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On the Minimum Variable Euler–Sombor Index

Mehdi Eliasi*

Department of Mathematics, Khansar Faculty, University of Isfahan, Isfahan, Iran

m.eliasi@khc.ui.ac.ir

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Abstract

The interplay between vertex degrees and edge-based topological invariants has recently gained significant traction in chemical graph theory. Among these, the Euler–Sombor index, defined by $\sum \sqrt{d_u^2 + d_v^2 + d_u d_v}$, has emerged as a promising descriptor. In this paper we study its one-parameter generalization $\mathrm{EU}_\lambda(G)$, where the edge-weight function is $\varphi_\lambda(x, y) = \sqrt{x^2 + y^2 + \lambda xy}$ for $\lambda \in [-2, 2)$.

For $\lambda \in [0, 2)$, we prove that every connected graph which attains the minimum of EU_λ among all connected graphs with fixed numbers of vertices and edges must be almost regular. This yields complete characterisations of the extremal graphs among trees, unicyclic graphs, bicyclic graphs, and more generally k-cyclic graphs for $k \geq 3$.

The case $\lambda \in [-2, 0)$ is more delicate: the kernel φ_λ induces a more subtle ordering on edge contributions. We give a complete classification of this ordering and use it to prove that the path P_n is the unique graph that attains the minimum of EU_λ among all trees of order n, and the cycle C_n is the unique graph for which the minimum is attained among all unicyclic graphs of order n, for every $\lambda \in [-2, 0)$.

*Corresponding author.

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Our results unify and extend several known extremal results, including those for the ordinary Euler–Sombor index ($\lambda = 1$) and the classical Sombor index ($\lambda = 0$).

1 Introduction

In chemical graph theory, topological indices are real-valued graph invariants. They play a fundamental role in quantitative structure–property relationships (QSPR) and quantitative structure–activity relationships (QSAR). They serve as numerical descriptors of molecular structure and are used to predict physicochemical properties and biological activities [3, 13]. A prominent class of such invariants consists of vertex-degree-based topological indices. For a graph G, they are generally defined as

$$TI(G) = \sum_{uv \in E(G)} I(d(u), d(v)),$$

where $d(u)$ denotes the degree of vertex u and I is a symmetric nonnegative real function.

In 2021, Gutman [6] introduced a novel geometric approach to constructing vertex-degree-based topological indices. He represented each edge uv as the degree-point $(d(u), d(v))$ in the coordinate plane and defined its degree-radius as the distance from this point to the origin. This led to the Sombor index:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

This index has since become one of the most extensively studied topological indices, with numerous mathematical properties and chemical applications documented in the literature [4, 14, 17].

Subsequently, Gutman, Furtula, and Oz [7] extended this geometric framework from points to ellipses and introduced the Elliptic Sombor index:

$$ESO(G) = \sum_{uv \in E(G)} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2}.$$

Building on this elliptic model, Gutman [8] shifted attention from the area of the ellipse to its perimeter. Using Euler's approximation formula for the perimeter of an ellipse (originally proposed in 1773 [5]), he defined the Euler–Sombor index as:

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

The basic properties of this index were determined in [8] and [19]. Gutman [8] also established several inequalities relating this new index to the original Sombor index. The chemical applicability of the Euler–Sombor index was validated through correlation studies with the properties of octane isomers, showing correlation coefficients exceeding 0.9 for properties such as acentric factor, entropy, and various thermodynamic descriptors [19].

More recently, Gutman, Redžepović, Kizilirmak, and Kulli [9] introduced a variable version of the Euler–Sombor index. It is denoted by $EU(\lambda)$ and defined as

$$EU(\lambda, G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + \lambda\, d(u)d(v)},$$

where λ is a real parameter. From the definition it is immediate that this variable index includes several well-known indices as special cases: $\lambda = 2$ gives the first Zagreb index $M_1(G) = \sum_{uv \in E(G)} (d(u) + d(v))$, introduced by Gutman and Trinajstić [10]; $\lambda = -2$ gives the Albertson index $Alb(G) = \sum_{uv \in E(G)} |d(u) - d(v)|$, introduced by Albertson [2]; $\lambda = 0$ gives the Sombor index $SO(G)$; and $\lambda = 1$ gives the ordinary Euler–Sombor index $EU(G)$. They also established that $EU(\lambda)$ is real-valued only for $\lambda \geq -2$, and provided bounds for $EU(\lambda)$ on various graph products [9]. The geometric justification for the parameter restriction $\lambda \in [-2, 2]$ was later provided by Nyauli and Buragohain [16], who derived the variable Euler–Sombor index via the law of cosines as a special case of the Cosine-Rule Generalized Sombor index.

The range $\lambda \in [-2, 0)$ is of particular interest, as the kernel penalises edges between high-degree vertices. This behavior is not present in the classical Sombor index ($\lambda = 0$) and has potential implications for model-

ing molecular structures with heterogeneous degree distributions. Mathematically, this range requires a completely different approach, based on a delicate classification of edge contributions.

The extremal properties of the Euler–Sombor index have attracted considerable attention. Tang, Li, and Deng [19] proved that among all trees of order n, the path P_n attains the minimum while the star S_n attains the maximum. They also showed that for general connected graphs the path attains the minimum and the complete graph attains the maximum. Khanra and Das [11] extended these results by determining the first through fifth minimum Euler–Sombor indices among trees and the first through third minimum indices among unicyclic graphs. They also established bounds for chemical graphs, chemical trees, and hexagonal systems, identifying the linear hexagonal chain as the graph for which the maximum is attained among hexagonal systems. Su and Tang [18] determined the extremal values of the Euler–Sombor index for all unicyclic and bicyclic graphs.

For tricyclic graphs, Kizilirmak [12] described the graphs for which the minimum and maximum Euler–Sombor index are attained among all tricyclic graphs of a given order. Albalahi et al. [1] subsequently extended these results to molecular tricyclic graphs (graphs with maximum degree at most 4), showing that the extremal structures identified by Kizilirmak also attain the minimum in the molecular setting.

The present paper is motivated by the recent work of Liu et al. [15], who developed a unified approach to the minimum of bond incident degree indices of k-cyclic graphs. Their framework applies to indices whose kernel satisfies three key properties: monotonicity in each variable, a decreasing difference property, and a certain convexity-like condition.

In this paper we study the variable version of the Euler–Sombor index from an extremal graph-theoretic perspective. Our main results are as follows.

For $\lambda \in [0, 2)$, we establish that the kernel φ_λ satisfies the three conditions of the Liu et al. framework. Consequently, any graph attaining the minimum value of the variable Euler–Sombor index within the class of connected graphs on a fixed number of vertices and edges must be almost

regular. Specializing this result to trees, unicyclic graphs, bicyclic graphs, and k-cyclic graphs with $k \geq 3$ yields complete characterizations of the extremal graphs in each class.

For $\lambda \in [-2, 0)$, the kernel exhibits a more complex behavior. For trees, we prove that P_n is the unique tree for which the minimum is attained, using an algebraic reformulation of the index and a case analysis. For unicyclic graphs, we completely determine the ordering of the first five smallest values of the squared kernel $f(x, y) = x^2 + y^2 + \lambda xy$ on $\mathbb{N}^2$; using this classification, we prove that C_n is the unique graph for which the minimum is attained.

The paper is organized as follows. Section 2 introduces the necessary notation and preliminary concepts. Section 3 establishes the almost regularity property for $\lambda \in [0, 2)$ and derives the extremal results for trees, unicyclic graphs, bicyclic graphs, and k-cyclic graphs. Section 4 treats the case $\lambda \in [-2, 0)$, beginning with a classification of the edge contribution ordering, followed by the characterization of the extremal trees and unicyclic graphs. Section 5 concludes with a summary of the results, a discussion of the methods employed, and some open problems.

