

New Lower Bounds for the Randić Energy of Graphs

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Abstract

New bounds on Randić energy of graphs are established. In addition, refinements of some earlier results pertaining to incidence energy, Randić energy, and Randić Estrada energy are reported.

1 Introduction and preliminaries

The signless Laplacian matrix $Q(G) = D(G) + A(G)$ of a graph G (where $D(G)$ is the degree matrix and $A(G)$ the adjacency matrix) has eigenvalues $q_1 \geq q_2 \geq \cdots \geq q_n$. The incidence energy $IE(G)$ is defined as $IE(G) = \sum_{i=1}^n \sqrt{q_i}$ [8]. More generally, for a real exponent α , we set $s_\alpha(G) = \sum_{i=1}^n q_i^\alpha$. The Randić matrix $R(G)$ of G has entries $r_{ij} = 1/\sqrt{d_i d_j}$ if $v_i v_j \in E(G)$, and $r_{ij} = 0$ otherwise. Its eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ give rise to the Randić energy $RE(G) = \sum_{i=1}^n |\rho_i|$ and the Randić Estrada index $REE(G) = \sum_{i=1}^n e^{\rho_i}$ [2].

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As defined in [2], we also set

$$R_{-1}(G) = \frac{1}{2} \sum_{i=1}^n \rho_i^2.$$

In Section 2 we establish new lower bounds for the Randić energy. In Section 3 we refine several earlier estimates concerning incidence energy, Randić energy, and Randić Estrada index.

2 New lower bounds for Randić energy

In this section we present our main results: new lower bounds for the Randić energy in terms of the determinant and spectral moments of the Randić matrix.

Theorem 1. *For any connected graph G of order $n \geq 2$, we have*

$$RE(G) \geq \frac{4R_{-1}(G)}{1 - \rho_n}. \quad (1)$$

Equality holds if and only if the spectrum of the Randić matrix of G consists of at most three distinct values: ρ_1 with multiplicity 1, 0 with multiplicity A_0 , and ρ_n with multiplicity $n - A_0 - 1$, where $\rho_1 = -(n - A_0 - 1)\rho_n$.

Proof. We have:

$$2R_{-1}(G) = \sum_{i=1}^n \rho_i^2 = \frac{1}{2} \sum_{i=1}^n (2\rho_i - \rho_1 - \rho_n)\rho_i.$$

Applying the triangle inequality and the bound $|2\rho_i - \rho_1 - \rho_n| \leq \rho_1 - \rho_n = 1 - \rho_n$ yields:

$$2R_{-1}(G) \leq \frac{1 - \rho_n}{2} \sum_{i=1}^n |\rho_i| = \frac{1 - \rho_n}{2} RE(G),$$

which is equivalent to (1).

Equality in (1) requires equality in both the triangle inequality and the bound $|2\rho_i - \rho_1 - \rho_n| \leq \rho_1 - \rho_n$ for every i . Equality in the bound forces

each non-zero eigenvalue to be either ρ_1 or ρ_n . Equality in the triangle inequality then demands that all terms $(2\rho_i - \rho_1 - \rho_n)\rho_i$ have the same sign. For $\rho_i = \rho_1$ we obtain $(\rho_1 - \rho_n)\rho_1 > 0$; for $\rho_i = \rho_n$ we obtain $(\rho_n - \rho_1)\rho_n = -(\rho_1 - \rho_n)\rho_n > 0$ (since $\rho_n < 0$). Hence the sign condition is automatically satisfied once $\rho_i \in \{\rho_1, \rho_n\}$.

Consequently, the spectrum of G can contain only ρ_1 , ρ_n , and possibly 0. Let A_0 be the multiplicity of 0. By the Perron–Frobenius theorem, ρ_1 has multiplicity 1. Let $k = n - A_0 - 1$ be the multiplicity of ρ_n . From $\sum_{i=1}^n \rho_i = 0$ we obtain $\rho_1 + k\rho_n = 0$, i.e. $\rho_1 = -k\rho_n$. Since $\rho_1 = 1$, we have $\rho_n = -1/k$. Thus, equality occurs precisely when the spectrum is of the form $\{1, 0^{(A_0)}, (-1/k)^{(k)}\}$. ■

Given that for any connected graph G of order $n \geq 2$, we have $1 - \rho_n \leq 2$ (see [7]), Theorem 1 yields the following corollary.

