

Minimal Trees for the Euler Sombor Index Revisited: A Correction and the First Thirteen Minima

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Abstract

The Euler Sombor index of a graph is the sum, over its edges uv , of the quantity $\sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}$, where d denotes the vertex degree. Khanra and Das [*MATCH Commun. Math. Comput. Chem.* **94** (2025) 525–548] classified the trees with the first five minimum Euler Sombor indices. Parts (c) and (d) of their Theorem 3 are incorrect: the family presented as fifth-minimal is in fact fourth-minimal for every $n \geq 7$, and the value printed for the fourth minimum, $4\sqrt{7} + 4\sqrt{19} - 18\sqrt{3} + 2\sqrt{3}n$, is not the Euler Sombor index of any tree. The error is a miscounted edge contribution in the proof, where the edge joining two adjacent vertices of degree three is given the weight $2\sqrt{3}$ of a $(2, 2)$ -edge instead of $\sqrt{27}$. We give the corrected classification of the first five minima for all $n \geq 7$ and, using a decomposition of the index over the threads of a tree that reduces the problem to a finite comparison of skeleton penalties, we extend it: for every $n \geq 11$ the fifth through thirteenth minimum Euler Sombor indices are attained exactly by the nine families of subdivided double stars $S(3, 3)$ determined by the number of subdivided leaf threads and the subdivision status of the central edge, in an explicit order governed by the quantities $\delta = \sqrt{13} + 2\sqrt{3} - \sqrt{7} - \sqrt{19}$ and $\varepsilon = 2\sqrt{19} - 5\sqrt{3}$, for which $0 < \varepsilon < \delta$. We also correct the range of validity of part (iii) of Theorem 4 of the same paper concerning

unicyclic graphs, which fails at $n = 4$. As an independent check, all statements are confirmed, within the stated ranges, by exact exhaustive computation over the 204,990 trees with $7 \leq n \leq 18$ and the 1,039 unicyclic graphs with $4 \leq n \leq 10$.

1 Introduction

Let G be a simple graph and let $d(u)$ denote the degree of a vertex u . The *Euler Sombor index*, introduced by Tang, Li and Deng [5] and belonging to the family of geometric degree-based indices initiated by Gutman's Sombor index [3], is

$$\text{EP}(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

Khanra and Das [4] determined, among other results, the trees with the first five minimum values of EP among all trees of a fixed order n (their Theorem 3), and the unicyclic graphs with the first three minimum values (their Theorem 4). The classification of the extremal trees has already been taken up by subsequent work; for instance, [2] invokes it as a starting point, and open problems posed in [4] are solved in [1].

The present note corrects two statements of [4].

First, parts (c) and (d) of Theorem 3 of [4] are incorrect. Part (c) asserts that, for $n \geq 10$, the fourth minimum is attained by the trees with two adjacent vertices of degree three, each adjacent to two vertices of degree two, with value

$$4\sqrt{7} + 4\sqrt{19} - 18\sqrt{3} + 2\sqrt{3}n, \quad (1)$$

and part (d) asserts that, for $n \geq 7$, the fifth minimum is attained by the trees with one vertex of degree three adjacent to one vertex of degree two and two leaves. Both claims fail for every admissible n : the family of part (d) is in fact *fourth*-minimal for all $n \geq 7$, the family of part (c) is *fifth*-minimal for all $n \geq 10$ with the corrected value

$$4\sqrt{7} + 4\sqrt{19} - 17\sqrt{3} + 2\sqrt{3}n, \quad (2)$$

and the printed value (1) is not the Euler Sombor index of any tree of any order $n \geq 7$ (Corollary 1). The source of the error is a single miscounted edge contribution in the proof, which we locate in Section 2.

Second, part (iii) of Theorem 4 of [4] asserts that for $n \geq 4$ a certain family of unicyclic graphs attains the third minimum of EP. At $n = 4$ there exist only two unicyclic graphs, so no third value exists, and the graph described is second-minimal; the statement is correct precisely for $n \geq 5$ (Section 6).

Beyond the correction, we prove a stronger classification. The proof method, a decomposition of EP over the *threads* of a tree (Section 3), reduces the ordering of the small values of EP to a finite comparison of skeleton penalties and yields, with little extra effort, the complete list of the first *thirteen* minimal values for every $n \geq 11$: after the four families inherited from [4] (suitably reordered), the next nine values are attained exactly by nine explicitly described families of subdivided double stars $S(3, 3)$, in an explicit order (Theorem 2).