2 Preliminaries and notation

We begin with some notation and terminology from graph theory. All graphs considered in this paper are finite, simple and undirected. For a graph G, we denote by $V(G)$ and $E(G)$ the sets of vertices and edges, respectively. The degree of a vertex v is denoted by d_v, and we write $\Delta(G)$ and $\delta(G)$ for the maximum and minimum degrees of G. A graph is called almost regular if $\Delta(G) - \delta(G) \leq 1$.

A connected graph is called k-cyclic if it has exactly $n - 1 + k$ edges, where n denotes the number of vertices. Thus, for $k = 0$, 1, and 2, this reduces to the classes of trees, unicyclic graphs, and bicyclic graphs, respectively.

For $\lambda \in [0, 2)$, we denote the variable version of the Euler–Sombor

index of a graph G by

$$\mathrm{EU}_\lambda(G) := \sum_{uv \in E(G)} \varphi_\lambda(d_u, d_v),$$

where

$$\varphi_\lambda(x, y) := \sqrt{x^2 + y^2 + \lambda xy}, \qquad x, y \geq 0.$$

The function φ_λ will be called the kernel of the index.

We shall use the standard notation P_n and C_n for the path and cycle on n vertices, respectively.

3 Minimal graphs for $\lambda \in [0, 2)$

In this section we study the kernel φ_λ for $\lambda \in [0, 2)$. Our aim is to determine the structure of all k-cyclic graphs for which the minimum of the index EU_λ is attained. We begin with the following lemma, which provides the elementary properties of the kernel needed for this purpose.

Lemma 1. *For $\lambda \in [0, 2)$, define*

$$\varphi_\lambda(x, y) := \sqrt{x^2 + y^2 + \lambda xy}, \qquad x, y \geq 0.$$

Then:

(i) φ_λ is strictly increasing in each variable;

(ii) for every $a > b \geq 0$, the function $x \mapsto \varphi_\lambda(a, x) - \varphi_\lambda(b, x)$ is decreasing on $(0, \infty)$;

(iii) for all integers $a > b + 1 \geq 2$,

$$a\big[\varphi_\lambda(a, a) - \varphi_\lambda(a - 1, a)\big] > b\big[\varphi_\lambda(b + 1, b) - \varphi_\lambda(b, b)\big].$$

Proof. Set $A := 2 + \lambda \in [2, 4)$.

(i) Differentiating gives

$$\partial_x \varphi_\lambda(x, y) = \frac{2x + \lambda y}{2\varphi_\lambda(x, y)}.$$

Since $x, y \geq 0$ and $\lambda \geq 0$, this is nonnegative, and strictly positive unless $(x, y) = (0, 0)$. By symmetry, the same holds for $\partial_y \varphi_\lambda$. Hence φ_λ is strictly increasing in each variable on $[0, \infty)^2$.

(ii) Fix $a > b \geq 0$, and define

$$h(y) := \varphi_\lambda(a, y) - \varphi_\lambda(b, y), \qquad y > 0.$$

Then

$$h'(y) = \frac{\partial \varphi_\lambda}{\partial y}(a, y) - \frac{\partial \varphi_\lambda}{\partial y}(b, y).$$

To prove that $h' < 0$, it suffices to show that for each fixed $y > 0$, the function

$$x \mapsto \frac{\partial \varphi_\lambda}{\partial y}(x, y)$$

is decreasing in x. Indeed,

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi_\lambda}{\partial y}(x, y)\right) = \frac{(\lambda^2 - 4)xy}{4\varphi_\lambda(x, y)^3} \leq 0,$$

since $\lambda^2 - 4 < 0$. Hence, because $a > b$,

$$\frac{\partial \varphi_\lambda}{\partial y}(a, y) < \frac{\partial \varphi_\lambda}{\partial y}(b, y),$$

so $h'(y) < 0$ on $(0, \infty)$. Thus h is decreasing.

(iii) First note that

$$\varphi_\lambda(x, x) = x\sqrt{A},$$

and

$$\varphi_\lambda(x - 1, x) = \sqrt{(x - 1)^2 + x^2 + \lambda x(x - 1)} = \sqrt{Ax^2 - Ax + 1},$$

while

$$\varphi_\lambda(x + 1, x) = \sqrt{(x + 1)^2 + x^2 + \lambda x(x + 1)} = \sqrt{Ax^2 + Ax + 1}.$$

Define

$$f(x) := x\big[\varphi_\lambda(x,x) - \varphi_\lambda(x-1,x)\big] = x\big[x\sqrt{A} - \sqrt{Ax^2 - Ax + 1}\big],$$

and

$$g(x) := x\big[\varphi_\lambda(x+1,x) - \varphi_\lambda(x,x)\big] = x\big[\sqrt{Ax^2 + Ax + 1} - x\sqrt{A}\big].$$

Then the desired inequality becomes $f(a) > g(b)$.

We prove two claims.

Claim 1. The function f is strictly increasing on $[1,\infty)$.

Differentiating gives

$$f'(x) = 2x\sqrt{A} - \frac{(2Ax - A)x}{2\sqrt{Ax^2 - Ax + 1}} - \sqrt{Ax^2 - Ax + 1}.$$

Combining the last two terms gives

$$f'(x) = 2x\sqrt{A} - \frac{4Ax^2 - 3Ax + 2}{2\sqrt{Ax^2 - Ax + 1}}.$$

Thus $f'(x) > 0$ is equivalent to

$$2x\sqrt{A} > \frac{4Ax^2 - 3Ax + 2}{2\sqrt{Ax^2 - Ax + 1}}.$$

For $x \geq 1$ and $A \geq 2$, we have $4Ax^2 - 3Ax + 2 > 0$, so both sides are positive. Squaring after multiplying by the positive denominators gives, after simplification,

$$P(x) := 8A^2x^3 - 9A^2x^2 + 12Ax - 4 > 0.$$

Now

$$P'(x) = 24A^2x^2 - 18A^2x + 12A = 6A(4Ax^2 - 3Ax + 2).$$

Since

$$4Ax^2 - 3Ax + 2 \geq 4Ax - 3Ax + 2 = Ax + 2 > 0 \qquad (x \geq 1,\ A \geq 2),$$

it follows that $P'(x) > 0$ for all $x \geq 1$. Hence P is strictly increasing on $[1, \infty)$. Moreover,

$$P(1) = 8A^2 - 9A^2 + 12A - 4 = -A^2 + 12A - 4.$$

The roots of the quadratic polynomial $-A^2 + 12A - 4$ are $6 \pm 4\sqrt{2}$, and since $A \in [2, 4)$, we get $P(1) > 0$. Therefore

$$P(x) \geq P(1) > 0 \qquad \text{for all } x \geq 1.$$

Consequently, $f'(x) > 0$ for all $x \geq 1$, so f is strictly increasing on $[1, \infty)$. This proves Claim 1.

Claim 2. For every integer $t \geq 1$,

$$f(t+1) > g(t).$$

Indeed,

$$f(t+1) = (t+1)\big[(t+1)\sqrt{A} - \sqrt{At^2 + At + 1}\big],$$

and

$$g(t) = t\big[\sqrt{At^2 + At + 1} - t\sqrt{A}\big].$$

Hence

$$f(t+1) - g(t) = \sqrt{A}(2t^2 + 2t + 1) - (2t+1)\sqrt{At^2 + At + 1}.$$

Thus $f(t+1) > g(t)$ is equivalent to

$$\sqrt{A}(2t^2 + 2t + 1) > (2t+1)\sqrt{At^2 + At + 1}.$$

Both sides are positive, so squaring gives

$$A(2t^2 + 2t + 1)^2 > (2t + 1)^2(At^2 + At + 1),$$

which simplifies to

$$A(3t^2 + 3t + 1) > 4t^2 + 4t + 1.$$

Since $A \geq 2$,

$$A(3t^2 + 3t + 1) \geq 2(3t^2 + 3t + 1) = 6t^2 + 6t + 2,$$

and for every $t \geq 1$,

$$6t^2 + 6t + 2 > 4t^2 + 4t + 1.$$

Thus

$$A(3t^2 + 3t + 1) > 4t^2 + 4t + 1,$$

which proves Claim 2.

