MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **97** (2027) 737–756
ISSN: 0340–6253
doi: 10.46793/match.97-2.16426

Euler Hyperbolic Sombor Index

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(Received May 5, 2026)

Abstract

In this paper, we introduce the Euler hyperbolic Sombor index and investigate its fundamental properties across various classes of graphs. We first establish general lower and upper bounds for this index and examine its relationships with several well-known topological indices, including the Euler Sombor index, the geometric-arithmetic index, the first Zagreb index, and the hyperbolic Sombor index. Furthermore, explicit formulas and extremal results are obtained for particular graph classes such as paths, cycles, stars, complete graphs, and trees, and the corresponding equality cases are characterized. In the application part, we present general combinatorial formulas for computing this index for benzenoid systems and phenylenes.

1 Introduction

Throughout the present paper, we use the following notation. By G is denoted a simple graph with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$, with

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$n = |\mathbf{V}(G)|$ vertices and $m = |\mathbf{E}(G)|$ edges. The edge connecting the vertices u and v is denoted by uv. The degree of the vertex $u \in V(G)$ is denoted by $d(u)$. The maximal and minimal vertex degree of G are Δ and δ, respectively.

The number of edges connecting a vertex of degree i with a vertex of degree j is denoted by m_{ij}. Thus, for a graph with n vertices and m edges,

$$\sum_{1 \le i \le j \le n-1} m_{ij} = m \, .$$

For additional concepts and definitions from graph theory and chemical graph theory, the readers are referred to the textbooks [4, 29–31, 33, 35].

In the recent and current literature, a large class of graph invariants is being considered, referred to as "vertex-degree-based" (VDB) or "bond incident degree" (BID) topological indices, defined as

$$TI = TI(G) = \sum_{uv \in \mathbf{E}(G)} f\big(d(u), d(v)\big) \tag{1}$$

where $f(x, y)$ is some properly chosen function, satisfying the condition $f(x, y) = f(y, x)$. The topological index defined via Eq. (1), can be calculated also as

$$TI = TI(G) = \sum_{1 \le i \le j \le n-1} m_{ij} \, f(i, j) \, . \tag{2}$$

If $f(x, y) = \sqrt{x^2 + y^2}$, then the respective graph invariant

$$SO = SO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{d(u)^2 + d(v)^2}$$

was named *Sombor index* [12]. It is worth noting that it has a geometry-based background. Its mathematical properties are studied in due detail, and numerous of its applications (in chemistry, pharmacology and elsewhere) were reported; see the surveys [21, 25] and the paper [26].

Since 2021, when the concept of Sombor index was conceived [12], a remarkably large number of its modifications were proposed and con-

sidered, see [13]. Among these, we mention here the *Euler Sombor index* [1, 15, 16, 28]

$$ESO = ESO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)\, d(v)} \tag{3}$$

and the *hyperbolic Sombor index* [2, 3, 6, 19, 22]

$$HSO = HSO(G) = \sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d(u)^2 + d(v)^2}}{\min\{d(u), d(v)\}} \,. \tag{4}$$

Within the below considerations, we shall need the following four well known and much studied BID indices, the *first Zagreb index* [18, 29, 30]

$$M_1 = M_1(G) = \sum_{u \in \mathbf{V}(G)} d(u)^2 = \sum_{uv \in \mathbf{E}(G)} \big[d(u) + d(v)\big]$$

the *second Zagreb index* [17, 29, 30]

$$M_2 = M_2(G) = \sum_{uv \in \mathbf{E}(G)} d(u)\, d(v)$$

the *forgotten topological index* [9, 29, 30]

$$F = F(G) = \sum_{u \in \mathbf{V}(G)} d(u)^3 = \sum_{uv \in \mathbf{E}(G)} \big[d(u)^2 + d(v)^2\big]$$

and the *geometric-arithmetic index* [29, 30, 32]

$$GA = GA(G) = \sum_{uv \in \mathbf{E}(G)} \frac{2\sqrt{d(u)\, d(v)}}{d(u) + d(v)} \,.$$

Comparing the definitions of the Euler and hyperbolic Sombor indices, Eqs. (3) and (4), and examining the list of all hitherto existing Sombor-type indices [13], we realized that their combination, namely

$$EHSO = EHSO(G) = \sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d(u)^2 + d(v)^2 + d(u)\, d(v)}}{\min\{d(u), d(v)\}} \tag{5}$$

has not been studied so far. It is appropriate to name the BID index $EHSO$, defined via Eq. (5), as the *Euler hyperbolic Sombor index*.