As an independent computational check, all results are confirmed, within the stated ranges, by exact exhaustive computation over all 204,990 pairwise non-isomorphic trees with $7 \leq n \leq 18$ and all 1,039 unicyclic graphs with $4 \leq n \leq 10$ (Section 7). The verification script is contained in the ancillary material.

2 The counterexample and the source of the error

Throughout, n denotes the number of vertices of a tree T , and an (i, j) -edge is an edge whose endpoints have degrees i and j .

Let A be the tree on 10 vertices consisting of two adjacent vertices u, v of degree three, each adjacent to two vertices of degree two, each vertex of degree two carrying one leaf. This tree belongs to the family of Theorem 3(c) of [4] at $n = 10$. Its edge types are one $(3, 3)$ -edge, four $(2, 3)$ -edges and four $(1, 2)$ -edges, so directly from the definition

$$\text{EP}(A) = \sqrt{27} + 4\sqrt{19} + 4\sqrt{7} = 3\sqrt{3} + 4\sqrt{19} + 4\sqrt{7} \approx 33.2148.$$

The value printed in Theorem 3(c) of [4] for $n = 10$ is

$$4\sqrt{7} + 4\sqrt{19} - 18\sqrt{3} + 20\sqrt{3} = 4\sqrt{7} + 4\sqrt{19} + 2\sqrt{3} \approx 31.4827,$$

which differs from $\text{EP}(A)$ by exactly $\sqrt{3}$. The origin of the discrepancy is visible in the proof of Theorem 3 in [4]: in the case of two adjacent vertices $x \sim y$ of degree three, the proof asserts

$$\text{EP}(T) = 4(\sqrt{9+4+6} + \sqrt{1+4+2}) + 2\sqrt{3}(n-9),$$

which accounts for the $(3,3)$ -edge xy with the weight $2\sqrt{3}$ of a $(2,2)$ -edge; its correct weight is $\sqrt{9+9+9} = 3\sqrt{3}$. Consequently every value derived from this display is too small by $\sqrt{3}$.

The rank claimed in part (d) is also incorrect. Let B be the tree on 10 vertices with one vertex of degree three adjacent to two leaves and to one vertex of degree two, the remaining vertices forming a pendent path (the family of Theorem 3(d) of [4] at $n = 10$). Then

$$\text{EP}(B) = 2\sqrt{13} + \sqrt{19} + \sqrt{7} + 10\sqrt{3} \approx 31.5363 < 33.2148 = \text{EP}(A),$$

so the family called fifth-minimal in [4] beats the family called fourth-minimal. The exhaustive computation of Section 7 confirms that for every $7 \leq n \leq 18$ the four smallest values of EP over n -vertex trees are attained, in order, by the path and by the three families of Theorem 3(a), (b) and (d) of [4], and that the value (1) is attained by no tree.

3 Thread decomposition of the Euler Sombor index

For integers $i, j \geq 1$ put

$$w(i, j) = \sqrt{i^2 + j^2 + ij}, \quad c = w(2, 2) = 2\sqrt{3}.$$

Note that $w(i, j)$ is strictly increasing in each argument.

Let T be a tree with $n \geq 3$ vertices. A *branch vertex* is a vertex of

degree at least three. A *thread* of T is a maximal path whose internal vertices all have degree two and whose two end vertices are leaves or branch vertices. Contracting every thread to a single edge yields the *skeleton* H of T : a tree without vertices of degree two whose vertices are the leaves and the branch vertices of T (for the path P_n , the skeleton is a single edge). A thread is *subdivided* if it has length at least two, i.e. if it contains at least one internal vertex of degree two.

Lemma 1 (Thread decomposition). *Let T be a tree with $n \geq 3$ vertices and skeleton H . For a thread f whose end vertices have degrees i and j in T , define the penalty*

$$p(f) = \begin{cases} w(i, j) - c & \text{if } f \text{ is not subdivided,} \\ w(i, 2) + w(2, j) - 2c & \text{if } f \text{ is subdivided.} \end{cases}$$

Then

$$\text{EP}(T) = 2\sqrt{3}(n-1) + \sum_{f \in E(H)} p(f).$$

In particular, the penalty of a subdivided thread does not depend on its length.