We now complete the proof of *(iii)*. Let $a > b+1 \geq 2$. Then $a \geq b+2$, and since f is strictly increasing on $[1, \infty)$, Claim 1 gives

$$f(a) \geq f(b + 2) > f(b + 1).$$

By Claim 2 with $t = b$,

$$f(b + 1) > g(b).$$

Combining these gives

$$f(a) > g(b),$$

which is exactly

$$a\big[\varphi_\lambda(a, a) - \varphi_\lambda(a - 1, a)\big] > b\big[\varphi_\lambda(b + 1, b) - \varphi_\lambda(b, b)\big].$$

This completes the proof. ■

We now apply the properties of the kernel φ_λ established in Lemma 1 to

characterize graphs attaining the minimum for the variable Euler–Sombor index.

Theorem 1. *Let $\lambda \in [0,2)$, and let G be a connected graph with n vertices and m edges. If G attains the minimum value of EU_λ among all connected graphs with the same numbers of vertices and edges, then G is almost regular.*

Proof. Lemma 1 shows that the kernel φ_λ satisfies the three conditions for every $\lambda \in [0,2)$. The conclusion follows from [15, Lemma 2.2], which states that if a symmetric function $I\colon \mathbb{Z}^2_{\geq 0} \to \mathbb{R}$ satisfies these conditions, then any connected graph minimizing

$$TI(G) = \sum_{uv \in E(G)} I(d_u, d_v)$$

must satisfy $\Delta(G) - \delta(G) \leq 1$. ■

As a consequence of Theorem 1, we may determine the trees attaining the minimum with respect to the variable Euler–Sombor index.

Corollary. *Let $\lambda \in [0,2)$ and $n \geq 3$. Among all trees of order n, the path P_n uniquely attains the minimum value of EU_λ.*

Proof. Let T be a tree of order n which attains the minimum of EU_λ over all trees on n vertices. Since every tree is connected and has $n-1$ edges, Theorem 1 implies that T is almost regular, so $\Delta(T) - \delta(T) \leq 1$.

Every tree has at least two leaves; hence $\delta(T) = 1$. Almost regularity then gives $\Delta(T) \leq 2$. Since T is connected and $n \geq 3$, we have $\Delta(T) = 2$.

A connected graph with maximum degree at most 2 is either a path or a cycle. Since T is a tree, it must be P_n. Consequently, $T \cong P_n$. ■

Similarly, Theorem 1 determines the unicyclic graphs attaining the minimum.

Corollary. *Let $\lambda \in [0,2)$ and $n \geq 3$. Among all unicyclic graphs of order n, the cycle C_n is the unique graph for which the minimum of EU_λ is attained.*

Proof. Let G be a unicyclic graph of order n which attains the minimum of EU_λ over all unicyclic graphs on n vertices. Every such graph is connected and has exactly n edges; hence Theorem 1 gives $\Delta(G) - \delta(G) \leq 1$.

The Handshaking Lemma gives $\sum_{v \in V(G)} d_v = 2n$. If $\delta(G) \geq 2$, then every vertex has degree at least 2; otherwise the degree sum would exceed $2n$. Hence all degrees equal 2, so G is 2-regular and therefore connected, which gives $G \cong C_n$.

It remains to exclude $\delta(G) = 1$. In that case almost regularity forces $\Delta(G) \leq 2$, so every vertex has degree 1 or 2. Let n_1 be the number of vertices of degree 1. Then

$$\sum_{v \in V(G)} d_v = n_1 + 2(n - n_1) = 2n - n_1.$$

Since the degree sum must be $2n$, we get $n_1 = 0$, which contradicts $\delta(G) = 1$. Thus $\delta(G) = 2$, and consequently $G \cong C_n$. ∎

We next determine the extremal bicyclic graphs for the variable version of the Euler–Sombor index.

Theorem 2. *Let $\lambda \in [0, 2)$ and $n \geq 4$. Among all bicyclic graphs of order n, a graph G minimises the variable version of the Euler–Sombor index if and only if G has exactly two vertices of degree 3, they are adjacent, and all remaining vertices have degree 2. The graphs with this property are precisely:*

(i) two cycles sharing a common edge, existing for all $n \geq 4$;

(ii) two cycles joined by a bridge, existing for $n \geq 6$.

In particular, for $n \in \{4, 5\}$ the minimum is uniquely attained by type (i), while for $n \geq 6$ both types attain the minimum simultaneously.

Proof. Let G be a bicyclic graph of order n minimizing EU_λ. Since every bicyclic graph is connected and has $m = n + 1$ edges, Theorem 1 gives $\Delta(G) - \delta(G) \leq 1$. Moreover, $\sum_v d_v = 2(n + 1) = 2n + 2$.

We first determine the minimum degree of G. If $\delta(G) \geq 3$, then $\sum_v d_v \geq 3n > 2n + 2$ for $n \geq 4$, impossible. On the other hand, if

$\delta(G) = 1$, almost regularity forces $\Delta(G) \leq 2$, so all degrees lie in $\{1, 2\}$ and consequently $\sum_v d_v \leq 2n < 2n + 2$, again a contradiction. Hence $\delta(G) = 2$ and therefore $\Delta(G) \leq 3$.

Let n_3 be the number of vertices of degree 3. The degree-sum condition gives

$$2(n - n_3) + 3n_3 = 2n + 2,$$

which immediately yields $n_3 = 2$. Thus G contains exactly two vertices of degree 3, while the remaining $n - 2$ vertices have degree 2.

A connected bicyclic graph with degree sequence $(3, 3, 2, \ldots, 2)$ falls into one of three isomorphism types, depending on the adjacency of the two degree-3 vertices u and v. If u and v are adjacent and the edge uv lies on two distinct cycles, we obtain the book graph $B(p, q)$, consisting of two cycles of lengths $p + 1$ and $q + 1$ sharing the common edge uv. If u and v are adjacent but each lies on a different cycle, the graph is the dumbbell graph, in which the two cycles are connected by the bridge uv. Finally, if u and v are not adjacent, they must be connected by three internally vertex-disjoint paths, giving the theta graph $\Theta(a, b, c)$, where a, b, c denote the lengths of these paths.

The book and dumbbell graphs have the same multiset of edge degree-pairs, namely one pair $(3, 3)$, four pairs $(3, 2)$, and $n - 4$ pairs $(2, 2)$, and therefore yield identical values of EU_λ. The degree-pair counts for the two cases are as follows:

$$\begin{aligned} \text{book, dumbbell} &: (3,3) : 1, \quad (3,2) : 4, \quad (2,2) : (n-4), \\ \text{theta} &: (3,3) : 0, \quad (3,2) : 6, \quad (2,2) : (n-5). \end{aligned}$$

Consequently, for graphs G_{bd} and G_θ of book/dumbbell and theta types, respectively,

$$\begin{aligned} \mathrm{EU}_\lambda(G_{\mathrm{bd}}) - \mathrm{EU}_\lambda(G_\theta) &= \varphi_\lambda(3,3) + \varphi_\lambda(2,2) - 2\varphi_\lambda(3,2) \\ &= \sqrt{18 + 9\lambda} + \sqrt{8 + 4\lambda} - 2\sqrt{13 + 6\lambda}. \end{aligned}$$

To determine the sign of this expression, note that

$$(18+9\lambda)(8+4\lambda) = 144+144\lambda+36\lambda^2 = (12+6\lambda)^2,$$

so $\sqrt{(18+9\lambda)(8+4\lambda)} = 12+6\lambda$. Squaring the inequality

$$\sqrt{18+9\lambda}+\sqrt{8+4\lambda} < 2\sqrt{13+6\lambda}$$

gives

$$(\sqrt{18+9\lambda}+\sqrt{8+4\lambda})^2 = 50+25\lambda, \qquad (2\sqrt{13+6\lambda})^2 = 52+24\lambda.$$

Hence

$$\mathrm{EU}_\lambda(G_{\mathrm{bd}}) < \mathrm{EU}_\lambda(G_\theta) \iff 50+25\lambda < 52+24\lambda \iff \lambda < 2.$$

This inequality holds for all $\lambda \in [0,2)$, showing that the theta graph cannot attain the minimum. Therefore the two vertices of degree 3 must be adjacent.

