

On the Degree-Ratio Sombor Index and Graph Energy

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Abstract

The energy of a simple graph G is defined as the sum of absolute values of all eigenvalues of its adjacency matrix and is denoted by $\mathcal{E}(G)$. The Degree-Ratio Sombor index of G is defined as

$$\text{DRSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v},$$

where d_u and d_v are the degrees of vertices u and v , respectively. In this paper, we investigate the relationship between $\text{DRSO}(G)$ and $\mathcal{E}(G)$.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with $m = |E(G)|$ edges, maximum degree $\Delta(G)$, and minimum degree $\delta(G)$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the adjacency matrix of G . The *energy* of G is defined in [8] as

$$\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|.$$

For recent bounds on graph energy, see [1, 6, 7].

Following [3], the *vertex energy* of a vertex v_i is the i -th diagonal entry of $|A| = (A^2)^{1/2}$, where A is the adjacency matrix of G :

$$\mathcal{E}_G(v_i) = |A|_{ii}. \quad (1)$$

Because $\text{tr } |A| = \sum_i |\lambda_i|$, we have the decomposition

$$\mathcal{E}(G) = \sum_{v \in V} \mathcal{E}_G(v) = \sum_{uv \in E(G)} \left(\frac{\mathcal{E}_G(u)}{d_u} + \frac{\mathcal{E}_G(v)}{d_v} \right). \quad (2)$$

The latter identity was proved in [2], together with the lower bound $\mathcal{E}_G(v) \geq \sqrt{d_v/\Delta}$. For a tutorial on vertex energies, see [4].

The *Degree-Ratio Sombor index* $\text{DRSO}(G)$ was recently introduced in [9] and further studied in [5]. It is originally defined as

$$\text{DRSO}(G) = \sum_{uv \in E(G)} \sqrt{\left(\frac{d_u}{d_v}\right)^2 + \left(\frac{d_v}{d_u}\right)^2},$$

which simplifies to

$$\text{DRSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v},$$

where d_u denotes the degree of vertex u .

In this paper we investigate the relationship between $\text{DRSO}(G)$ and $\mathcal{E}(G)$. We establish several sharp inequalities. In particular, we combine $\text{DRSO}(G)$ with the Randić index $R(G)$ to obtain a refined lower bound for

the energy. We then derive general upper and lower bounds for $\mathcal{E}(G)$ in terms of $\text{DRSO}(G)$, the maximum degree Δ , the minimum degree δ , and the number of edges m . For regular graphs these bounds simplify nicely.

2 Auxiliary Lemmas

In this section we collect all the technical lemmas that will be used throughout the paper.

Lemma 1. *For any positive integers r, s with $r + s \geq 3$,*

$$\sqrt{r} + \sqrt{s} \geq \sqrt{r + s} + c,$$

where $c = 1 + \sqrt{2} - \sqrt{3} \approx 0.6822$. The constant c is optimal; equality holds for $(r, s) \in \{(1, 2), (2, 1)\}$.

Proof. Fix $t = r + s \geq 3$ and define $f(r) = \sqrt{r} + \sqrt{t-r} - \sqrt{t}$ for $r \in (0, t)$. Since $f''(r) = -\frac{1}{4}r^{-3/2} - \frac{1}{4}(t-r)^{-3/2} < 0$, f is concave on $(0, t)$. Consequently, on the integer set $\{1, \dots, t-1\}$ the minimum of f is attained at an endpoint. By symmetry in r and s , we may assume $r \leq s$, so the minimum occurs at $r = 1$ (and $s = t - 1 \geq 2$). Thus it suffices to minimize $g(s) = 1 + \sqrt{s} - \sqrt{s+1}$ over integers $s \geq 2$. For $s > 0$,

$$g'(s) = \frac{1}{2\sqrt{s}} - \frac{1}{2\sqrt{s+1}} > 0,$$

hence g is strictly increasing. Therefore the minimum over $s \geq 2$ is attained at $s = 2$, yielding $c = g(2) = 1 + \sqrt{2} - \sqrt{3}$. ■

Lemma 2. *For any edge $uv \in E(G)$,*

$$\frac{d_u}{d_v} + \frac{d_v}{d_u} \leq d_u + d_v.$$

Consequently,

$$\text{DRSO}(G) \leq \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq M_1(G),$$

where $M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{v \in V(G)} d_v^2$.