Proof. Since T has $n-1$ edges,

$$\text{EP}(T) - c(n-1) = \sum_{uv \in E(T)} (w(d(u), d(v)) - c),$$

and the edge set of T is partitioned by the threads. An edge interior to a thread is a $(2, 2)$ -edge and contributes $w(2, 2) - c = 0$. A thread of length one contributes $w(i, j) - c$. A thread of length $\ell \geq 2$ with end degrees i, j contributes $(w(i, 2) - c) + (w(2, j) - c)$ from its two extreme edges and 0 from its $\ell - 2$ interior edges. ■

Only a handful of penalties are needed. A *leaf thread* is a thread with one leaf end; an *internal thread* joins two branch vertices. Writing α_d and β_d for the penalties of a subdivided and a non-subdivided leaf thread at a

branch vertex of degree d , and using Lemma 1,

$$\begin{aligned}\alpha_3 &= \sqrt{7} + \sqrt{19} - 4\sqrt{3} \approx 0.076447, & \beta_3 &= \sqrt{13} - 2\sqrt{3} \approx 0.141450, \\ \alpha_4 &= 3\sqrt{7} - 4\sqrt{3} \approx 1.009051, & \beta_4 &= \sqrt{21} - 2\sqrt{3} \approx 1.118474,\end{aligned}$$

and for internal threads with end degrees $(3, 3)$ and $(3, 4)$,

$$\begin{aligned}\gamma^0 &= \sqrt{27} - 2\sqrt{3} = \sqrt{3} \approx 1.732051, & \gamma^1 &= 2\sqrt{19} - 4\sqrt{3} \approx 1.789595, \\ \pi^0 &= \sqrt{37} - 2\sqrt{3} \approx 2.618661, & \pi^1 &= \sqrt{19} + 2\sqrt{7} - 4\sqrt{3} \approx 2.722198,\end{aligned}$$

where the superscript 0 (resp. 1) refers to the non-subdivided (resp. subdivided) thread. The path P_n ($n \geq 3$) consists of a single subdivided thread between two leaves, with penalty $2\sqrt{7} - 4\sqrt{3} \approx -1.636701$.

We abbreviate $a = \alpha_3$, $b = \beta_3$ and set

$$\delta = b - a = \sqrt{13} + 2\sqrt{3} - \sqrt{7} - \sqrt{19}, \quad \varepsilon = \gamma^1 - \gamma^0 = 2\sqrt{19} - 5\sqrt{3}.$$

Lemma 2. $0 < \varepsilon < \delta$; $0 < a < b < 2a$; $3a < 2b$; $\sqrt{3} > 2b$.

Proof. $\varepsilon > 0$ is equivalent, after squaring, to $4 \cdot 19 > 25 \cdot 3$, i.e. $76 > 75$.

$\delta > 0$ is equivalent to $\sqrt{13} + 2\sqrt{3} > \sqrt{7} + \sqrt{19}$; squaring both sides, to $25 + 4\sqrt{39} > 26 + 2\sqrt{133}$, i.e. to $4\sqrt{39} > 2\sqrt{133} + 1$; squaring again, to $624 > 533 + 4\sqrt{133}$, i.e. to $91 > 4\sqrt{133}$, and $91^2 = 8281 > 2128 = 16 \cdot 133$. Since $\delta = b - a$ this also gives $a < b$; and $a > 0$ follows from $\sqrt{7} + \sqrt{19} > 4\sqrt{3} \Leftrightarrow 26 + 2\sqrt{133} > 48 \Leftrightarrow \sqrt{133} > 11 \Leftrightarrow 133 > 121$.

$\delta > \varepsilon$ is equivalent to $\sqrt{13} + 7\sqrt{3} > \sqrt{7} + 3\sqrt{19}$; squaring, to $160 + 14\sqrt{39} > 178 + 6\sqrt{133}$, i.e. to $7\sqrt{39} > 3\sqrt{133} + 9$; squaring again, to $1911 > 1278 + 54\sqrt{133}$, i.e. to $633 > 54\sqrt{133}$, and $633^2 = 400689 > 387828 = 54^2 \cdot 133$.