It follows that the minimum is attained exactly by the book and dumbbell graphs. For $n \in \{4,5\}$ only the book graph exists, yielding a unique graph attaining the minimum, whereas for $n \geq 6$ both types occur and attain the same minimum value of EU_λ. ■

Our analysis for $\lambda \in [0,2)$ concludes with the following result for k-cyclic graphs.

Theorem 3. *Let $\lambda \in [0,2)$, $k \geq 3$, and $n \geq 5(k-1)$. Among all k-cyclic graphs of order n, a graph G attains the minimum value of $\mathrm{EU}_\lambda(G)$ if and only if G is bidegreed with degrees in $\{2,3\}$ and the multiset of its edge degree-pairs consists of:*

(i) two edges of type $(2,3)$;

(ii) $n-2k+1$ edges of type $(2,2)$;

(iii) $3k-4$ edges of type $(3,3)$.

The minimum value is

$$\mathrm{EU}_\lambda(G) = 2\sqrt{13+6\lambda} + (n-2k+1)\sqrt{8+4\lambda} + (3k-4)\sqrt{18+9\lambda}.$$

Proof. Lemma 1 shows that the kernel φ_λ satisfies the three required conditions for every $\lambda \in [0,2)$. Hence Theorem 2.1 of [15] applies, which establishes the structure of the extremal graphs and the minimum value as stated above. ■

4 Minimal graphs for $\lambda \in [-2,0)$

We now turn to the case $\lambda \in [-2,0)$. Unlike the nonnegative range, the behavior of the kernel φ_λ in this interval is substantially different, and a general result analogous to Theorem 1 is not available. Consequently, we restrict our attention to trees and unicyclic graphs. The characterization of graphs for which the minimum is attained for k-cyclic graphs with $k \geq 2$ in this range remains an open problem for future research.

The following technical lemma will be used in the proof of the main theorem for trees.

Lemma 2. *For all integers $i,j \geq 3$ with $(i,j) \neq (3,3),(3,4)$, we have $C(i,j) \leq 0$, where*

$$C(i,j) := 4B(i,j)^2 - ij - 2A(i,j)^2,$$

with

$$A(i,j) = 2 - \frac{2}{i} - \frac{2}{j}, \qquad B(i,j) = \frac{2}{i} + \frac{2}{j} - 3.$$

Proof. Let $u = 1/i$, $v = 1/j$, so $0 < v \leq u \leq 1/3$. A straightforward computation gives

$$C = 8(u+v)^2 - 32(u+v) + 28 - ij.$$

We split the proof into three exhaustive cases.

Case $i = j \geq 4$: Here $u = v = 1/i$, giving

$$C(i,i) = \frac{32}{i^2} - \frac{64}{i} + 28 - i^2.$$

For $i = 4$, $C(4,4) = -2 < 0$. For $i > 4$,

$$\frac{d}{di}C(i,i) = -2i + \frac{64(i-1)}{i^3} < -2i + 4 \leq -4,$$

so $C(i,i)$ is strictly decreasing, hence $C(i,i) \leq C(4,4) < 0$ for all $i \geq 4$.

Case $i = 3$, $j \geq 5$: Substituting $u = 1/3$, $v = 1/j$ gives

$$C(3,j) = \frac{164}{9} - \frac{80}{3j} + \frac{8}{j^2} - 3j.$$

For $j \geq 5$,

$$\frac{\partial C}{\partial j} = \frac{80}{3j^2} - \frac{16}{j^3} - 3 < 0.$$

Since $C(3,5) = -403/225 < 0$, it follows that $C(3,j) < 0$ for all $j \geq 5$.

Case $i \geq 4$, $j > i$: Differentiating with respect to j gives

$$\frac{\partial C}{\partial j} = -\frac{16}{j^2}(u+v) + \frac{32}{j^2} - i < 0,$$

because $u + v > 0$ and $\frac{32}{j^2} < i$ (since $j \geq i+1$ and $i(i+1)^2 > 32$). Thus $C(i,j)$ is strictly decreasing in j for fixed i, and from the first case $C(i,i) < 0$, so $C(i,j) < 0$ for all $j > i$.

The three cases cover every pair except $(3,3)$ and $(3,4)$, proving the lemma. ■

We now prove the main result for trees.

Theorem 4. *Let $n \geq 3$ and $\lambda \in (-2,0)$. Among all trees of order n, the path P_n is the unique tree for which the minimum of the variable version of the Euler–Sombor index EU_λ is attained.*

Proof. For a tree T of order n, let n_i denote the number of vertices of degree i ($1 \leq i \leq n-1$), and let $x_{i,j}$ denote the number of edges joining a vertex of degree i with a vertex of degree j ($1 \leq i \leq j \leq n-1$).

The standard identities for trees are

$$n = \sum_i n_i, \quad 2n-2 \quad = \sum_i i n_i, \quad i n_i = 2x_{i,i} + \sum_{j \neq i} x_{i,j}. \tag{1}$$

Define the auxiliary quantities

$$f_1 = x_{1,3} + x_{1,4} + \cdots = n_1 - x_{1,2},$$

$$f_2 = 2n_2 - x_{1,2} - 2x_{2,2}$$

$$\big(\text{by (1) for } i=2,\ 2n_2 = 2x_{2,2} + x_{1,2} + \sum_{j \geq 3} x_{2,j},\ \text{so } f_2 = \sum_{j \geq 3} x_{2,j}\big),$$

$$f_k = k n_k \quad (k \geq 3).$$

Summing f_i and $\frac{1}{i} f_i$ and using (1) gives

$$\sum_{i=1}^{n-1} f_i = 2m - 2x_{1,2} - 2x_{2,2}, \qquad \sum_{i=1}^{n-1} \frac{1}{i} f_i = n - \frac{3}{2} x_{1,2} - x_{2,2},$$

where $m = n-1$ is the number of edges. Solving for $x_{1,2}$ and $x_{2,2}$ gives

$$x_{1,2} = 2 + \sum\nolimits^{*} \Big(2 - \frac{2}{i} - \frac{2}{j}\Big) x_{i,j}, \quad x_{2,2} \quad = n - 3 + \sum\nolimits^{*} \Big(\frac{2}{i} + \frac{2}{j} - 3\Big) x_{i,j}, \tag{2}$$

where $\sum^{*}$ runs over all pairs (i,j) with $1 \leq i \leq j \leq n-1$, except $(1,2)$ and $(2,2)$.

Substituting (2) and using $\varphi_\lambda(1,2) = \sqrt{5+2\lambda}$ and $\varphi_\lambda(2,2) = \sqrt{8+4\lambda}$ gives, after simplification,

$$\mathrm{EU}_\lambda(T) = 2\varphi_\lambda(1,2) + (n-3)\varphi_\lambda(2,2) + \sum\nolimits^{*} F(i,j)\, x_{i,j}, \tag{3}$$

where for every $(i,j) \neq (1,2), (2,2)$,

$$F(i,j) = \varphi_\lambda(i,j) + \Big(2 - \frac{2}{i} - \frac{2}{j}\Big)\varphi_\lambda(1,2) + \Big(\frac{2}{i} + \frac{2}{j} - 3\Big)\varphi_\lambda(2,2). \tag{4}$$

Since $\mathrm{EU}_\lambda(P_n) = 2\varphi_\lambda(1,2) + (n-3)\varphi_\lambda(2,2)$, (3) shows that P_n is the unique tree attaining the minimum of EU_λ if and only if $F(i,j) > 0$ for every admissible pair $(i,j) \neq (1,2), (2,2)$.