In the present paper we establish the main mathematical properties of this new member of the Sombor-index family, and report some of its possible chemical applications.

In what follows, for each edge $uv \in \mathbf{E}(G)$, it will be assumed that $d(u) \leq d(v)$. Thus, without loss of generality, $\min\{d(u), d(v)\} = d(u)$ will be taken unless otherwise stated.

2 Main results

In the following theorem, we obtain a lower bound for the Euler hyperbolic Sombor index with respect to the size of a graph.

Theorem 1. *Let G be a simple and connected graph with m edges. Then,*

$$EHSO(G) \geq \sqrt{3}\, m\,.$$

Equality holds if G is a regular graph.

Proof. It is a known fact that

$$(x-y)^2 = x^2 - 2xy + y^2 \geq 0\,.$$

Then,

$$\begin{aligned} & x^2 + y^2 \geq 2xy \Rightarrow x^2 + y^2 + xy \geq 3xy \\ \Rightarrow \quad & \sqrt{\frac{x^2+y^2+xy}{x^2}} \geq \sqrt{\frac{3xy}{x^2}} = \sqrt{3}\sqrt{\frac{y}{x}} \\ \Rightarrow \quad & \frac{\sqrt{x^2+y^2+xy}}{x} \geq \sqrt{3}\sqrt{\frac{y}{x}}\,. \end{aligned}$$

If we consider that $\min\{x, y\} = x$ and also x, y are positive integers, we get

$$\frac{\sqrt{x^2+y^2+xy}}{x} \geq \sqrt{3}\sqrt{\frac{y}{x}} \geq \sqrt{3}.$$

Now, by taking the positive integers $d(u)$ and $d(v)$ instead of x and y, respectively, we get

$$\begin{aligned} EHSO(G) &= \sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d(u)^2+d(v)^2+d(u)\,d(v)}}{d(u)} \\ &\geq \sum_{uv\in\mathbf{E}(G)} \sqrt{3} = \underbrace{\sqrt{3}+\sqrt{3}+\cdots+\sqrt{3}}_{m \text{ times}} = \sqrt{3}\,m\,. \end{aligned}$$

■

In the next theorem, we provide an upper and lower bound for the Euler hyperbolic Sombor index in terms of Euler Sombor index.

Theorem 2. *Let G be a simple connected graph. Then*

$$\frac{1}{\Delta} ESO(G) \leq EHSO(G) \leq \frac{1}{\delta} ESO(G)\,. \tag{6}$$

Equality on both sides holds if G is a regular graph.

Proof. By the definitions of δ and Δ, it is clear that $\delta \leq d(u) \leq \Delta$ for any $u \in \mathbf{V}(G)$. Therefore, $\frac{1}{\Delta} \leq \frac{1}{d(u)} \leq \frac{1}{\delta}$ and thus,

$$\begin{aligned} \frac{1}{\Delta}\sqrt{d(u)^2+d(v)^2+d(u)\,d(v)} &\leq \frac{1}{d(u)}\sqrt{d(u)^2+d(v)^2+d(u)\,d(v)} \\ &\leq \frac{1}{\delta}\sqrt{d(u)^2+d(v)^2+d(u)\,d(v)}\,. \end{aligned}$$

Summing over all edges, we obtain

$$\begin{aligned} &\frac{1}{\Delta}\sum_{uv\in\mathbf{E}(G)} \sqrt{d(u)^2+d(v)^2+d(u)\,d(v)} \leq EHSO(G) \\ \leq\ &\frac{1}{\delta}\sum_{uv\in\mathbf{E}(G)} \sqrt{d(u)^2+d(v)^2+d(u)\,d(v)} \end{aligned}$$

which is same as Eq. (6).

