{7} + 2(n-3)\sqrt{3}$. Subtracting this from the formula in Theorem 12 gives the required identity. Nonnegativity follows from Theorem 3: among trees with p leaves, the broom is maximal, and the case $p = 2$ is exactly the path. If $p > 2$, the path is not in the

same fixed-leaf class, and the strict monotonicity in Theorem 14 gives a strict gap. ■

This formula corrects the common off-by-one term in the path gap, as the coefficient is $2(p-1)\sqrt{3}$, not $2p\sqrt{3}$.

The following proposition compares a broom with the unrestricted star.

Proposition 15. *For $n \geq p+2$ and $2 \leq p \leq n-2$,*

$$\text{EU}(S_n) - \text{EU}(B_{n,p}) > 0.$$

Moreover, the exact gap is

$$(n-1)\sqrt{n^2-n+1} - (p-1)\sqrt{p^2+p+1} - \sqrt{p^2+2p+4} - 2(n-p-2)\sqrt{3} - \sqrt{7}.$$

Proof. The exact expression is obtained by subtracting Theorem 12 from $\text{EU}(S_n) = (n-1)\sqrt{(n-1)^2+n} = (n-1)\sqrt{n^2-n+1}$. The positivity follows from Theorem 1, as $B_{n,p}$ is not a star when $p \leq n-2$. ■

The exact star gap is a convenient stability measure: it quantifies how much is lost when the number of leaves is forced below $n-1$.

Example 3. For $n=10$, the broom values increase with p . In particular,

$$\text{EU}(B_{10,2}) = 2\sqrt{7} + 14\sqrt{3} \approx 29.5402,$$

whereas

$$\text{EU}(B_{10,5}) = 4\sqrt{31} + \sqrt{39} + 6\sqrt{3} + \sqrt{7} \approx 41.5541.$$

For more numerical comparison of brooms of order n , see Table 2

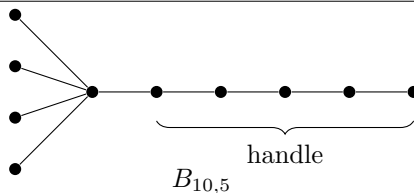


Figure 3. The broom $B_{10,5}$.

Table 2. Broom values for fixed order $n = 10$.

p	Exact expression	Decimal value
2	$2\sqrt{7} + 14\sqrt{3}$	29.5402
3	$2\sqrt{13} + \sqrt{19} + 10\sqrt{3} + \sqrt{7}$	31.5363
4	$3\sqrt{21} + \sqrt{28} + 8\sqrt{3} + \sqrt{7}$	35.5414
5	$4\sqrt{31} + \sqrt{39} + 6\sqrt{3} + \sqrt{7}$	41.5541
9	$9\sqrt{91}$	85.8545

6 Subdivided stars and matching number

We now study the structured tree that appears in the fixed-matching-number theorem.

Definition 4. Let $n \geq 2a$ and $a \geq 1$. The matching tree $M_{n,a}$ is obtained from the star S_{n-a+1} by subdividing $a - 1$ pendent edges once. See $M_{10,3}$ in Figure 4.

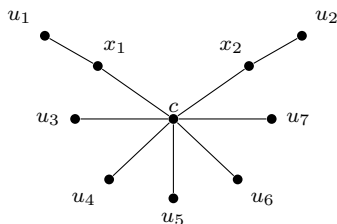


Figure 4. The matching tree $M_{10,3}$. It is obtained from the star S_8 by subdividing two pendent edges once.

The following structural facts justify the notation.

Proposition 16. *The tree $M_{n,a}$ has order n , maximum degree $n - a$, exactly $n - a$ pendent vertices, and matching number a .*

Proof. The star S_{n-a+1} has $n - a + 1$ vertices. Subdividing $a - 1$ pendent edges adds $a - 1$ new vertices, so the order is n . The center remains incident with all $n - a$ original pendent edges or their first subdivision vertices, hence its degree is $n - a$. The leaves are the $n - 2a + 1$ unsubdivided original leaves together with the $a - 1$ terminal leaves of the subdivided branches, giving $n - a$ leaves. A matching of size a is obtained by taking each terminal edge in the $a - 1$ subdivided branches and one unsubdivided pendent edge. No larger matching exists because every matching can use at most one edge incident with the center and at most one terminal edge from each subdivided branch. Hence $\alpha'(M_{n,a}) = a$. ■

The fixed-matching extremal tree is defined in [25]. The proposition records all parameters that are needed when comparing it with brooms and stars.