Set $q = \sqrt{2+\lambda}$. Since $\lambda \in (-2, 0)$ we have $q \in (0, \sqrt{2})$. Then

$$\varphi_\lambda(x,y) = \sqrt{(x-y)^2 + q^2 xy}, \quad S := \varphi_\lambda(1,2) = \sqrt{1+2q^2}, \quad \varphi_\lambda(2,2) = 2q.$$

Consequently,

$$F(i,j;q) = \sqrt{(i-j)^2 + q^2 ij} + A(i,j)\, S + 2B(i,j)\, q, \tag{5}$$

where

$$A(i,j) = 2 - \frac{2}{i} - \frac{2}{j}, \qquad B(i,j) = \frac{2}{i} + \frac{2}{j} - 3.$$

One checks directly that $F(1,2;q) = F(2,2;q) = 0$. We must prove $F(i,j;q) > 0$ for every admissible pair $(i,j) \neq (1,2), (2,2)$. For $F(i,j;q)$ defined in (4), we prove positivity for all admissible pairs.

Case 1: Pairs of the form $(1,j)$, $j \geq 3$. Here $A = -2/j$ and $B = 2/j - 1 < 0$, so

$$F(1,j;q) = \sqrt{(j-1)^2 + q^2 j} - \frac{2S}{j} + \Big(\frac{4}{j} - 2\Big) q.$$

$j = 3$: $F(1,3;q) = \sqrt{4+3q^2} - \frac{2S+2q}{3}$. The function $q \mapsto F(1,3;q)$ is continuous on $[0, \sqrt{2})$. Assuming $F(1,3;q) = 0$ leads, after squaring twice, to $97q^4 + 896q^2 + 1024 = 0$, which has no real solution. Since $F(1,3;0) = 4/3 > 0$, the intermediate value theorem implies $F(1,3;q) > 0$ for all $q \in (0, \sqrt{2})$.

$j \geq 4$: Since the second term under the radical is positive,

$$F(1,j;q) \geq (j-1) - \frac{2S}{j} - \Big(2 - \frac{4}{j}\Big) q.$$

Define

$$g(j,q) := j - 1 - \frac{2S}{j} - \Big(2 - \frac{4}{j}\Big) q.$$

Multiplying by $j > 0$ gives

$$j g(j,q) = j(j-1) - 2S - 2jq + 4q =: \Phi(q).$$

For $q \in (0, \sqrt{2})$,

$$\Phi''(q) = -\frac{4}{(1+2q^2)^{3/2}} < 0,$$

so Φ is strictly concave. Hence

$$\Phi(q) > \min\{\lim_{q\to 0^+} \Phi(q),\ \lim_{q\to\sqrt{2}^-} \Phi(q)\}.$$

Now,

$$\lim_{q\to 0^+} \Phi(q) = j(j-1) - 2 = (j-2)(j+1) > 0 \quad (j \geq 4),$$

and

$$\lim_{q\to\sqrt{2}^-} \Phi(q) = j^2 - (1+2\sqrt{2})j + 4\sqrt{2} - 2\sqrt{5}.$$

The larger root of this quadratic is approximately 3.489, so for every integer $j \geq 4$ the limit is positive. Thus $\Phi(q) > 0$ for all $q \in (0, \sqrt{2})$, and consequently $g(j,q) > 0$. Therefore $F(1,j;q) > 0$ for all $j \geq 4$.

Thus $F(1,j;q) > 0$ for all $j \geq 3$.

Case 2: Pairs of the form $(2,j)$, $j \geq 3$. Now $A = 1 - 2/j > 0$, $B = 2/j - 2 < 0$, and

$$F(2,j;q) = \sqrt{(j-2)^2 + 2q^2 j} + \left(1 - \frac{2}{j}\right) S - \left(4 - \frac{4}{j}\right) q.$$

By the inequalities $\sqrt{(j-2)^2 + 2q^2 j} \geq q\sqrt{2j}$ and $S \geq \sqrt{2}\, q$, we get

$$F(2,j;q) \geq q\left(\sqrt{2j} + \sqrt{2} - \frac{2\sqrt{2}}{j} - 4 + \frac{4}{j}\right).$$

The expression in parentheses is positive for every integer $j \geq 3$: indeed, the function

$$h(j) := \sqrt{2j} + \sqrt{2} - \frac{2\sqrt{2}}{j} - 4 + \frac{4}{j}$$

is increasing on $[3,\infty)$, since

$$h'(j) = \frac{1}{\sqrt{2j}} + \frac{2\sqrt{2}-4}{j^2} > 0.$$

Thus $h(j) \geq h(3) = \sqrt{6} + \sqrt{2} - \frac{2\sqrt{2}}{3} - \frac{8}{3} > 0$. Since $q > 0$, we conclude that $F(2, j; q) > 0$ for all $j \geq 3$ and all $q \in (0, \sqrt{2})$.

Case 3: Pairs with $3 \leq i \leq j$. Here $A > 0$, $B < 0$, and we write

$$F(i,j;q) = P + AS - 2|B|q, \qquad P = \sqrt{(i-j)^2 + q^2 ij}.$$

To prove $F > 0$, it suffices to show $P + AS > 2|B|q$. Since both sides are non-negative, squaring gives

$$(i-j)^2 + A^2 + 2APS > q^2(4B^2 - ij - 2A^2).$$

Define $C(i,j) = 4B^2 - ij - 2A^2$. If $C \leq 0$, the inequality holds automatically because the left-hand side is strictly positive (since $(i-j)^2 + A^2 > 0$ for $i, j \geq 3$). By Lemma 2, this covers all pairs except $(3,3)$ and $(3,4)$.
Verification of $(3,3)$*:* Here $i = j = 3$, so

$$P = 3q, \quad A = \frac{2}{3}, \quad B = -\frac{5}{3}, \quad S = \sqrt{1+2q^2},$$

and

$$F(3,3;q) = 3q + \frac{2S}{3} - \frac{10q}{3} = \frac{1}{3}(2S - q) > 0,$$

since $2S > q$.
Verification of $(3,4)$*:*

$$A = \frac{5}{6}, \quad B = -\frac{11}{6}, \quad P = \sqrt{1+12q^2}, \quad S = \sqrt{1+2q^2},$$

$$F(3,4;q) = \sqrt{1+12q^2} + \frac{5S}{6} - \frac{11q}{3}.$$

If $\frac{11q}{3} \leq \frac{5S}{6}$ (i.e., $q^2 \leq 25/434$), then $F > 0$ trivially. Otherwise, squaring

the equivalent inequality $\sqrt{1+12q^2} > \frac{11q}{3} - \frac{5S}{6}$ yields

$$220qS > 102q^2 - 11. \tag{6}$$

If $102q^2 \leq 11$, (6) holds trivially. If $102q^2 > 11$, then since $S > 1$ and $q < \sqrt{2} < q_0$, where $q_0 \approx 2.206$ is the larger root of $102q^2 - 220q - 11 = 0$, we have

$$220qS > 220q > 102q^2 - 11,$$

which proves (6). Hence $F(3,4;q) > 0$ for all $q \in (0, \sqrt{2})$.

Thus $F(i,j;q) > 0$ for all admissible pairs $(i,j) \neq (1,2), (2,2)$.

Consequently, by (3), $\mathrm{EU}_\lambda(T) > \mathrm{EU}_\lambda(P_n)$ for every tree $T \neq P_n$, and therefore P_n is the unique tree attaining the minimum. This completes the proof.