Corollary ([3]). *For any connected graph G of order $n \geq 2$,*

$$RE(G) \geq 2R_{-1}(G). \quad (2)$$

Now, let us explore a lower bound for the Randić energy of a graph.

Theorem 2. *Let G be a non-empty graph of order n . Then*

$$RE(G) \geq 1 + (t+1)(n-1) \left| \det R \right|^{\frac{n(t+1)-t}{n(n-1)(t+1)}} - t(n-1) \left| \det R \right|^{\frac{1}{n}}, \quad (3)$$

where $t \geq 0$ is a real parameter. Equality in (3) holds if and only if $G \cong \frac{n}{2}K_2$ (i.e., n is even).

Proof. This bound is derived from a known inequality [6]. Let x_i and r_i be non-negative numbers with $\sum_{i=1}^n r_i = 1$. The inequality states:

$$\sum_{i=1}^n x_i r_i - \prod_{i=1}^n x_i^{r_i} \geq s n \left(\frac{\sum_{i=1}^n x_i}{n} - \prod_{i=1}^n x_i^{\frac{1}{n}} \right), \quad (4)$$

where $s = \min\{r_1, \dots, r_n\}$.

Take $x_i = |\rho_i|$ for $i = 1, \dots, n$, and set

$$r_1 = \frac{t}{n(t+1)}, \quad r_i = \frac{n(t+1)-t}{n(n-1)(t+1)} \quad (i = 2, \dots, n),$$

with $t \geq 0$. Substituting these choices into (4) gives:

$$\begin{aligned} & \frac{t|\rho_1|}{n(t+1)} + \frac{n(t+1)-t}{n(n-1)(t+1)} \sum_{i=2}^n |\rho_i| - |\rho_1|^{\frac{t}{n(t+1)}} \prod_{i=2}^n |\rho_i|^{\frac{n(t+1)-t}{n(n-1)(t+1)}} \\ & \geq \frac{t}{n(t+1)} \sum_{i=1}^n |\rho_i| - \frac{t}{t+1} \prod_{i=1}^n |\rho_i|^{\frac{1}{n}}. \end{aligned}$$

Rearranging terms and using $RE(G) = \sum_{i=1}^n |\rho_i|$, we obtain:

$$RE(G) \geq |\rho_1| + (t+1)(n-1) \frac{|\det R|^{\frac{n(t+1)-t}{n(n-1)(t+1)}}}{|\rho_1|^{\frac{1}{(t+1)(n-1)}}} - t(n-1) |\det R|^{\frac{1}{n}}.$$

For the Randić matrix, the largest eigenvalue satisfies $\rho_1 = 1$ (see [7]). Hence $|\rho_1| = 1$, and the bound simplifies to (3).

Equality in (3) requires equality in (4) with the chosen r_i . According to the original inequality [6], equality occurs if and only if all x_i are equal. In our setting $x_i = |\rho_i|$, so equality demands $|\rho_1| = |\rho_2| = \cdots = |\rho_n|$. Since $\rho_1 = 1$, this forces $|\rho_i| = 1$ for every i . Consequently, all eigenvalues of the Randić matrix are ± 1 . Moreover, because $\sum_{i=1}^n \rho_i = 0$, the numbers of $+1$ and -1 must be equal. The only graphs whose Randić eigenvalues are all ± 1 with equal multiplicities are those consisting of disjoint copies of K_2 . For a connected graph, this forces $n = 2$ and $G \cong K_2$. For a disconnected graph, the condition is satisfied precisely by $G \cong \frac{n}{2} K_2$ when n is even. ■

Setting $t = 0$ in Theorem 2 yields a simpler bound.