The exact Euler–Sombor value is obtained by counting two edge types.

Theorem 17. *Let $n \geq 2a$ with $a \geq 1$, and let $q = n - a$. Then*

$$\text{EU}(M_{n,a}) = (n - 2a + 1)\sqrt{q^2 + q + 1} + (a - 1)(\sqrt{q^2 + 2q + 4} + \sqrt{7}).$$

Proof. The center has degree $q = n - a$. There are $n - 2a + 1$ unsubdivided pendent edges of type $(q, 1)$. Each of the $a - 1$ subdivided branches contributes one edge of type $(q, 2)$ and one terminal edge of type $(2, 1)$. Summing these contributions proves the formula. ■

The maximum value decreases when the matching number is forced upward.

Theorem 18. *Let $n \geq 4$ and $1 \leq a < \lfloor \frac{n}{2} \rfloor$. Then $\text{EU}(M_{n,a}) > \text{EU}(M_{n,a+1})$. Consequently,*

$$\text{EU}(M_{n,1}) > \text{EU}(M_{n,2}) > \cdots > \text{EU}(M_{n,\lfloor n/2 \rfloor}).$$

Proof. For $1 \leq a < \lfloor \frac{n}{2} \rfloor$, we have $a+1 \leq \lfloor \frac{n}{2} \rfloor$, and therefore $2(a+1) \leq n$. Thus Definition 4 applies to both $M_{n,a}$ and $M_{n,a+1}$. Let $q = n - a$, and write $A_q = \sqrt{q^2 + q + 1}$, $C_q = \sqrt{q^2 + 2q + 4}$. From Theorem 17, we obtain

$$\begin{aligned} \text{EU}(M_{n,a}) - \text{EU}(M_{n,a+1}) &= (n - 2a - 1)(A_q - A_{q-1}) + 2A_q \\ &\quad + (a - 1)(C_q - C_{q-1}) - C_{q-1} - \sqrt{7}. \end{aligned}$$

As $n \geq 2a + 2$, so $n - 2a - 1 \geq 1$. Moreover, $A_q > A_{q-1}$, and $C_q > C_{q-1}$. Therefore, it suffices to show $2A_q > C_{q-1} + \sqrt{7}$. Since

$$C_{q-1} = \sqrt{(q-1)^2 + 2(q-1) + 4} = \sqrt{q^2 + 3},$$

the required inequality becomes

$$2\sqrt{q^2 + q + 1} > \sqrt{q^2 + 3} + \sqrt{7}.$$

Here the hypothesis implies $q \geq 3$, and the inequality is strict for every such q . Consequently,

$$\text{EU}(M_{n,a}) > \text{EU}(M_{n,a+1}).$$

With successive application of this comparison, we obtain

$$\text{EU}(M_{n,1}) > \text{EU}(M_{n,2}) > \cdots > \text{EU}(M_{n,\lfloor n/2 \rfloor}). \quad \blacksquare$$

Theorem 18 compares the extremal values from Theorem 2 as the prescribed matching number changes.

At fixed matching number, increasing the order increases the value.

Proposition 19. *If $n \geq 2a$, then*

$$\text{EU}(M_{n+1,a}) > \text{EU}(M_{n,a}).$$

Proof. The tree $M_{n+1,a}$ is obtained from $M_{n,a}$ by adding one new leaf at the center and increasing the center degree from $n - a$ to $n - a + 1$. Proposition 6 shows that the Euler–Sombor index strictly increases. $\blacksquare$

The above proposition identifies the direction of growth of the fixed-matching extremal value with respect to order.

The next comparison links the matching tree with the broom having the same number of leaves.

Theorem 20. *Let $n \geq 2a$ and let $p = n - a$. Then*

$$\text{EU}(B_{n,p}) \geq \text{EU}(M_{n,a}).$$

Equality holds if and only if $a = 1$ or $a = 2$.