Proof. From $d_u, d_v \geq 1$ we obtain $d_u/d_v \leq d_u$ and $d_v/d_u \leq d_v$, hence

$$\frac{d_u}{d_v} + \frac{d_v}{d_u} \leq d_u + d_v \quad \text{for every edge } uv.$$

Summing over all edges yields the right-hand inequality in the statement.

For the left-hand inequality, apply $\sqrt{a^2 + b^2} \leq a + b$ with $a = d_u^2$, $b = d_v^2$ to get

$$\frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \leq \frac{d_u^2 + d_v^2}{d_u d_v} = \frac{d_u}{d_v} + \frac{d_v}{d_u}.$$

Summing this over all edges and using the previously established bound gives $\text{DRSO}(G) \leq \sum_{uv} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq M_1(G)$. ■

Lemma 3. *For any edge xy of G ,*

$$\sqrt{d_x + d_y} \geq \frac{\sqrt{d_x^4 + d_y^4}}{\Delta(G)\sqrt{\Delta(G)}}.$$

Proof. Using $d_x, d_y \leq \Delta$,

$$\sqrt{d_x^4 + d_y^4} \leq \sqrt{\Delta^2(d_x^2 + d_y^2)} = \Delta\sqrt{d_x^2 + d_y^2} \leq \Delta\sqrt{\Delta(d_x + d_y)} = \Delta\sqrt{\Delta}\sqrt{d_x + d_y}.$$

Rearranging gives the desired inequality. ■

Lemma 4. *For any graph G ,*

$$\sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq \sqrt{2} \text{DRSO}(G).$$

Proof. The well-known inequality $\sqrt{a^2 + b^2} \geq (a + b)/\sqrt{2}$ for $a, b \geq 0$ applied to $a = d_u^2$, $b = d_v^2$ yields

$$\sqrt{d_u^4 + d_v^4} \geq \frac{d_u^2 + d_v^2}{\sqrt{2}}.$$

Consequently,

$$\frac{d_u^2 + d_v^2}{d_u d_v} \leq \sqrt{2} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v}.$$

Since $\frac{d_u^2 + d_v^2}{d_u d_v} = \frac{d_u}{d_v} + \frac{d_v}{d_u}$, summing over all edges completes the proof. ■

Lemma 5. *For any graph G ,*

$$M_1(G) \leq \Delta \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right).$$

Proof. Since $d_u \leq \Delta$ we have $d_u \leq \Delta \cdot \frac{d_u}{d_v}$, hence $d_u + d_v = \frac{d_u}{d_v} d_v + \frac{d_v}{d_u} d_u \leq \Delta \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right)$. Summation yields the claim. ■

3 Relations between graph energy and the Degree-ratio Sombor index

In this section we present our main results, establishing various sharp inequalities between $\mathcal{E}(G)$ and $\text{DRSO}(G)$.

Theorem 1. *For any graph G with at least one edge and maximum degree Δ , we have*

$$\text{DRSO}(G) \leq \mathcal{E}(G) \Delta^2(G) - \sum_{uv \in E(G)} \frac{d_u^2(d_v - 1) + d_v^2(d_u - 1)}{d_u d_v}.$$

Proof. For any edge $uv \in E(G)$, using $\sqrt{a^2 + b^2} \leq a + b$ with $a = d_u/d_v$ and $b = d_v/d_u$, we have

$$\text{DRSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \leq \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right).$$