Corollary (Theorem 3.2 of [9]). *Let G be a graph of order n . Then*

$$RE(G) \geq 1 + (n-1) |\det R|^{\frac{1}{n-1}}. \quad (5)$$

2.1 Further bounds via power sums

Our next result establishes a lower bound for the Randić energy based on powers of the Randić eigenvalues. For a real number $p \geq 0$, define

$$N_p(G) = \sum_{j=1}^k |\rho_j|^p,$$

where $\rho_1, \dots, \rho_k$ are the non-zero Randić eigenvalues of G .

Theorem 3. *Let G be a non-empty graph, and let a, b, c be non-negative real numbers such that $4a = b + c + 2$. Then*

$$RE(G) \geq \frac{(N_a(G))^2}{\sqrt{N_b(G) N_c(G)}}. \quad (6)$$

If $(b, c) \neq (1, 1)$, then equality holds if and only if $G \cong \frac{n}{2}K_2$ (i.e., n is even).

Proof. We employ the following inequality [11]: for positive real numbers $z_1, z_2, \dots, z_n$ and non-negative a, b, c satisfying $4a = b + c + 2$,

$$\left(\sum_{j=1}^n z_j^a\right)^4 \leq \left(\sum_{j=1}^n z_j\right)^2 \left(\sum_{j=1}^n z_j^b\right) \left(\sum_{j=1}^n z_j^c\right). \quad (7)$$

Equality in (7) occurs if and only if $z_1 = z_2 = \dots = z_n$, provided $(b, c) \neq (1, 1)$.

Let $\rho_1 \geq \rho_2 \geq \dots \geq \rho_k$ be the non-zero Randić eigenvalues of G . Since G is non-empty, $\rho_1 > 0$. Set $z_j = |\rho_j|$ for $j = 1, \dots, k$ in (7). Then

$$\left(\sum_{j=1}^k |\rho_j|^a\right)^4 \leq \left(\sum_{j=1}^k |\rho_j|\right)^2 \left(\sum_{j=1}^k |\rho_j|^b\right) \left(\sum_{j=1}^k |\rho_j|^c\right).$$

Taking square roots and noting that $RE(G) = \sum_{j=1}^k |\rho_j|$ (because zero eigenvalues do not contribute to the energy) yields (6).

If equality holds in (6), then equality holds in (7), which forces $|\rho_1| = |\rho_2| = \dots = |\rho_k|$. Since $\rho_1 = 1$ for the Randić matrix, we obtain $|\rho_j| = 1$ for every non-zero eigenvalue. Consequently, all non-zero eigenvalues are ± 1 , and because $\sum_{j=1}^k \rho_j = 0$, the numbers of $+1$ and -1 must be equal. The only graphs whose Randić eigenvalues are all ± 1 with equal multiplicities are those consisting of disjoint copies of K_2 . Hence $G \cong \frac{n}{2}K_2$ when n is even. ■

Special cases

Choosing $a = 1$, $b = 0$, $c = 2$ gives $N_0(G) = k$ (the number of non-zero eigenvalues) and $N_2(G) = 2R_{-1}(G)$. Then (6) becomes

$$RE(G) \geq \frac{(N_1(G))^2}{\sqrt{k \cdot 2R_{-1}(G)}} = \frac{(RE(G))^2}{\sqrt{2k R_{-1}(G)}},$$

which, after simplification, yields the upper bound

$$RE(G) \leq \sqrt{2k R_{-1}(G)}.$$

Let n_0 be the multiplicity of the zero eigenvalue ($k = n - n_0$). Thus we obtain the following refinement of a known result.

Corollary. *For any graph G with k non-zero Randić eigenvalues,*

$$RE(G) \leq \sqrt{2k R_{-1}(G)}.$$

In particular, if G has no isolated vertices, then $k = n$ and $RE(G) \leq \sqrt{2n R_{-1}(G)}$, which coincides with Theorem 4 of [1] and the bound in [3].