Proof. By Proposition 16, $M_{n,a}$ has exactly $p = n - a$ pendent vertices. By Theorem 3, $B_{n,p}$ is the unique tree of order n with p pendent vertices maximizing EU. Therefore $\text{EU}(B_{n,p}) \geq \text{EU}(M_{n,a})$. If $a = 1$, both trees are the star S_n . If $a = 2$, both trees have one pendent path of length two and all other leaves adjacent to the center, so they are isomorphic. If $a \geq 3$, then $M_{n,a}$ has at least two length-two branches, while $B_{n,p}$ has only one nontrivial handle; hence the trees are not isomorphic and uniqueness gives strict inequality. ■

The above result compares two different extremal regimes. It also corrects the equality case, as the two constructions agree for $a = 1$ and $a = 2$, but not generally at $n = 2a$.

The following corollary locates the matching tree between the star and the path when the parameters are nontrivial.

Corollary 4. *For $n \geq 2a \geq 4$,*

$$\text{EU}(P_n) < \text{EU}(M_{n,a}) < \text{EU}(S_n)$$

unless $M_{n,a} \cong P_n$ at $n = 4$ and $a = 2$, where the left equality holds.

Proof. The upper inequality follows from Theorem 1, since $M_{n,a}$ is not a star for $a \geq 2$. The lower inequality follows from the same theorem unless the matching tree is the path. The only such case with $n \geq 2a \geq 4$ is $M_{4,2} \cong P_4$. ■

The above consequence gives a quick consistency check for the numerical range of the matching-tree formula.

Example 4. For $n = 10$, the matching-tree values strictly decrease as a increases. For instance,

$$\text{EU}(M_{10,2}) = 7\sqrt{73} + \sqrt{84} + \sqrt{7} \approx 71.6189,$$

whereas

$$\text{EU}(M_{10,5}) = \sqrt{31} + 4\sqrt{39} + 4\sqrt{7} \approx 41.1308.$$

The value of $B_{10,5}$ is larger, namely 41.5541, so $M_{10,5}$ and $B_{10,5}$ do not coincide.

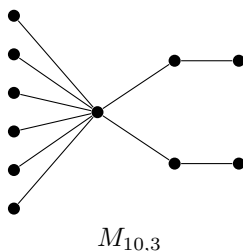


Figure 5. The matching tree $M_{10,3}$.

Figure 5 shows the matching tree $M_{10,3}$, and Table 3 gives the value of EU for order 10.

Table 3. Matching-tree values for fixed order $n = 10$.

a	Structure	Decimal value of $\text{EU}(M_{10,a})$
1	star S_{10}	85.8545
2	one subdivided pendent edge	71.6189
3	two subdivided pendent edges	59.4114
4	three subdivided pendent edges	49.2429
5	four subdivided pendent edges and one direct leaf	41.1308

7 One-hub trees with fixed diameter

The fixed-diameter extremal tree is a one-hub caterpillar whose hub is adjacent to an endpoint of a diametrical path. This section gives the exact formula for every hub position and the corresponding stability gap.

Definition 5. Let $n \geq d + 1$ and $1 \leq i \leq d - 1$. Let $D_{n,d}^i$ be the tree obtained from the path $u_0 u_1 \dots u_d$ by attaching $n - d - 1$ new leaves to u_i . Put $q = n - d + 1$; then $d_{D_{n,d}^i}(u_i) = q$. See $D_{10,5}^2$ in Figure 6.

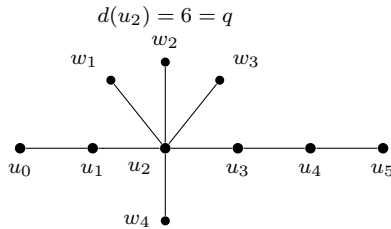


Figure 6. The diameter-hub tree $D_{10,5}^2$.

The complete one-hub formula is the following.

Theorem 21. Let $2 \leq d \leq n - 2$, and put $q = n - d + 1$. For the one-hub tree $D_{n,d}^i$ defined in Definition 5, the following formulas hold.