Also, from the elementary inequality $\frac{d_u}{d_v} + \frac{d_v}{d_u} \leq d_u + d_v$ (which follows from $d_u^2 + d_v^2 \leq d_u d_v (d_u + d_v)$), we get

$$\text{DRSO}(G) \leq \sum_{uv \in E(G)} (d_u + d_v) = M_1(G). \quad (3)$$

Now, since $d_v \leq \Delta(G)$ for every vertex v ,

$$M_1(G) = \sum_{v \in V(G)} d_v^2 \leq \Delta(G) \sum_{v \in V(G)} d_v = 2m\Delta(G).$$

For any eigenvalue λ_i of the adjacency matrix we have $|\lambda_i| \leq \lambda_1 \leq \Delta(G)$, hence $\lambda_1 |\lambda_i| \geq \lambda_i^2$. Summing over all eigenvalues yields

$$2m = \sum_i \lambda_i^2 \leq \lambda_1 \sum_i |\lambda_i| = \lambda_1 \mathcal{E}(G) \leq \Delta(G) \mathcal{E}(G),$$

so $2m \leq \Delta(G) \mathcal{E}(G)$. Therefore

$$M_1(G) \leq \Delta^2(G) \mathcal{E}(G). \quad (4)$$

To obtain the refined bound, observe that

$$M_1(G) - \text{DRSO}(G) = \sum_{uv \in E(G)} \left[(d_u + d_v) - \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \right].$$

Since $\frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \leq \frac{d_u}{d_v} + \frac{d_v}{d_u}$, it follows that

$$M_1(G) - \text{DRSO}(G) \geq \sum_{uv \in E(G)} \left[(d_u + d_v) - \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \right].$$

A direct algebraic simplification gives

$$(d_u + d_v) - \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) = \frac{d_u^2(d_v - 1) + d_v^2(d_u - 1)}{d_u d_v}.$$

Therefore,

$$\text{DRSO}(G) \leq M_1(G) - \sum_{uv \in E(G)} \frac{d_u^2(d_v - 1) + d_v^2(d_u - 1)}{d_u d_v}. \quad (5)$$

Combining (4) and (5) yields the desired inequality:

$$\text{DRSO}(G) \leq \mathcal{E}(G)\Delta^2(G) - \sum_{uv \in E(G)} \frac{d_u^2(d_v - 1) + d_v^2(d_u - 1)}{d_u d_v}. \quad \blacksquare$$

Theorem 2. *Let $G \neq K_2$ be a connected graph with maximum degree Δ . Then*

$$\mathcal{E}(G) \geq \frac{\text{DRSO}(G)}{\Delta^2} + \frac{c}{\sqrt{\Delta}} R(G),$$

where $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$ is the Randić index. The constant $c = 1 + \sqrt{2} - \sqrt{3} \approx 0.6822$ is defined in Lemma 1.

Proof. Using the vertex energy definition (1) and the identity (2) together with the bound $\mathcal{E}_G(v) \geq \sqrt{d_v/\Delta}$ from [2], we have

$$\mathcal{E}(G) \geq \frac{1}{\sqrt{\Delta}} \sum_{uv \in E(G)} \left(\frac{1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_v}} \right) = \frac{1}{\sqrt{\Delta}} \sum_{uv \in E(G)} \frac{\sqrt{d_u} + \sqrt{d_v}}{\sqrt{d_u d_v}}.$$

Since G is connected and $G \neq K_2$, every edge satisfies $d_u + d_v \geq 3$, so Lemma 1 applies and gives

$$\sqrt{d_u} + \sqrt{d_v} \geq \sqrt{d_u + d_v} + c.$$

Hence

$$\mathcal{E}(G) \geq \frac{1}{\sqrt{\Delta}} \sum_{uv \in E(G)} \frac{\sqrt{d_u + d_v}}{\sqrt{d_u d_v}} + \frac{c}{\sqrt{\Delta}} \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