$b < 2a$ is equivalent to $\sqrt{13} + 6\sqrt{3} < 2\sqrt{7} + 2\sqrt{19}$; squaring, to $121 + 12\sqrt{39} < 104 + 8\sqrt{133}$, i.e. to $17 + 12\sqrt{39} < 8\sqrt{133}$; squaring again, to $5905 + 408\sqrt{39} < 8512$, i.e. to $408\sqrt{39} < 2607$, and $408^2 \cdot 39 = 6492096 < 6796449 = 2607^2$.

$3a < 2b$ is equivalent to $3\sqrt{7} + 3\sqrt{19} < 2\sqrt{13} + 8\sqrt{3}$; squaring, to $234 + 18\sqrt{133} < 244 + 32\sqrt{39}$, i.e. to $18\sqrt{133} < 10 + 32\sqrt{39}$; squaring again, to $43092 < 40036 + 640\sqrt{39}$, i.e. to $3056 < 640\sqrt{39}$, and $3056^2 = 9339136 <$

$$15974400 = 640^2 \cdot 39.$$

Finally $\sqrt{3} > 2b = 2\sqrt{13} - 4\sqrt{3}$ is equivalent to $5\sqrt{3} > 2\sqrt{13}$, i.e. to $75 > 52$. ■

The remaining comparisons in this note involve the eight quantities $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$, $\sqrt{21}$, $\sqrt{27}$, $\sqrt{28}$, $\sqrt{37}$. The six-decimal values displayed above are correct to $\pm 10^{-6}$: rational enclosures for the individual radicals, certified at sufficient precision by squaring the enclosing rationals (e.g. $1732050^2 < 3 \cdot 10^{12} < 1732051^2$ for $\sqrt{3}$) and propagated to the displayed linear combinations by interval arithmetic, establish them. Each remaining comparison has a margin far larger than the enclosure error—the smallest margins, occurring in the comparisons $\alpha_4 > 1$ and $\pi^0 > 2.6$, are about 0.009 and 0.0187—so the displayed certified approximations establish all of them rigorously. Only the tight inequalities of Lemma 2 require exact arithmetic.

4 The first five minima, corrected

For $s \in \{1, 2, 3\}$ let $\mathcal{K}_s(n)$ denote the family of n -vertex trees whose skeleton is the star $K_{1,3}$ with exactly s subdivided leaf threads; for $r \in \{0, 1, 2, 3, 4\}$ and $c \in \{0, 1\}$ let $\mathcal{D}_{r,c}(n)$ denote the family of n -vertex trees whose skeleton is the double star $S(3, 3)$ (two adjacent branch vertices of degree three, four leaf threads) with exactly r *non*-subdivided leaf threads and with the internal thread subdivided if and only if $c = 1$. By Lemma 1, all members of a fixed family on a fixed n have the same Euler Sombor index; a family may nevertheless contain several pairwise non-isomorphic trees, both because thread lengths may vary and because the subdivided leaf threads may be distributed differently between the two branch vertices (for instance, $\mathcal{D}_{2,0}(8)$ contains exactly two non-isomorphic trees). The family $\mathcal{K}_s(n)$ is non-empty if and only if $n \geq 4 + s$; the family $\mathcal{D}_{r,c}(n)$ is non-empty if and only if $n \geq 6 + (4 - r) + c$, except that $\mathcal{D}_{4,0}$ requires $n = 6$ exactly.

The families of Theorem 3(a), (b), (d) of [4] are $\mathcal{K}_3$, $\mathcal{K}_2$, $\mathcal{K}_1$, and the family of Theorem 3(c) is $\mathcal{D}_{0,0}$.