If $d = 2$, then necessarily $i = 1$ and $D_{n,2}^1 \cong S_n$. Consequently,

$$\text{EU}(D_{n,2}^1) = (n - 1)\sqrt{n^2 - n + 1}.$$

Equivalently, since $q = n - 1$,

$$\text{EU}(D_{n,2}^1) = q\sqrt{q^2 + q + 1}.$$

If $3 \leq d \leq n - 2$ and $i = 1$ or $i = d - 1$, then

$$\text{EU}(D_{n,d}^i) = (q - 1)\sqrt{q^2 + q + 1} + \sqrt{q^2 + 2q + 4} + 2(d - 3)\sqrt{3} + \sqrt{7}.$$

If $4 \leq d \leq n - 2$ and $2 \leq i \leq d - 2$, then

$$\text{EU}(D_{n,d}^i) = (q - 2)\sqrt{q^2 + q + 1} + 2\sqrt{q^2 + 2q + 4} + 2(d - 4)\sqrt{3} + 2\sqrt{7}.$$

Proof. First suppose that $d = 2$. Then the underlying diametrical path is $u_0u_1u_2$, and all $n - 3$ additional leaves are attached to u_1 . Hence every vertex other than u_1 is pendent and $d(u_1) = 2 + (n - 3) = n - 1 = q$. Thus $D_{n,2}^1 \cong S_n$. There are exactly $n - 1 = q$ edges, and every edge has degree type $(q, 1)$. Since $\omega(q, 1) = \sqrt{q^2 + q + 1}$, so we obtain

$$\text{EU}(D_{n,2}^1) = q\sqrt{q^2 + q + 1}.$$

Using $q = n - 1$, we obtain

$$\text{EU}(D_{n,2}^1) = (n - 1)\sqrt{n^2 - n + 1}.$$

Now assume that $d \geq 3$ and take $i = 1$. The vertex u_1 has degree $d(u_1) = q$. Its pendent neighbors consist of u_0 together with the $n - d - 1$ newly attached leaves, and their number is

$$1 + (n - d - 1) = n - d = q - 1.$$

Therefore these edges contribute $(q - 1)\sqrt{q^2 + q + 1}$. Since $d \geq 3$, the vertex u_2 has degree 2, and hence the edge u_1u_2 is of type $(q, 2)$ and contributes $\sqrt{q^2 + 2q + 4}$. Along the remainder of the diametrical path there is one terminal edge of type $(2, 1)$, contributing $\sqrt{7}$, and exactly $d - 3$ edges of type $(2, 2)$, each contributing $\omega(2, 2) = 2\sqrt{3}$. It follows that

$$\text{EU}(D_{n,d}^1) = (q - 1)\sqrt{q^2 + q + 1} + \sqrt{q^2 + 2q + 4} + 2(d - 3)\sqrt{3} + \sqrt{7}.$$

The case $i = d - 1$ follows by symmetry.

Finally, suppose that $4 \leq d \leq n - 2$ and $2 \leq i \leq d - 2$. The hub u_i has exactly $n - d - 1 = q - 2$ new pendent neighbors, and their total contribution is

$$(q - 2)\sqrt{q^2 + q + 1}.$$

Both path neighbors u_{i-1} and u_{i+1} have degree 2, so the two edges incident with the hub along the diametrical path contribute $2\sqrt{q^2 + 2q + 4}$. The two terminal path edges u_0u_1 and $u_{d-1}u_d$ are of type $(2, 1)$ and contribute $2\sqrt{7}$. After removing these two terminal edges and the two path edges incident with the hub, exactly $d - 4$ edges of the diametrical path remain. They are all of type $(2, 2)$, and hence contribute $2(d - 4)\sqrt{3}$. Adding these four contributions, we have

$$\text{EU}(D_{n,d}^i) = (q - 2)\sqrt{q^2 + q + 1} + 2\sqrt{q^2 + 2q + 4} + 2(d - 4)\sqrt{3} + 2\sqrt{7}.$$

■

Remark. The separate case $d = 2$ is essential. Direct substitution of $d = 2$ into the endpoint-near formula would give terms corresponding to edges of types $(q, 2)$ and $(2, 1)$, although no vertex of degree 2 occurs in $D_{n,2}^1$. In fact, $D_{n,2}^1 = S_n$, so its Euler–Sombor index must be $(n - 1)\sqrt{n^2 - n + 1}$. For $d = 3$, the endpoint-near formula is valid because the edge joining the hub to the next path vertex is of type $(q, 2)$, while the coefficient $2(d - 3)\sqrt{3}$ vanishes, as it should. The interior-hub formula begins only at $d = 4$, since the range $2 \leq i \leq d - 2$ is empty for $d \leq 3$.

The endpoint-near case is the fixed-diameter extremal value in [25]. The formula above includes every one-hub position.

The endpoint-near advantage has a clean closed form.