Since $\Delta = d(u) = \delta$ for a regular graph, equality holds in that case. ■

We now present an upper bound for the Euler hyperbolic Sombor index via the first Zagreb and geometric-arithmetic indices.

Theorem 3. *Let G be a simple connected graph. Then,*

$$\frac{\sqrt{3}}{2}\, GA(G) \leq EHSO(G) \leq \frac{1}{\delta} M_1(G)$$

where GA and M_1 stand for the geometric-arithmetic and first Zagreb topological indices.

Proof. For all positive values of x and y, we have

$$3xy \leq x^2 + y^2 + xy < x^2 + y^2 + 2xy = (x+y)^2 .$$

Also, the inequality $1/d(u) \leq 1/\delta$ is evident.

If we take $d(u)$ and $d(v)$ instead of x and y, respectively, we obtain the following statements from these inequalities.

$$\frac{3d(u)\, d(v)}{d(u)+d(v)} \leq \frac{d(u)^2 + d(v)^2 + d(u)\, d(v)}{d(u)+d(v)} < \frac{(d(u)+d(v))^2}{\delta} .$$

So, we obtain

$$\begin{aligned} \frac{\sqrt{3}}{2} \sum_{uv \in \mathbf{E}(G)} \frac{2\sqrt{d(u)\, d(v)}}{d(u)+d(v)} &\leq \sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d(u)^2 + d(v)^2 + d(u)\, d(v)}}{d(u)} \\ &\leq \frac{1}{\delta} \sum_{uv \in \mathbf{E}(G)} (d(u)+d(v)) . \end{aligned}$$

Thus,

$$\frac{\sqrt{3}}{2} GA(G) \leq EHSO(G) \leq \frac{1}{\delta} M_1(G) .$$

Here note that

$$(x-y)^2 \geq 0 \Rightarrow x^2 + y^2 - 2xy \geq 0 \Rightarrow 2xy \leq x^2 + y^2 \Rightarrow 3xy \leq x^2 + y^2 + xy .$$

■

Theorem 4. *Let G be a simple connected graph, then*

$$EHSO(G) \leq \sqrt{\frac{3}{2}}\, HSO(G)\,. \tag{7}$$

Equality holds when G is regular.

Proof. For any positive integer x and y, it is clear that

$$\begin{aligned}
(x-y)^2 &= x^2+y^2-2xy \geq 0 \\
&\Rightarrow\ 2xy \leq x^2+y^2 \\
&\Rightarrow\ xy \leq \frac{1}{2}(x^2+y^2) \\
&\Rightarrow\ x^2+y^2+xy \leq (x^2+y^2)+\frac{1}{2}(x^2+y^2) = \frac{3}{2}(x^2+y^2) \\
&\Rightarrow\ \frac{\sqrt{x^2+y^2+xy}}{x} \leq \frac{\sqrt{\frac{3}{2}}\sqrt{x^2+y^2}}{x}\,.
\end{aligned}$$

Now, taking $x = d(u)$ and $y = d(v)$ we get

$$\sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{\min\{d(u)\,d(v)\}} \leq \sqrt{\frac{3}{2}} \sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d^2(u)+d^2(v)}}{\min\{d(u)\,d(v)\}}$$

implying Eq. (7).

■

Theorem 5. *Let P_n, C_n, S_n, and K_n denote, respectively, the path graph, cycle graph, star graph, and complete graph of order n. Then for $n \geq 2$,*

$$\begin{aligned}
EHSO(P_n) &= 2\sqrt{7}+(n-3)\sqrt{3} \\
EHSO(C_n) &= \sqrt{3}\,n \\
EHSO(S_n) &= (n-1)\sqrt{n^2-n+1} \\
EHSO(K_n) &= \frac{\sqrt{3}}{2}\,n(n-1)\,.
\end{aligned}$$

Proof. We first note that by Theorem 1, $EHSO(G) = \sqrt{3}\,m$ for every regular graph with m vertices. Then the second and fourth equality result

from the fact that C_n and K_n are regular graphs with n and $n(n-1)/2$ edges.