■

Remark 5. *The boundary case $\lambda = -2$ can also be treated directly. In this case the kernel becomes*

$$f(x,y) = x^2 + y^2 - 2xy = (x-y)^2,$$

so the index reduces to

$$EU_{-2}(T) = \sum_{uv \in E(T)} |d_u - d_v|.$$

For the path P_n, only the two edges incident with leaves contribute, hence

$$EU_{-2}(P_n) = 2.$$

Now let T be any tree on n vertices different from P_n. Then T has at least one vertex of degree at least 3. Starting from such a vertex, there are at least three edge-disjoint paths ending at distinct leaves. On each of these paths, the first edge whose endpoints have different degrees contributes at least 1 to $SO_{-2}(T)$. These three edges are distinct, so

$$EU_{-2}(T) \geq 3 > 2 = EU_{-2}(P_n).$$

Thus the path P_n uniquely attains the minimum also at $\lambda = -2$.

We now turn to unicyclic graphs. The proof here is different from that for trees, and the following lemma plays a key role.

Lemma 3. *Throughout, let $f(x, y) = x^2 + y^2 + \lambda xy$ with $\lambda \in (-2, 0)$, and let the domain be $S = \mathbb{N}^2 \setminus \{(1, 1)\}$. Write*

$$A = f(1, 2) = 5 + 2\lambda, \quad B = f(2, 2) = 8 + 4\lambda, \quad C = f(1, 3) = 10 + 3\lambda,$$

$$D = f(2, 3) = 13 + 6\lambda, \quad E = f(3, 3) = 18 + 9\lambda.$$

These five values are the only candidates for the five smallest values of f on S. Their complete ordering, together with the resulting first five minima $m_1(\lambda) \le m_2(\lambda) \le \cdots \le m_5(\lambda)$, is given in Table 1.

Table 1. Complete ordering of A, B, C, D, E and the five smallest values of f on S, as λ ranges over $(-2, 0)$. Tied entries share a rank.

Region	Order (min to max)	m_1	m_2	m_3	m_4	m_5
$-2 < \lambda < -\frac{13}{7}$	$B < E < A < D < C$	B	E	A	D	C
$\lambda = -\frac{13}{7}$	$B < A = E < D < C$	B	A	E	D	C
$-\frac{13}{7} < \lambda < -\frac{5}{3}$	$B < A < E < D < C$	B	A	E	D	C
$\lambda = -\frac{5}{3}$	$B < A < D = E < C$	B	A	D	E	C
$-\frac{5}{3} < \lambda < -\frac{3}{2}$	$B < A < D < E < C$	B	A	D	E	C
$\lambda = -\frac{3}{2}$	$A = B < D < E < C$	A	D	E	C	-
$-\frac{3}{2} < \lambda < -\frac{4}{3}$	$A < B < D < E < C$	A	B	D	E	C
$\lambda = -\frac{4}{3}$	$A < B < D < C = E$	A	B	D	C	E
$-\frac{4}{3} < \lambda < -1$	$A < B < D < C < E$	A	B	D	C	E
$\lambda = -1$	$A < B < C = D < E$	A	B	C	D	E
$-1 < \lambda < 0$	$A < B < C < D < E$	A	B	C	D	E

In particular, the minimum and second minimum are

$$m_1(\lambda) = \begin{cases} 8+4\lambda, & -2 < \lambda \le -\frac{3}{2}, \\ 5+2\lambda, & -\frac{3}{2} \le \lambda < 0, \end{cases}$$

$$m_2(\lambda) = \begin{cases} 18+9\lambda, & -2 < \lambda < -\frac{13}{7}, \\ \frac{9}{7}, & \lambda = -\frac{13}{7}, \\ 5+2\lambda, & -\frac{13}{7} < \lambda < -\frac{3}{2}, \\ 13+6\lambda, & \lambda = -\frac{3}{2}, \\ 8+4\lambda, & -\frac{3}{2} < \lambda < 0. \end{cases}$$

Proof. We first determine the relative order of A, B, C, D, E as λ varies over $(-2, 0)$. The pairwise comparisons that change sign on this interval are

$$A \lesseqgtr B \iff \lambda \gtreqless -\tfrac{3}{2}, \qquad A \lesseqgtr E \iff \lambda \gtreqless -\tfrac{13}{7}, \tag{7}$$

$$D \lesseqgtr E \iff \lambda \gtreqless -\tfrac{5}{3}, \qquad C \lesseqgtr D \iff \lambda \gtreqless -1, \tag{8}$$

$$C \lesseqgtr E \iff \lambda \gtreqless -\tfrac{4}{3}. \tag{9}$$

All other comparisons remain constant throughout $(-2, 0)$:

$$A < C, \quad A < D, \quad B < C, \quad B < D, \quad B < E.$$

Each of these is a linear inequality in λ that holds for all $\lambda \in (-2, 0)$ and may be checked directly. In particular, B is always below D and E, so the global minimum is either A or B depending on the sign of $A - B = -3 - 2\lambda$.

Combining (7)–(9) with the fixed inequalities yields the ordering in Table 1. For completeness, we briefly indicate the argument on each region.

For $-2 < \lambda < -\frac{13}{7}$ we have $\lambda < -\frac{13}{7} < -\frac{5}{3} < -\frac{3}{2}$, hence $A > B$, $E > A$, and $D > E$. Together with $A < D$ and $D < C$ (which holds since $\lambda < -1$) this gives $B < E < A < D < C$.

At $\lambda = -\frac{13}{7}$ we find $A = 5 - \frac{26}{7} = \frac{9}{7}$, $E = 18 - \frac{117}{7} = \frac{9}{7}$, and $B = 8 - \frac{52}{7} = \frac{4}{7}$, so $B < A = E < D < C$.

For $-\frac{13}{7} < \lambda < -\frac{5}{3}$ we have $\lambda < -\frac{3}{2}$ so $A > B$; $\lambda > -\frac{13}{7}$ so $E > A$; $\lambda < -\frac{5}{3}$ so $D > E$; and $\lambda < -1$ so $D < C$. Thus $B < A < E < D < C$.

At $\lambda = -\frac{5}{3}$ we obtain $D = 13-10 = 3$, $E = 18-15 = 3$, $A = 5-\frac{10}{3} = \frac{5}{3}$, $B = 8 - \frac{20}{3} = \frac{4}{3}$, and $C = 10 - 5 = 5$, whence $B < A < D = E < C$.

If $-\frac{5}{3} < \lambda < -\frac{3}{2}$ then $\lambda < -\frac{3}{2}$ gives $A > B$, $\lambda > -\frac{13}{7}$ gives $E > A$, $\lambda > -\frac{5}{3}$ gives $D < E$, and $\lambda < -1$ gives $D < C$. Hence $B < A < D < E < C$.

At $\lambda = -\frac{3}{2}$ we have $A = B = 2$, $D = 4$, $E = \frac{9}{2}$, and $C = \frac{11}{2}$, so $A = B < D < E < C$.

For $-\frac{3}{2} < \lambda < -\frac{4}{3}$: $\lambda > -\frac{3}{2}$ gives $A < B$; also $B < D$ holds for all λ. Moreover, $\lambda < -\frac{4}{3}$ gives $C > E$; $\lambda < -1$ gives $D < C$; and $\lambda > -\frac{5}{3}$ gives $D < E$. Hence $A < B < D < E < C$.

At $\lambda = -\frac{4}{3}$: $C = 10 - 4 = 6$, $E = 18 - 12 = 6$, $A = 5 - \frac{8}{3} = \frac{7}{3}$, $B = 8 - \frac{16}{3} = \frac{8}{3}$, $D = 13 - 8 = 5$. Thus $A < B < D < C = E$.

For $-\frac{4}{3} < \lambda < -1$: $\lambda > -\frac{4}{3}$ gives $C < E$; $\lambda < -1$ gives $D < C$; and $\lambda > -\frac{5}{3}$ gives $D < E$. Hence $A < B < D < C < E$.