Theorem 22. *Let $2 \leq i \leq d - 2$ and $q = n - d + 1 \geq 3$. Then*

$$\text{EU}(D_{n,d}^1) - \text{EU}(D_{n,d}^i) = 2\sqrt{3} - \sqrt{7} - (\sqrt{q^2 + 2q + 4} - \sqrt{q^2 + q + 1}) > 0.$$

Thus, within the one-hub family, the largest Euler–Sombor index is attained by $D_{n,d}^1$ and $D_{n,d}^{d-1}$.

Proof. Subtracting the two formulae in Theorem 21 gives the expression for $\text{EU}(D_{n,d}^1) - \text{EU}(D_{n,d}^i)$. Let $g(q) = \sqrt{q^2 + 2q + 4} - \sqrt{q^2 + q + 1}$. Then as in the proof of Theorem 10, $g(q)$ is decreasing for $q \geq 3$. Hence $g(q) \leq g(3) = \sqrt{19} - \sqrt{13} < 2\sqrt{3} - \sqrt{7}$. So, the difference is positive, and the result follows. ■

The above theorem extracts the numerical gap behind the global fixed-diameter theorem of [25].

The endpoint-near tree is also a broom.

Corollary 5. *For $2 \leq d \leq n - 2$, $D_{n,d}^1 \cong B_{n,n-d+1}$. Consequently, the fixed-diameter extremal value is the broom value with $p = n - d + 1$.*

Proof. The tree $D_{n,d}^1$ has a center of degree $n - d + 1$, exactly $n - d$ leaf neighbors at that center, and one handle from u_1 to u_d of length $d - 1$. This is precisely the broom $B_{n,n-d+1}$. ■

The above consequence identifies the overlap between the leaf-constrained and diameter-constrained extremal families.

The next theorem gives monotonicity of the fixed-diameter extremal value as the diameter changes.

Theorem 23. *For $n \geq 4$, the values $\text{EU}(D_{n,d}^1)$ are strictly decreasing as d increases from 2 to $n - 2$.*

Proof. By Corollary 5, $D_{n,d}^1 \cong B_{n,n-d+1}$. As d increases by one, the corresponding number of leaves $p = n - d + 1$ decreases by one. The strict monotonicity now follows from Theorem 14. ■

The fixed-diameter theorem identifies the best tree for each diameter. Theorem 23 compares those best values across diameters.

The stability gap increases with the hub degree.

Proposition 24. *Let*

$$\delta(q) = 2\sqrt{3} - \sqrt{7} - (\sqrt{q^2 + 2q + 4} - \sqrt{q^2 + q + 1}).$$

Then for $q \geq 3$, $\delta(q) > 0$ and $\delta(q)$ is strictly increasing in q .

Proof. The positivity was proved in Theorem 22. Since $g(q) = \sqrt{q^2 + 2q + 4} - \sqrt{q^2 + q + 1}$ is strictly decreasing for $q \geq 3$, the function $\delta(q) = 2\sqrt{3} - \sqrt{7} - g(q)$ is strictly increasing. ■

The above proposition quantifies stability, as moving the hub away from the endpoint becomes more costly when the hub degree is larger.

Example 5. Let $n = 10$ and $d = 5$. Then $q = 6$. The endpoint-near one-hub tree has value

$$5\sqrt{43} + \sqrt{52} + 4\sqrt{3} + \sqrt{7} \approx 49.5722.$$

An interior one-hub tree has value

$$4\sqrt{43} + 2\sqrt{52} + 2\sqrt{3} + 2\sqrt{7} \approx 49.4076.$$

The endpoint-near advantage is approximately 0.1647, see Figure 7 and Table 4.

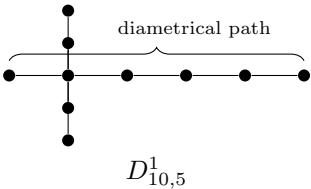


Figure 7. The endpoint-near one-hub diameter tree $D_{10,5}^1$.

Table 4. One-hub diameter values for $n = 10$ and $d = 5$.

Hub position	Exact expression	Decimal value
$i = 1$ or 4	$5\sqrt{43} + \sqrt{52} + 4\sqrt{3} + \sqrt{7}$	49.5722
$i = 2$ or 3	$4\sqrt{43} + 2\sqrt{52} + 2\sqrt{3} + 2\sqrt{7}$	49.4076
Deficit	$2\sqrt{3} - \sqrt{7} - (\sqrt{52} - \sqrt{43})$	0.1647

Here, we collect the computations in one place. The purpose is not to replace the proofs, but to make the edge-type counts and parameter behavior transparent.