For the path graph P_n, it is known that $|\mathbf{V}(P_n)| = n$ and $|\mathbf{E}(P_n)| = n-1$. Let $\mathbf{V} = \{u_1, u_2, \ldots, u_n\}$. Then $d(u_1) = d(u_n) = 1$ and $d(u_2) = d(u_3) = \cdots = d(u_{n-1}) = 2$. It follows that there are exactly 2 edges connecting the vertices of degree 1 and 2, and there are exactly $(n-3)$ edges connecting the vertices of $E_{2,2}$. Therefore,

$$\begin{aligned} EHSO(P_n) &= \sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d^2(u) + d^2(v) + d(u)\,d(v)}}{d(u)} \\ &= 2 \cdot \frac{\sqrt{1^2 + 2^2 + 1 \cdot 2}}{1} + (n-3) \cdot \frac{\sqrt{2^2 + 2^2 + 2 \cdot 2}}{2} \\ &= 2\sqrt{7} + (n-3)\sqrt{3}\,. \end{aligned}$$

Finally, for the star graph S_n, $|\mathbf{V}(S_n)| = n$, $|\mathbf{E}(S_n)| = n-1$. One of the vertices has degree $n-1$ and the remaining vertices have degree 1. Therefore, $m(S_n) = m_{1,n-1}$ and by Eq. (2),

$$\begin{aligned} EHSO(S_n) &= m(S_n)\,\frac{\sqrt{1^2 + (n-1)^2 + 1 \cdot (n-1)}}{1} \\ &= (n-1)\sqrt{n^2 - n + 1}\,. \end{aligned}$$

■

At this point we note that the complete bipartite graph $K_{a,b}$ has ab edges, all connecting vertices of degree a and b. Therefore, if $a \leq b$, then

$$EHSO(K_{a,b}) = ab\,\frac{\sqrt{a^2 + b^2 + ab}}{a} = b\,\sqrt{a^2 + b^2 + ab}\,.$$

The formula for $EHSO(S_n)$ stated in Theorem 5 is a special case for $a = 1$, $b = n-1$.

Theorem 6. *Let G be a simple graph with m edges. Then*

$$EHSO(G) \leq \frac{\sqrt{m}}{\delta}\sqrt{F(G) + M_2(G)}$$

where F and M_2 are the forgotten and second Zagreb indices, respectively. Equality holds if G is a regular graph.

Proof. As $\delta \le d(u) \le \Delta$, we have

$$\frac{1}{\Delta^2} \le \frac{1}{d^2(u)} \le \frac{1}{\delta^2}$$

and using the Cauchy–Schwarz inequality,

$$\begin{aligned}
EHSO(G) &= \sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \\
&\le \sqrt{\sum_{uv\in\mathbf{E}(G)} 1^2} \sqrt{\sum_{uv\in\mathbf{E}(G)} \frac{d^2(u)+d^2(v)+d(u)\,d(v)}{d^2(u)}} \\
&= \sqrt{m\left(\sum_{uv\in\mathbf{E}(G)} \frac{d^2(u)+d^2(v)}{d^2(u)} + \sum_{uv\in\mathbf{E}(G)} \frac{d(u)\,d(v)}{d^2(u)}\right)} \\
&\le \sqrt{\frac{m}{\delta^2}\left(\sum_{uv\in\mathbf{E}(G)} \big(d^2(u)+d^2(v)\big) + \sum_{uv\in\mathbf{E}(G)} d(u)\,d(v)\right)} \\
&= \frac{\sqrt{m}}{\delta}\sqrt{F(G)+M_2(G)}\,.
\end{aligned}$$

■

Lemma 1. *Let G be a graph and Δ its maximal vertex degree. Then,*

(i) For any pendent edge $uv \in \mathbf{E}(G)$,

$$\sqrt{7} \le \sqrt{d^2(u)+d^2(v)+d(u)\,d(v)} \le \sqrt{\Delta^2+\Delta+1}\,.$$

The equality on the left-hand side holds whenever $d(u)=1$, $d(v)=2$, and the equality on the right-hand side holds whenever $d(u)=1$, $d(v)=\Delta$.