At $\lambda = -1$: $C = 10 - 3 = 7$, $D = 13 - 6 = 7$, $A = 5 - 2 = 3$, $B = 8 - 4 = 4$, $E = 18 - 9 = 9$. Thus $A < B < C = D < E$.

Finally, for $-1 < \lambda < 0$: $\lambda > -1$ gives $C < D$; $\lambda > -\frac{4}{3}$ gives $C < E$; and $\lambda > -\frac{5}{3}$ gives $D < E$. Hence $A < B < C < D < E$.

The above discussion covers all regions and gives the ordering recorded in Table 1.

We next show that all other values of f on S are strictly larger than $m_2(\lambda)$. By symmetry $f(x, y) = f(y, x)$ it is enough to consider $x \le y$. The pairs not yet considered are those with $\max(x, y) \ge 4$ or $x \ge 4$, and we check that in every λ-region the smallest such value exceeds m_2.

For $x = 1$ and $y \ge 4$ we have $\partial_y f(1, y) = 2y + \lambda > 0$ for $y \ge 1$ and $\lambda > -2$, so $y \mapsto f(1, y)$ is strictly increasing and $f(1, y) \ge f(1, 4) = 17+4\lambda$. If $-2 < \lambda < -\frac{13}{7}$ then $f(1, 4) - m_2 = -1 - 5\lambda > 0$ since $\lambda < -\frac{1}{5}$. If $-\frac{13}{7} \le \lambda < -\frac{3}{2}$ then $f(1, 4) - m_2 = 12 + 2\lambda > 0$ since $\lambda > -6$. At $\lambda = -\frac{3}{2}$ we have $f(1, 4) = 11 > 4 = m_2$. If $-\frac{3}{2} < \lambda < 0$ then $f(1, 4) - m_2 = 9 > 0$. Thus $f(1, y) \ge f(1, 4) > m_2(\lambda)$ in all cases.

For $x = 2$ and $y \ge 4$ we have $\partial_y f(2, y) = 2y + 2\lambda > 0$ for $y \ge 2$ and $\lambda > -2$, so $f(2, y) \ge f(2, 4) = 20 + 8\lambda$. If $-2 < \lambda < -\frac{13}{7}$ then $f(2, 4) - m_2 = 2 - \lambda > 0$. If $-\frac{13}{7} \le \lambda < -\frac{3}{2}$ then $f(2, 4) - m_2 = 15 + 6\lambda > 0$ since $\lambda > -\frac{5}{2}$. At $\lambda = -\frac{3}{2}$ we have $f(2, 4) = 8 > 4 = m_2$. If $-\frac{3}{2} < \lambda < 0$ then $f(2, 4) - m_2 = 12 + 4\lambda > 0$ since $\lambda > -3$. Hence

$f(2,y) \geq f(2,4) > m_2(\lambda)$ throughout.

For $x = 3$ and $y \geq 4$ we have $\partial_y f(3,y) = 2y + 3\lambda \geq 8 + 3\lambda > 8 - 6 = 2 > 0$, so $f(3,y) \geq f(3,4) = 25 + 12\lambda$. If $-2 < \lambda < -\frac{13}{7}$ then $f(3,4) - m_2 = 7 + 3\lambda > 0$ since $\lambda > -\frac{7}{3}$. If $-\frac{13}{7} \leq \lambda < -\frac{3}{2}$ then $f(3,4) - m_2 = 20 + 10\lambda > 0$ since $\lambda > -2$. At $\lambda = -\frac{3}{2}$ we have $f(3,4) = 7 > 4 = m_2$. If $-\frac{3}{2} < \lambda < 0$ then $f(3,4) - m_2 = 17 + 8\lambda > 0$ since $\lambda > -2 > -\frac{17}{8}$. Thus $f(3,y) \geq f(3,4) > m_2(\lambda)$ in all cases.

Finally, consider $x \geq 4$ and $y \geq x$. For fixed $x \geq 4$,

$$\partial_y f(x,y) = 2y + \lambda x \geq 2x + \lambda x = x(2+\lambda) > 0,$$

so $f(x,y) \geq f(x,x) = x^2(2+\lambda) \geq 16(2+\lambda)$. If $-2 < \lambda < -\frac{13}{7}$ then $16(2+\lambda) - m_2 = 14 + 7\lambda > 0$ since $\lambda > -2$. If $-\frac{13}{7} \leq \lambda < -\frac{3}{2}$ then $16(2+\lambda) - m_2 = 27 + 14\lambda$, and since $\lambda \geq -\frac{13}{7} > -\frac{27}{14}$ we have $27 + 14\lambda > 27 - 26 = 1 > 0$. At $\lambda = -\frac{3}{2}$ we obtain $16(2+\lambda) = 8 > 4 = m_2$. If $-\frac{3}{2} < \lambda < 0$ then $16(2+\lambda) - m_2 = 24 + 12\lambda > 0$ since $\lambda > -2$. Thus $f(x,y) \geq 16(2+\lambda) > m_2(\lambda)$ throughout.

We conclude that every $(x,y) \in S$ outside the set

$$\{(1,2),(2,1),(2,2),(1,3),(3,1),(2,3),(3,2),(3,3)\}$$

satisfies $f(x,y) > m_2(\lambda)$. Hence neither m_1 nor m_2, nor indeed any of the five smallest values, can be attained outside this set. The values of f at these eight pairs are precisely $\{A,B,C,D,E\}$ (using $f(x,y) = f(y,x)$), and the ordering established above gives Table 1 together with the explicit formulae for m_1 and m_2. ■

Theorem 6. *Let $n \geq 3$ and $\lambda \in \left[-2, -\frac{3}{2}\right]$. Among all unicyclic graphs of order n, the cycle C_n is the unique graph for which the minimum of the variable version of the Euler–Sombor index EU_λ is attained.*

Proof. For $n = 3$, the only unicyclic graph is C_3, so the result is trivial. Assume henceforth that $n \geq 4$.

For $\lambda \in (-2, -\frac{3}{2}]$, Lemma 3 gives

$$m_1 = f(2,2) = 8 + 4\lambda,$$

and every edge type (i,j) with $i+j>4$ satisfies $f(i,j)>m_1$. At the boundary point $\lambda=-\frac{3}{2}$ the type $(1,2)$ also attains $f(1,2)=m_1$, but all other types still satisfy $f(i,j)>m_1$.

Let $m_2(\lambda)$ denote the second smallest value of f on the set of all degree pairs. By Lemma 3,

$$m_2(\lambda)>m_1(\lambda) \quad \text{for all } \lambda\in(-2,-\tfrac{3}{2}],$$

and in particular

$$\varepsilon(\lambda):=\sqrt{m_2(\lambda)}-\sqrt{m_1(\lambda)}>0.$$

Every edge whose type is not $(2,2)$ (and, when $\lambda=-\frac{3}{2}$, also not $(1,2)$) has weight at least $\sqrt{m_2(\lambda)}=\sqrt{m_1(\lambda)}+\varepsilon(\lambda)$.

At the boundary point $\lambda=-\frac{3}{2}$ we also have $f(1,2)=m_1$, so a priori edges of type $(1,2)$ might occur. We claim that in a unicyclic graph G, the presence of an edge of type $(1,2)$ forces another edge of weight strictly greater than $\sqrt{m_1}$. Indeed, let uv be an edge with $d(u)=1$ and $d(v)=2$. Since G is connected and unicyclic, every vertex outside the unique cycle belongs to a tree attached to the cycle. In particular, the vertex v has a neighbour w (distinct from u) on the unique path from v to the cycle. If $d(w)=2$, we continue along this path; it must eventually reach the cycle at a vertex x with $d(x)\geq 3$, and the edge on the cycle incident with x has type $(2,3)$ or $(3,3)$, hence kernel strictly greater than m_1. If instead $d(w)\geq 3$, then vw itself is of type $(2,3)$ or $(2,k)$ with $k\geq 3$ and again has kernel strictly greater than m_1. In either case G contains at least one edge whose kernel exceeds m_1. Thus, even when $\lambda=-\frac{3}{2}$, no edge of type $(1,2)$ can occur in a unicyclic graph all of whose edges have kernel at most m_1.