For the first sum, Lemma 3 yields

$$\frac{1}{\sqrt{\Delta}} \sum_{uv \in E(G)} \frac{\sqrt{d_u + d_v}}{\sqrt{d_u d_v}} \geq \frac{1}{\sqrt{\Delta}} \sum_{uv \in E(G)} \frac{\sqrt{d_u^4 + d_v^4}}{\Delta \sqrt{\Delta} d_u d_v} = \frac{\text{DRSO}(G)}{\Delta^2}.$$

Therefore,

$$\mathcal{E}(G) \geq \frac{\text{DRSO}(G)}{\Delta^2} + \frac{c}{\sqrt{\Delta}} R(G). \quad \blacksquare$$

Corollary. *Let $G \neq K_2$ be a connected graph with m edges and maximum*

degree Δ . Then

$$\text{DRSO}(G) \leq \Delta^2 \mathcal{E}(G) - c\sqrt{\Delta} m.$$

Proof. Because $d_u d_v \leq \Delta^2$, we have $\frac{1}{\sqrt{d_u d_v}} \geq \frac{1}{\Delta}$ for every edge, so $R(G) \geq m/\Delta$. Substituting this into Theorem 2 and rearranging yields the claimed inequality. ■

Corollary. Let G be an r -regular graph with $r \geq 2$ and m edges. Then

$$\text{DRSO}(G) \leq r^2 \mathcal{E}(G) - c\sqrt{r} m.$$

Proof. The result follows immediately by substituting $\Delta(G) = r$ into Corollary 3. ■

Theorem 3. Let G be a graph with minimum degree $\delta \geq 1$ and maximum degree Δ . Then

$$\frac{\sqrt{2}\delta}{\Delta^2} \text{DRSO}(G) \leq \mathcal{E}(G) \leq \frac{\sqrt{2}\Delta}{\delta^{3/2}} \text{DRSO}(G).$$

Proof. Lower bound. By the Cauchy–Schwarz inequality,

$$(\text{DRSO}(G))^2 = \left(\sum_{uv \in E} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \right)^2 \leq \left(\sum_{uv \in E} \frac{d_u^4 + d_v^4}{d_u d_v} \right) \left(\sum_{uv \in E} \frac{1}{d_u d_v} \right).$$

For each edge,

$$\frac{d_u^4 + d_v^4}{d_u d_v} = \frac{d_u^3}{d_v} + \frac{d_v^3}{d_u} \leq \Delta^2 \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right).$$

Summing over all edges and applying Lemma 4 gives

$$\sum_{uv \in E} \frac{d_u^4 + d_v^4}{d_u d_v} \leq \Delta^2 \sum_{uv \in E} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq \Delta^2 \sqrt{2} \text{DRSO}(G).$$

Thus

$$(\text{DRSO}(G))^2 \leq \Delta^2 \sqrt{2} \text{DRSO}(G) \sum_{uv \in E} \frac{1}{d_u d_v},$$

which implies

$$\sum_{uv \in E} \frac{1}{d_u d_v} \geq \frac{\text{DRSO}(G)}{\Delta^2 \sqrt{2}}.$$

Since $d_u, d_v \geq \delta$, we have $\frac{1}{d_u d_v} \leq \frac{1}{\delta \sqrt{d_u d_v}}$. Summation yields

$$\sum_{uv \in E} \frac{1}{d_u d_v} \leq \frac{1}{\delta} \sum_{uv \in E} \frac{1}{\sqrt{d_u d_v}} = \frac{1}{\delta} R(G).$$

Combining the two estimates,

$$\frac{\text{DRSO}(G)}{\Delta^2 \sqrt{2}} \leq \frac{1}{\delta} R(G).$$

Hence,

$$R(G) \geq \frac{\delta}{\sqrt{2} \Delta^2} \text{DRSO}(G).$$

Finally, the well-known inequality $\mathcal{E}(G) \geq 2R(G)$ (which holds for graphs without isolated vertices, see [8]) gives

$$\mathcal{E}(G) \geq 2R(G) \geq \frac{\sqrt{2} \delta}{\Delta^2} \text{DRSO}(G).$$

Upper bound. We start from the bound $\mathcal{E}(G) \leq \sum_{v \in V} \sqrt{d_v}$ (see [3]).