(ii) For any non-pendent edge $uv \in \mathbf{E}(G)$,

$$\sqrt{3} \leq \frac{\sqrt{d^2(u) + d^2(v) + d(u)\,d(v)}}{d(u)} \leq \frac{\sqrt{\Delta^2 + 2\Delta + 4}}{2}.$$

The equality on the left-hand side holds whenever $d(u) = d(v)$, *and the equality on the right-hand side holds whenever* $d(u) = 2$, $d(v) = \Delta$.

Corollary 1. *Let* G *be a graph of order* $n \geq 2$. *Then,*

(i) For any pendent edge $uv \in \mathbf{E}(G)$, *we get*

$$\sqrt{7} \leq \sqrt{d^2(v) + d(v) + 1} \leq \sqrt{n^2 - n + 1}\,.$$

The equality on the left-hand side holds whenever $d(u) = 1$, $d(v) = 2$, *and the equality on the right-hand side holds whenever* $d(u) = 1$, $d(v) = n - 1$.

(ii) For any non-pendent edge $uv \in \mathbf{E}(G)$, *we get*

$$\sqrt{3} \leq \frac{\sqrt{d^2(u) + d^2(v) + d(u)\,d(v)}}{d(u)} \leq \frac{\sqrt{n^2 + 3}}{2}\,.$$

The equality on the left-hand side holds whenever $d(u) = d(v)$, *and the equality on the right-hand side holds whenever* $d(u) = 2$, $d(v) = n - 1$.

Theorem 7. *Let* T_n *be a tree of order* $n \geq 2$. *Then,*

$$2\sqrt{7} + (n-3)\sqrt{3} \leq EHSO(T_n) \leq (n-1)\sqrt{n^2 - n + 1}\,.$$

The equalities hold if and only if $T_n \cong P_n$ *(for left) and* $T_n \cong S_n$ *(for right), respectively.*

Proof. Let uv be an arbitrary edge in T_n. For $n \geq 2$, it is clear that

$$\begin{aligned} & (3n-4)(n-1) \geq 0 \Rightarrow 3n^2 - 4n + 1 \geq 0 \\ \Rightarrow \quad & n^2 + 3 \leq 4n^2 - 4n + 4 \Rightarrow \frac{n^2+3}{4} \leq n^2 - n + 1 \\ \Rightarrow \quad & \frac{\sqrt{n^2+3}}{2} \leq \sqrt{n^2 - n + 1}\,. \end{aligned}$$

According to this, by Corollary 1, for any edge of T_n

$$\frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \leq \sqrt{n^2-n+1}\,.$$

It follows that

$$EHSO(T_n) = \sum_{uv\in\mathbf{E}(T_n)} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \leq (n-1)\sqrt{n^2-n+1}$$

and, the equality holds here if and only if $d(u)=1$, $d(v)=n-1$ for any edge in T_n, that is, $T_n \cong S_n$.

Now, let w be the number of pendent vertices in T_n, and $\mathbf{E}_p$ the set of pendent edges. Then, there exist $(n-1-w)$ non-pendent edges in T_n. By Corollary 1, we have

$$\begin{aligned} EHSO(T_n) &= \sum_{uv\in\mathbf{E}_p(T_n)} \sqrt{d^2(v)+d(v)+1} \\ &+ \sum_{uv\in(\mathbf{E}(T_n)\setminus\mathbf{E}_p(T_n))} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \\ &\geq w\sqrt{7}+(n-w-1)\sqrt{3} \\ &= (n-1)\sqrt{3}+w(\sqrt{7}-\sqrt{3}); \quad w\geq 2 \\ &\geq (n-1)\sqrt{3}+2(\sqrt{7}-\sqrt{3}) = 2\sqrt{7}+(n-3)\sqrt{3}\,. \end{aligned}$$

Consequently,

$$2\sqrt{7}+(n-3)\sqrt{3} \leq EHSO(T_n) \leq (n-1)\sqrt{n^2-n+1}$$

is obtained. Note that the equality holds on the left if and only if $T_n \cong P_n$. ■

Theorem 8. *Let G be a connected graph with $|\mathbf{V}(G)|=n$ and $|\mathbf{E}(G)|=m$. Then*

$$EHSO(G) \geq \sqrt{3}\,n\,.$$

Moreover, the equality holds for $G \cong C_n$, where C_n is the cycle graph.