Theorem 1 (Corrected classification of the first five minima). *Let $n \geq 7$. Among all trees with n vertices,*

- (a) *the unique minimum of EP is attained by the path P_n , with value $2\sqrt{7} - 6\sqrt{3} + 2\sqrt{3}n$;*
- (b) *the second minimum is attained exactly by $K_3(n)$, with value $3\sqrt{19} + 3\sqrt{7} - 14\sqrt{3} + 2\sqrt{3}n$;*
- (c) *the third minimum is attained exactly by $K_2(n)$, with value $\sqrt{13} + 2\sqrt{19} + 2\sqrt{7} - 12\sqrt{3} + 2\sqrt{3}n$;*
- (d) *the fourth minimum is attained exactly by $K_1(n)$, with value $2\sqrt{13} + \sqrt{19} + \sqrt{7} - 10\sqrt{3} + 2\sqrt{3}n$;*
- (e) *the fifth minimum is attained exactly by $\mathcal{D}_{4-s,0}(n)$ with $s = \min\{4, n - 6\}$ subdivided leaf threads. Its value is*

$$3\sqrt{3} + s(\sqrt{19} + \sqrt{7}) + (4 - s)\sqrt{13} + 2\sqrt{3}(n - 6 - s);$$

in particular, for $n \geq 10$ it equals $4\sqrt{7} + 4\sqrt{19} - 17\sqrt{3} + 2\sqrt{3}n$.

Parts (a)–(c) agree with [4]; part (d) promotes the family called fifth-minimal there; part (e) replaces Theorem 3(c) of [4], with the constant $-18\sqrt{3}$ corrected to $-17\sqrt{3}$ and the rank corrected from fourth to fifth.

Proof. By Lemma 1 it suffices to order the total penalties $P(T) = \text{EP}(T) - 2\sqrt{3}(n - 1)$. The path is the unique tree whose skeleton has no branch vertex, and $P(P_n) = 2\sqrt{7} - 4\sqrt{3} < 0$. Every other tree has at least three leaves; let $k \geq 1$ be its number of branch vertices.

Case $k = 1$, degree 3. The skeleton is $K_{1,3}$ and $P = sa + (3 - s)b$ where s is the number of subdivided leaf threads. Since $n \geq 7 > 4$, at least one thread is subdivided, so $s \in \{1, 2, 3\}$, and all three values occur for every $n \geq 7$. By $a < b$ (Lemma 2),

$$3a < 2a + b < a + 2b,$$

with $3a \approx 0.229341$, $2a + b \approx 0.294344$, $a + 2b \approx 0.359346$, all positive.

Case $k = 1$, degree $d \geq 4$. Each of the d leaf threads has penalty at least $\min\{\alpha_d, \beta_d\} \geq \min\{\alpha_4, \beta_4\}$ by the monotonicity of w , and $\min\{\alpha_4, \beta_4\} > 1$ by the certified values above, so $P > 4$.

Case $k = 2$. The two branch vertices have degrees $d_1, d_2 \geq 3$ and are joined by one internal thread; there are $d_1 + d_2 - 2 \geq 4$ leaf threads, each of positive penalty. If $(d_1, d_2) = (3, 3)$, the tree lies in some $\mathcal{D}_{r,c}(n)$ and

$$P = \gamma^0 + c\varepsilon + 4a + r\delta, \quad (3)$$

which is minimised, among the patterns non-empty for the given n , by taking $c = 0$ and r as small as possible, i.e. $r = 4 - s$ with $s = \min\{4, n - 6\}$: by (3), lowering r by one lowers P by δ , while trading a leaf-thread subdivision for a subdivision of the internal thread raises P by $\varepsilon + \delta > 0$. The minimal double-star penalty is thus $\gamma^0 + 4a \approx 2.037839$ for $n \geq 10$, and $\gamma^0 + (n - 6)a + (10 - n)b \leq \gamma^0 + a + 3b \approx 2.232847$ for $n \in \{7, 8, 9\}$. If $\max\{d_1, d_2\} \geq 4$, the internal thread alone has penalty at least $\min\{\pi^0, \pi^1\}$ by monotonicity, and $\min\{\pi^0, \pi^1\} > 2.6$ by the certified values above, so $P > 2.6$.

Case $k \geq 3$. The skeleton contains at least two internal threads, each with both end degrees at least three, hence each of penalty at least $\min\{\gamma^0, \gamma^1\} = \gamma^0 = \sqrt{3}$ by monotonicity and Lemma 2; thus $P \geq 2\sqrt{3} \approx 3.464102$.