Now take $a = b = 2$, $c = 4$. Then $N_2(G) = 2R_{-1}(G)$ and $N_4(G) = \text{tr}(R^4)$. Inequality (6) gives

$$RE(G) \geq \frac{(2R_{-1}(G))^2}{\sqrt{2R_{-1}(G) \cdot \text{tr}(R^4)}} = \sqrt{\frac{(2R_{-1}(G))^3}{\text{tr}(R^4)}} = \sqrt{\frac{(\text{tr}(R^2))^3}{\text{tr}(R^4)}}.$$

Corollary (Theorem 5 of [1]). *For any graph G of order n with at least one edge,*

$$RE(G) \geq \sqrt{\frac{(\text{tr}(R^2))^3}{\text{tr}(R^4)}}.$$

3 Refinements of earlier results

In this section we refine the bounds on incidence energy and Randić Estrada index reported in [10] by providing sharper characterizations of

the extremal graphs and clarifying the ranges of parameters for which the inequalities are valid.

3.1 Theorem 2.4 in [10]

In [10], Theorem 2.4 states $IE(G) \leq \sqrt{2mn}$ for a connected graph, with equality supposedly when $q_1 = \dots = q_n$. This equality condition, however, cannot occur when $m \geq 1$. Indeed, if G is bipartite then $q_n = 0$ forces all $q_i = 0$, contradicting $m \geq 1$. If G is non-bipartite, then $q_n > 0$ and $Q(G) = \lambda I_n$ would imply $A(G) = 0$, again a contradiction. We sharpen the characterization as follows.

Theorem 4. *For a graph G of order n and size m ,*

$$IE(G) \leq \sqrt{2mn}.$$

Equality holds if and only if $G \cong \overline{K_n}$.

Proof. Apply the Cauchy–Schwarz inequality to the vectors

$$\mathbf{a} = (\sqrt{q_1}, \dots, \sqrt{q_n}), \quad \mathbf{b} = (1, \dots, 1) :$$

$$\left(\sum_{i=1}^n \sqrt{q_i} \right)^2 \leq \left(\sum_{i=1}^n q_i \right) \left(\sum_{i=1}^n 1 \right) = (2m)n.$$

Taking square roots gives $IE(G) \leq \sqrt{2mn}$.

Equality in Cauchy–Schwarz occurs exactly when $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, i.e. when all $\sqrt{q_i}$ are equal. Hence $q_1 = q_2 = \dots = q_n = c$. Since $Q(G) = D(G) + A(G)$ has all eigenvalues equal to c , we must have $Q(G) = cI_n$. The off-diagonal entries of $Q(G)$ are precisely the adjacency entries of G , so $A(G) = 0$ and $c = 0$. Thus G has no edges, i.e. $G \cong \overline{K_n}$. ■

3.2 Theorem 2.6 [10]

In [10], Theorem 2.6 states that $s_\alpha \geq \sqrt{s_{2\alpha}}$ holds for every bipartite graph, with no restriction on α . Notice, however, that for $\alpha = -1$ and $G = K_2$ one obtains $s_{-1} = 2^{-1} + 0^{-1}$, which is undefined. The inequality therefore requires $\alpha > 0$.

Theorem 5. *For a bipartite graph G and $\alpha > 0$,*

$$s_\alpha = \sum_{i=1}^n q_i^\alpha \geq \sqrt{s_{2\alpha}}.$$

3.3 Theorem 2.9 in [10]

Let Δ and δ denote the maximum and minimum degrees of G , and set

$$T = \frac{\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}}{2}.$$

In [10], Theorem 2.9 states that for a connected bipartite graph G and any $\alpha \neq 0, 1$,

$$s_\alpha \geq T^\alpha,$$

with equality iff $G \cong K_{1,n-1}$. Taking the star $K_{1,3}$ ($n = 4$, $\Delta = 3$, $\delta = 1$) and $\alpha = 2$, one finds $T = \frac{3+1+\sqrt{4+12}}{2} = 4$ and $s_2 = 4^2 + 1^2 + 1^2 + 0^2 = 18 > 16 = T^2$, so the inequality is strict. We show that the bound is actually sharp only for K_2 .