Proof. In this case $m = n - 1$ or $m \geq n$.

Case 1: Let $m = n - 1$, then G is a tree. By Theorem 7,

$$EHSO(G) = EHSO(T_n) \geq 2\sqrt{7}+(n-3)\sqrt{3} = \sqrt{28}-\sqrt{27}+\sqrt{3}\,n > \sqrt{3}\,n\,.$$

Case 2: Let $m \geq n$. By Theorem 1, we get

$$EHSO(G) \geq \sqrt{3}\,m \geq \sqrt{3}\,n\,.$$

Moreover, equality will hold for $G \cong C_n$, since $m = n$ and G is regular. ■

In the following theorem we obtain upper and lower bounds for Euler hyperbolic Sombor index in terms of maximum vertex degree (Δ) and minimum vertex degree (δ).

Theorem 9. *Let G be a graph with $|\mathbf{E}(G)| = m$, and Δ and δ represent, respectively, the maximum and minimum degrees of its vertices. Then,*

$$\left(1 + \frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}\right) m \leq EHSO(G) \leq \left(\frac{2\Delta + (\Delta+\delta)(\sqrt{3}-1)}{2\delta}\right) m\,.$$

Equalities on both sides hold whenever G is a regular graph.

Proof. Without loss of generality, assume that $d(u) \leq d(v)$. It is evident that $1 \leq d(v)/d(u) \leq \Delta/\delta$. Then it follows that

$$\begin{aligned} \sqrt{3}+1 &\leq \sqrt{\frac{d^2(v)}{d^2(u)} + \frac{d(v)}{d(u)} + 1} + \frac{d(v)}{d(u)} \leq \sqrt{\frac{\Delta^2}{\delta^2} + \frac{\Delta}{\delta} + 1} + \frac{\Delta}{\delta} \\ &= \frac{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}{\delta} \end{aligned} \tag{8}$$

From inequality (8), we get

$$\sqrt{\frac{d^2(v)}{d^2(u)} + \frac{d(v)}{d(u)} + 1} - \frac{d(v)}{d(u)} = \frac{\frac{d(v)}{d(u)} + 1}{\sqrt{\frac{d^2(v)}{d^2(u)} + \frac{d(v)}{d(u)} + 1} + \frac{d(v)}{d(u)}}$$

$$\geq \frac{2}{\sqrt{\frac{d^2(v)}{d^2(u)}+\frac{d(v)}{d(u)}+1}+\frac{d(v)}{d(u)}} \geq \frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}$$

$$\Rightarrow \sqrt{\frac{d^2(v)}{d^2(u)}+\frac{d(v)}{d(u)}+1} \geq \frac{d(v)}{d(u)}+\frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}$$

$$\geq 1+\frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}\,.$$

Finally we get the lower bound such that:

$$\begin{aligned} EHSO(G) &= \sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \\ &\geq \sum_{uv\in\mathbf{E}(G)} \left(1+\frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}\right) \\ &= \left(1+\frac{2\delta}{\sqrt{\Delta^2+\delta^2+\Delta\delta}+\Delta}\right) m\,. \end{aligned}$$

The equality is clear whenever G is regular.

Now, we use the inequality (8) to get the upper bound with respect to $\{\Delta,\delta\}$.

$$\begin{aligned} \sqrt{\frac{d^2(v)}{d^2(u)}+\frac{d(v)}{d(u)}+1}-\frac{d(v)}{d(u)} &= \frac{\frac{d(v)}{d(u)}+1}{\sqrt{\frac{d^2(v)}{d^2(u)}+\frac{d(v)}{d(u)}+1}+\frac{d(v)}{d(u)}} \leq \frac{\frac{\Delta}{\delta}+1}{\sqrt{3}+1} \\ &= \frac{(\Delta+\delta)(\sqrt{3}-1)}{2\delta}\,. \end{aligned}$$