It remains to sort the values. The three $K_{1,3}$ penalties are less than every double-star penalty: $\gamma^0 + 4a > a + 2b$ follows from $\sqrt{3} > 2b$ and $3a > 0$ (Lemma 2), and every non-minimal double-star pattern has an even larger penalty by (3). Every penalty from the remaining cases exceeds 2.6, while the minimal double-star penalty is at most $\gamma^0 + a + 3b \approx 2.232847$ for every $n \geq 7$. Therefore the five smallest penalties are, in order: $2\sqrt{7} - 4\sqrt{3}$; $3a$; $2a + b$; $a + 2b$; and the minimal double-star penalty, attained exactly by $\mathcal{D}_{4-s,0}(n)$. Adding $2\sqrt{3}(n-1)$ and expanding gives the displayed values. ■

Corollary 1. *For every $n \geq 7$, the value $4\sqrt{7} + 4\sqrt{19} - 18\sqrt{3} + 2\sqrt{3}n$ printed in Theorem 3(c) of [4] is not the Euler Sombor index of any tree with n vertices.*

Proof. Subtracting $2\sqrt{3}(n-1)$, the corresponding penalty is $4\sqrt{7} + 4\sqrt{19} -$

$16\sqrt{3} = 4a \approx 0.305788$. By the proof of Theorem 1, the penalty of an n -vertex tree ($n \geq 7$) is either $2\sqrt{7} - 4\sqrt{3} < 0$, or one of $3a$, $2a + b$, $a + 2b$, or at least $\gamma^0 + 4a > 2$. By Lemma 2, $b < 2a$ gives $2a + b < 4a$, and $3a < 2b$ gives $4a < a + 2b$; hence $4a$ lies strictly between the third and the fourth penalty and is attained by no tree. (Indeed $4a$ is exactly the double-star penalty $P_{0,0} = \gamma^0 + 4a$ of Section 5 with the internal-thread contribution $\gamma^0 = \sqrt{3}$ omitted, which is precisely the term dropped in the proof in [4].) ■

5 The first thirteen minima for $n \geq 11$

For $n \geq 11$ every family $\mathcal{D}_{r,c}(n)$ with $(r, c) \neq (4, 0)$ is non-empty, since $6 + (4 - r) + c \leq 11$. By (3) their penalties are

$$P_{r,c} = (\gamma^0 + 4a) + r\delta + c\varepsilon,$$

and $0 < \varepsilon < \delta$ (Lemma 2) yields the strict ordering

$$P_{0,0} < P_{0,1} < P_{1,0} < P_{1,1} < P_{2,0} < P_{2,1} < P_{3,0} < P_{3,1} < P_{4,1},$$

the consecutive gaps being ε , $\delta - \varepsilon$, ε , $\delta - \varepsilon$, ε , $\delta - \varepsilon$, ε , δ .

Theorem 2 (The fifth through thirteenth minima). *Let $n \geq 11$. Among all trees with n vertices, the fifth through thirteenth minimum values of EP are attained exactly by the families*

$$\mathcal{D}_{0,0}, \mathcal{D}_{0,1}, \mathcal{D}_{1,0}, \mathcal{D}_{1,1}, \mathcal{D}_{2,0}, \mathcal{D}_{2,1}, \mathcal{D}_{3,0}, \mathcal{D}_{3,1}, \mathcal{D}_{4,1}$$

in this order, with values

$$\text{EP} = 2\sqrt{3}n + \left(4\sqrt{7} + 4\sqrt{19} - 17\sqrt{3}\right) + r\delta + c\varepsilon.$$

Proof. By the case analysis in the proof of Theorem 1, every n -vertex tree other than the path, the trees with skeleton $K_{1,3}$ and the double stars has

penalty larger than 2.6. The largest double-star penalty is

$$P_{4,1} = \gamma^1 + 4b = 4\sqrt{13} + 2\sqrt{19} - 12\sqrt{3} \approx 2.355393 < 2.6,$$

so *all nine* double-star penalties lie strictly below every penalty from the remaining cases, and strictly above the three $K_{1,3}$ penalties (Theorem 1). The nine values are pairwise distinct and ordered as displayed; each is attained, since $n \geq 11$ makes each family non-empty; and no other tree attains any of them, because every other tree belongs to a case whose penalties avoid the open interval $(a + 2b, 2.6) \supseteq [P_{0,0}, P_{4,1}]$. Ranks 1–4 are settled by Theorem 1, so these nine families occupy ranks 5–13. ■

Remark. For $n = 10$ the family $\mathcal{D}_{0,1}$ is empty (it needs $n \geq 11$) and the list must be shortened accordingly; for $n \in \{7, 8, 9\}$ further patterns become infeasible. We do not spell out these boundary cases; the computational data of Section 7 contains them.