Using $\sqrt{d_v} \leq d_v / \sqrt{\delta}$ (because $d_v \geq \delta$), we obtain

$$\mathcal{E}(G) \leq \frac{1}{\sqrt{\delta}} \sum_{v \in V} d_v = \frac{2m}{\sqrt{\delta}}.$$

Now $2m = \sum_v d_v \leq \frac{1}{\delta} \sum_v d_v^2 = \frac{M_1(G)}{\delta}$, hence

$$\mathcal{E}(G) \leq \frac{M_1(G)}{\delta^{3/2}}.$$

From Lemma 5 we have $M_1(G) \leq \Delta \sum_{uv} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right)$, and Lemma 4 gives $\sum_{uv} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq \sqrt{2} \text{DRSO}(G)$. Therefore,

$$\mathcal{E}(G) \leq \frac{\Delta \cdot \sqrt{2} \text{DRSO}(G)}{\delta^{3/2}} = \frac{\sqrt{2} \Delta}{\delta^{3/2}} \text{DRSO}(G). \quad \blacksquare$$

Theorem 4. Let G be a graph of order n with minimum degree $\delta \geq 1$ and maximum degree Δ . Then

$$\mathcal{E}(G) \leq \sqrt{\frac{\sqrt{2} n \Delta}{\delta}} \text{DRSO}(G).$$

Proof. From Lemma 5 and Lemma 4 we obtain $M_1(G) \leq \Delta\sqrt{2} \text{DRSO}(G)$. Since $M_1(G) = \sum_{v \in V} d_v^2 \geq \delta \sum_{v \in V} d_v = 2m\delta$, it follows that $2m \leq \frac{M_1(G)}{\delta} \leq \frac{\Delta\sqrt{2}}{\delta} \text{DRSO}(G)$. By the Cauchy–Schwarz inequality applied to the eigenvalues of G , $\mathcal{E}(G) \leq \sqrt{n \sum_{i=1}^n \lambda_i^2} = \sqrt{2mn}$. Substituting the upper bound for $2m$ completes the proof. ■

Theorem 5. Let G be a graph with m edges and maximum degree Δ . Then

$$\mathcal{E}(G) \geq \frac{\text{DRSO}^2(G)}{m \Delta^3}.$$

Proof. By the Cauchy–Schwarz inequality,

$$(\text{DRSO}(G))^2 = \left(\sum_{uv \in E} \frac{\sqrt{d_u^4 + d_v^4}}{d_u d_v} \right)^2 \leq m \sum_{uv \in E} \frac{d_u^4 + d_v^4}{d_u^2 d_v^2}.$$

For any edge uv ,

$$\frac{d_u^4 + d_v^4}{d_u^2 d_v^2} = \frac{d_u^2}{d_v^2} + \frac{d_v^2}{d_u^2} \leq \Delta \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right),$$

where we used $\frac{d_u^2}{d_v^2} = \frac{d_u}{d_v} \cdot \frac{d_u}{d_v} \leq \Delta \cdot \frac{d_u}{d_v}$ together with the symmetric estimate for the second term. Summing over all edges and applying Lemma 2 yields

$$\sum_{uv \in E} \frac{d_u^4 + d_v^4}{d_u^2 d_v^2} \leq \Delta \sum_{uv \in E} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right) \leq \Delta M_1(G).$$

As shown in the proof of Theorem 1, $M_1(G) \leq \Delta^2 \mathcal{E}(G)$. Consequently,

$$(\text{DRSO}(G))^2 \leq m \cdot \Delta \cdot \Delta^2 \mathcal{E}(G) = m \Delta^3 \mathcal{E}(G),$$

and taking square roots completes the proof. ■

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