For the proof we need the following known relation between q_1 , T , and Δ .

Lemma 1 ([4]). *Let G be a connected graph with $n \geq 3$ vertices and let Δ be the maximum vertex degree of G . Then*

$$q_1 \geq T \geq \Delta + 1,$$

with either equality if and only if G is a star graph $K_{1,n-1}$.

Theorem 6. *For any connected bipartite graph G and $\alpha > 0$,*

$$s_\alpha \geq T^\alpha,$$

with equality if and only if $G \cong K_2$.

Proof. For a connected bipartite graph, $q_n = 0$ [5] and $q_1 \geq T$ (by Lemma

1). Using the power-mean inequality for $\alpha > 0$,

$$s_\alpha = \sum_{i=1}^n q_i^\alpha \geq q_1^\alpha \geq T^\alpha.$$

Equality $s_\alpha = T^\alpha$ holds if and only if $q_1^\alpha = T^\alpha$ (hence $q_1 = T$) and $q_i = 0$ for every $i = 2, \dots, n$. From $q_1 = T$ together with Lemma 1, we obtain $q_1 = \Delta + 1$, which forces G to be a star $K_{1,n-1}$. However, for $n > 2$, the star $K_{1,n-1}$ possesses eigenvalues $\{\Delta + 1, 1^{(n-2)}, 0\}$; consequently $q_2 = \dots = q_{n-1} = 1 \neq 0$, contradicting the requirement that $q_2, \dots, q_n = 0$. Thus the only admissible case is $n = 2$, i.e., $G \cong K_2$, whose eigenvalues are $\{2, 0\}$ and indeed satisfy $q_1 = T = 2$ and $q_2 = 0$. ■

3.4 Theorem 2.24 in [10]

For connected bipartite graphs, Theorem 2.24 of [10] gives $REE(G) \geq e + e^{-1}$, with equality supposedly at every complete bipartite graph. However, $K_{2,2}$ has Randić eigenvalues $\{1, -1, 0, 0\}$, so

$$REE(K_{2,2}) = e + e^{-1} + 2 > e + e^{-1}.$$

More generally, any connected bipartite graph with $n \geq 3$ satisfies $\rho_1 = 1$, $\rho_n = -1$, and has $n - 2$ additional eigenvalues. Since $e^{\rho_i} > 0$,

$$REE(G) = e + e^{-1} + \sum_{i=2}^{n-1} e^{\rho_i} > e + e^{-1},$$

so the original bound is sharp only for $G \cong K_2$. The next theorem sharpens this estimate.

Theorem 7. *For any connected bipartite graph G of order $n \geq 2$,*

$$REE(G) \geq e^1 + e^{-1} + n - 2.$$

Equality holds if and only if $G \cong K_{a,b}$.

Proof. For a connected bipartite graph G , the Randić eigenvalues satisfy $\rho_1 = 1$ and $\rho_n = -1$. Consider the remaining $n - 2$ eigenvalues

$\{\rho_2, \dots, \rho_{n-1}\}$. By Jensen's inequality applied to $f(x) = e^x$,

$$\frac{1}{n-2} \sum_{i=2}^{n-1} e^{\rho_i} \geq e^{\left(\frac{1}{n-2} \sum_{i=2}^{n-1} \rho_i\right)}.$$

Since the trace of the Randić matrix is zero,

$$\sum_{i=2}^{n-1} \rho_i = -(\rho_1 + \rho_n) = -(1 + (-1)) = 0.$$

Hence $\frac{1}{n-2} \sum_{i=2}^{n-1} e^{\rho_i} \geq 1$, which gives $\sum_{i=2}^{n-1} e^{\rho_i} \geq n-2$. Therefore,

$$REE(G) = e^{\rho_1} + e^{\rho_n} + \sum_{i=2}^{n-1} e^{\rho_i} \geq e + e^{-1} + n - 2.$$

Equality in Jensen's inequality forces $\rho_2 = \dots = \rho_{n-1} = 0$, which holds precisely for complete bipartite graphs. ■

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