Remark. The classification cannot be prolonged past rank 13 without new families: the double-star skeleton yields only the nine distinct penalty values $P_{r,c}$, so the fourteenth value belongs to a different skeleton. The natural candidates are the skeletons with three branch vertices of degree three (penalties from $2\sqrt{3} + 5a \approx 3.846$) and the spiders $K_{1,4}$ (penalties at least $4 \min\{\alpha_4, \beta_4\} > 4$); prolonging the list is a routine application of the same method, and we stop at thirteen.

6 A boundary correction to Theorem 4 of Khanra and Das

Theorem 4(iii) of [4] states that for $n \geq 4$ the unicyclic graphs with exactly one vertex of degree three adjacent to two vertices of degree two and one leaf attain the *third* minimum of EP among n -vertex unicyclic graphs, with value $2\sqrt{19} + \sqrt{13} - 6\sqrt{3} + 2\sqrt{3}n$.

At $n = 4$ there are exactly two unicyclic graphs: the cycle C_4 , with $\text{EP}(C_4) = 8\sqrt{3} \approx 13.856$, and the triangle with a pendent edge, which is the graph described in Theorem 4(iii), with $\text{EP} = 2\sqrt{19} + \sqrt{13} + 2\sqrt{3} \approx 15.788$.

Hence at $n = 4$ the described graph is the *second* minimum and no third minimum exists. Part (iii) is correct precisely for $n \geq 5$; part (ii), stated for $n \geq 5$, is unaffected. (Exhaustive verification for $4 \leq n \leq 10$ is reported in Section 7.)

7 Computational verification

As an independent check—the theorems above are proved for every admissible n , while a computation necessarily covers a finite range—all statements of this note were confirmed by a direct exhaustive computation that does not use the thread decomposition: for each $7 \leq n \leq 18$, all pairwise non-isomorphic trees with n vertices were generated (11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320, 48629, 123867 trees; 204,990 in total) and their Euler Sombor indices computed from the definition. Every index is handled exactly, as an integer vector of coefficients of $\sqrt{q}$ over squarefree q (such vectors determine the value uniquely, since square roots of distinct square-free integers are linearly independent over the rationals); no floating-point comparison enters the verification. Exact arithmetic is necessary here: the minimal gap between distinct Euler Sombor values of 18-vertex trees is about $3.8 \cdot 10^{-8}$. In every case:

- (i) the four smallest values are attained exactly by P_n , $\mathcal{K}_3(n)$, $\mathcal{K}_2(n)$, $\mathcal{K}_1(n)$, confirming Theorem 1(a)–(d) and refuting the rank claimed in Theorem 3(d) of [4];
- (ii) the value (1) of Theorem 3(c) of [4] is attained by no tree, confirming Corollary 1;
- (iii) the fifth smallest value and its attaining trees are as in Theorem 1(e);
- (iv) for $11 \leq n \leq 18$, the fifth through thirteenth smallest values, together with the full lists of attaining trees, are exactly as in Theorem 2.

Likewise, all unicyclic graphs with $4 \leq n \leq 10$ were generated (2, 5, 13, 33, 89, 240, 657 graphs; 1,039 in total): for $n = 4$ the two graphs and their ranks are as in Section 6, and for $5 \leq n \leq 10$ parts (i)–(iii) of Theorem 4

of [4] hold as stated, including the descriptions of the graphs attaining each of the three values.

The attainment checks compare the multiset of degree-typed edges of every attaining graph against the profile of the stated family; for each of the profiles occurring here, the profile determines the family (for example, a tree profile with six incidences at degree three has exactly two branch vertices, hence skeleton $S(3,3)$, and the counts of $(1,3)$ - and $(3,3)$ -edges recover the parameters r and c), so this is equivalent to checking the families themselves. The verification script (Python, using `networkx` for the generation of non-isomorphic trees) is included in the ancillary files and reruns in minutes on a desktop computer.

Data availability

The verification script and its full output are included with this article as ancillary material and are archived at [10.5281/zenodo.21854189](https://zenodo.org/record/21854189).

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