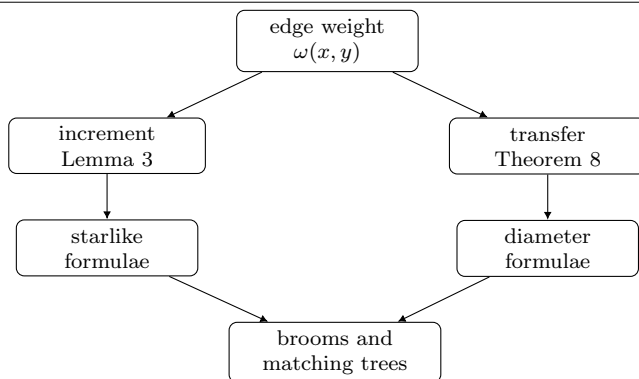


Figure 8. Block diagram of the method.

Figure 8 shows the block diagram of the methods given in above sections. The local weight calculus feeds exact formulae for structured tree families. Table 5 gives the comparison of Euler–Sombor values. All values are rounded to four decimals.

Table 5. Comparison of representative Euler–Sombor values.

Constraint	Candidate	Exact value	Decimal value
Order 10 only	P_{10}	$2\sqrt{7} + 14\sqrt{3}$	29.5402
Order 10, $p = 4$ leaves	$B_{10,4}$	$3\sqrt{21} + \sqrt{28} + 8\sqrt{3} + \sqrt{7}$	35.5414
Order 10, $a = 3$ matching	$M_{10,3}$	$5\sqrt{57} + 2\sqrt{67} + 2\sqrt{7}$	59.4114
Order 10, $d = 5$ diameter	$D_{10,5}^1$	$5\sqrt{43} + \sqrt{52} + 4\sqrt{3} + \sqrt{7}$	49.5722
Order 10 only	S_{10}	$9\sqrt{91}$	85.8545

Table 5 shows that the same order can lead to very different Euler–Sombor values once additional constraints are imposed. The star is the unrestricted maximum, but it is inadmissible under most nontrivial matching or diameter constraints. Brooms and matching trees both concentrate degree at one center, yet their branch structures are different, which explains the strict comparison in Theorem 20. Based on our results and numerical computations, we pose the following problems

Open Problem 25. *Determine the maximum Euler–Sombor index among chemical trees of order n with a prescribed number of pendent vertices, where chemical means maximum degree at most 4.*

Open Problem 26. For $0 \leq \lambda \leq 2$, define

$$\text{EU}_\lambda(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + \lambda d(u)d(v)}.$$

Which degree-transfer and stability results in this paper remain valid uniformly in λ ?

8 Conclusion and future work

This paper developed a validated degree-transfer framework for the Euler–Sombor index of structured trees. The main contributions are as follows. First, the local increment $\Delta_y(x) = \omega(x, y) - \omega(x-1, y)$ was proved with full algebraic details, and exact subdivision, leaf-extension, and leaf-transfer identities were stated. Second, a strict branch-concentration theorem was proved by checking every affected edge weight. Third, a closed formula for starlike trees was obtained, together with an invariance principle for long branches and a strict ordering by the number of length-one branches. Fourth, brooms were analyzed through formulae, order increments, monotonicity in the number of leaves, and explicit gaps from the path and star. Fifth, matching trees and one-hub diameter trees were compared through exact formulae, monotonicity laws, and stability gaps.

The main limitation is that the paper focuses on structured tree families. When a new constraint is imposed, one must still prove that an extremal tree belongs to one of these compressed families or to a related family. This is not automatic. For example, a chemical maximum-degree restriction prevents unrestricted branch concentration, and the resulting extremal trees may require different transformations.

Future work may proceed in three directions. The first is to solve the chemical-tree version of the fixed-leaf and fixed-diameter problems. The second is to study constraints such as domination number, number of branch vertices, or independence number. The third is to develop stability theorems for arbitrary trees: if $\text{EU}(T)$ is close to the sharp upper bound under a constraint, then T should be close, in a suitable edit distance, to the corresponding broom, matching tree, or endpoint-near diameter tree.

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