The cycle C_n is 2-regular, so

$$\mathrm{EU}_\lambda(C_n)=n\sqrt{m_1(\lambda)}=n\sqrt{8+4\lambda}. \tag{10}$$

Now let $G\ncong C_n$ be a unicyclic graph of order n. Then G has at least one edge whose type is not $(2,2)$. If $\lambda\in(-2,-\frac{3}{2})$, such an edge necessarily has weight at least $\sqrt{m_2(\lambda)}=\sqrt{m_1(\lambda)}+\varepsilon(\lambda)$. If $\lambda=-\frac{3}{2}$, we

have just seen that no edge of type $(1,2)$ can occur in a graph attaining the minimum, so G has at least one edge whose type is neither $(2,2)$ nor $(1,2)$, and this edge has weight at least $\sqrt{m_2(\lambda)} = \sqrt{m_1(\lambda)} + \varepsilon(\lambda)$. In all cases, there exists an edge e of G with

$$w(e) \geq \sqrt{m_1(\lambda)} + \varepsilon(\lambda).$$

Every other edge of G has weight at least $\sqrt{m_1(\lambda)}$, so if $E(G) = \{e_1, \ldots, e_n\}$ with $e_1 = e$, then

$$\begin{aligned} \mathrm{EU}_\lambda(G) = \sum_{i=1}^{n} w(e_i) &\geq \big(\sqrt{m_1(\lambda)} + \varepsilon(\lambda)\big) \\ &\quad + (n-1)\sqrt{m_1(\lambda)} = n\sqrt{m_1(\lambda)} + \varepsilon(\lambda). \end{aligned}$$

Combining this with (10) gives

$$\mathrm{EU}_\lambda(G) \geq \mathrm{EU}_\lambda(C_n) + \varepsilon(\lambda) > \mathrm{EU}_\lambda(C_n).$$

For $\lambda = -2$, the kernel becomes $f(x,y) = (x-y)^2$, so $EU_{-2}(G) = \sum_{uv \in E(G)} |d_u - d_v|$. For C_n this sum is 0. Any other unicyclic graph has at least one edge with endpoints of different degrees, hence its index is strictly positive. Thus C_n uniquely attains the minimum also at $\lambda = -2$.

Hence C_n uniquely attains the minimum of EU_λ among unicyclic graphs of order n. ■

It remains to consider the second parameter range given in Lemma 3.

Theorem 7. *Let $n \geq 3$ and $\lambda \in \big(-\frac{3}{2}, 0\big)$. Among all unicyclic graphs of order n, the cycle C_n uniquely attains the minimum of the variable version of the Euler–Sombor index EU_λ.*

Proof. For $n = 3$, the only unicyclic graph is C_3, so the result is trivial. Assume henceforth that $n \geq 4$.

For $\lambda \in (-\frac{3}{2}, 0)$, Lemma 3 gives

$$A := f(1,2) = 5 + 2\lambda, \quad B := f(2,2) = 8 + 4\lambda, \quad D := f(2,3) = 13 + 6\lambda,$$

with

$$A < B < D.$$

Moreover, for every edge type $(i,j) \notin \{(1,2),(2,2)\}$ we have:

$$f(1,k) \geq C := f(1,3) = 10 + 3\lambda > B \quad (k \geq 3),$$

and

$$f(i,j) \geq D > B \quad (i,j \geq 2,\ \max\{i,j\} \geq 3).$$

Thus, edges of type $(1,2)$ have weight $\sqrt{A}$, edges of type $(2,2)$ have weight $\sqrt{B}$, edges of type $(1,k)$ with $k \geq 3$ have weight at least $\sqrt{C}$, and edges with both endpoints of degree at least 2 and at least one of degree ≥ 3 have weight at least $\sqrt{D}$.

We first record the key inequality

$$\sqrt{A} + \sqrt{D} > 2\sqrt{B}. \tag{11}$$

Indeed, squaring twice (all terms are positive) gives

$$4AD - (4B - A - D)^2 = 16(4 - \lambda^2) > 0,$$

since $|\lambda| < 2$.

The cycle C_n is 2-regular, so all its edges are of type $(2,2)$ and

$$\mathrm{EU}_\lambda(C_n) = n\sqrt{B}. \tag{12}$$

Now let $G \ncong C_n$ be a unicyclic graph of order n. Let t be the number of edges of type $(1,2)$ in G. For each such edge, consider the maximal path starting at its leaf endpoint. Since G is not a cycle, this path must end at a vertex of degree at least 3, and the first edge incident with that vertex on this path has type $(2,k)$ with $k \geq 3$. These paths are edge-disjoint, so the corresponding heavy edges of type $(2,k)$ are distinct. Thus G contains at least t edges of type $(2,k)$ with $k \geq 3$; call these the special heavy edges.

We now split into two cases.

Case 1: $t > 0$**.** Let $h_{\mathrm{sp}} = t$ be the number of special heavy edges. Every

other edge of G either is of type $(2,2)$ (weight $\sqrt{B}$) or is a non-special heavy edge (weight at least $\sqrt{B}$, since B is the smallest weight after A). Therefore

$$\mathrm{EU}_\lambda(G) \geq t\sqrt{A} + (n-2t)\sqrt{B} + t\sqrt{D}.$$

Subtracting (12) gives

$$\mathrm{EU}_\lambda(G) - \mathrm{EU}_\lambda(C_n) \geq t(\sqrt{A} + \sqrt{D} - 2\sqrt{B}) > 0,$$

by (11). Hence $\mathrm{EU}_\lambda(G) > \mathrm{EU}_\lambda(C_n)$.

Case 2: $t = 0$**.** Then G has no edges of type $(1,2)$. Since $G \ncong C_n$, it has at least one vertex of degree ≥ 3, so at least one edge of G has weight strictly greater than $\sqrt{B}$: indeed, any edge incident with a vertex of degree ≥ 3 is either of type $(1,k)$ with $k \geq 3$ (weight $\geq \sqrt{C} > \sqrt{B}$) or of type (i,j) with $i,j \geq 2$ and $\max\{i,j\} \geq 3$ (weight $\geq \sqrt{D} > \sqrt{B}$). All edges of G have weight at least $\sqrt{B}$, because the only type with weight below $\sqrt{B}$ is $(1,2)$, and $t = 0$. Therefore

$$\mathrm{EU}_\lambda(G) > n\sqrt{B} = \mathrm{EU}_\lambda(C_n).$$

Thus in all cases, for every $G \ncong C_n$, we have $\mathrm{EU}_\lambda(G) > \mathrm{EU}_\lambda(C_n)$. Therefore C_n uniquely attains the minimum. ■

5 Discussion

Methodological trade-offs. The classification lemma used for unicyclic graphs could in principle be applied to trees, but would require splitting λ into many more subintervals, making the analysis considerably longer. Conversely, the algebraic method used for trees would be significantly more involved for unicyclic graphs. Each approach has its advantages: the classification method is more systematic and has the potential to determine the second minimum as well, while the algebraic method is more efficient for trees. Determining the second minimum for trees and unicyclic graphs remains an open problem.

Negative parameter regime: obstacles and open problems. We

did not extend our analysis to bicyclic or k-cyclic graphs ($k \geq 3$) in the range $\lambda \in [-2, 0)$. The main obstacle is that both available methods would require a substantial increase in the number of cases, making the proofs impractically long. Characterizing the minimum graphs in these classes for $\lambda \in [-2, 0)$ remains open. It would also be interesting to study the variable Euler–Sombor index on other graph classes, such as chemical graphs or graphs with bounded maximum degree.

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