From this, we have

$$\sqrt{\frac{d^2(v)}{d^2(u)}+\frac{d(v)}{d(u)}+1} \leq \frac{d(v)}{d(u)}+\frac{(\Delta+\delta)(\sqrt{3}-1)}{2\delta}$$

$$\leq \frac{\Delta}{\delta}+\frac{(\Delta+\delta)(\sqrt{3}-1)}{2\delta} = \frac{2\Delta+(\Delta+\delta)(\sqrt{3}-1)}{2\delta}\,.$$

Hence,

$$\begin{aligned} EHSO(G) &= \sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d^2(u)+d^2(v)+d(u)\,d(v)}}{d(u)} \\ &\leq \sum_{uv\in\mathbf{E}(G)} \left(\frac{2\Delta+(\Delta+\delta)(\sqrt{3}-1)}{2\delta}\right) \\ &= \left(\frac{2\Delta+(\Delta+\delta)(\sqrt{3}-1)}{2\delta}\right) m\,. \end{aligned}$$

And the equality holds for a regular graph. ■

3 Applications

Benzenoid and phenylene compounds hold considerable importance across chemistry, materials science, and environmental protection. Their topological properties were extensively studied [8, 10, 11, 14]. The respective molecular graphs of these structures are depicted in Fig. 1.

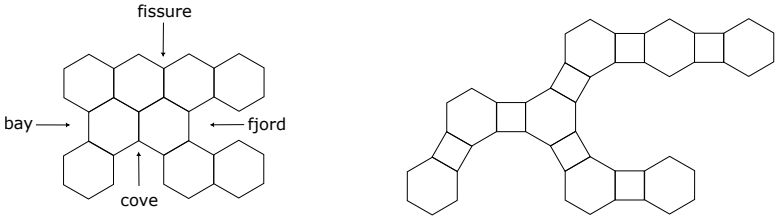

Figure 1. Left: A benzenoid graph B with structural features along its perimeter. In this example, B contains four fissures, one bay, one cove, and one fjord, giving a total number of inlets $r = 4 + 1 + 1 + 1 = 7$. Right: A phenylene graph PH.

As observed, these graphs contain only vertices of degree 2 and degree 3. Consequently, all their edges are of type $(2, 2)$, $(2, 3)$, or $(3, 3)$. Hence, in line with Eq. (2), the $EHSO$ for this class of molecular graphs takes

the following simple form (cf. Refs. [5, 20, 27]):

$$EHSO(B) = EHSO(PH) = m_{2,2}\,F(2,2) + m_{2,3}\,F(2,3) + m_{3,3}\,F(3,3)\,. \tag{9}$$

The values of $F(2,2)$, $F(2,3)$, and $F(3,3)$ follow immediately from the definition of $EHSO$:

$$F(2,2) = \frac{\sqrt{2^2+2^2+2\cdot 2}}{2} = \frac{2\sqrt{3}}{2}$$
$$F(2,3) = \frac{\sqrt{2^2+3^2+2\cdot 3}}{2} = \frac{\sqrt{19}}{2}$$
$$F(3,3) = \frac{\sqrt{3^2+3^2+3\cdot 3}}{3} = \frac{3\sqrt{3}}{3}\,.$$

The main topological features of the molecular graphs of benzenoids and phenylenes are determined by the three structural parameters: number of vertices (n), number of hexagons (h), and number of inlets (r). The number of inlets is the total number of structural features on the perimeter, shown in Fig. 1.

The number of edges of the type (2,2), (2,3), and (3,3) in a benzenoid graph B with n vertices, h hexagons, and r inlets is [24]:

$$m_{2,2} = n - 2h - r + 2 \tag{10}$$
$$m_{2,3} = 2r \tag{11}$$
$$m_{3,3} = 3h - r - 3\,. \tag{12}$$

Let n_i be the number of internal vertices of B, i.e., the number of vertices not lying on the perimeter. Then [10, 14]:

$$n = 4h - n_i + 2 \tag{13}$$

By incorporating Eqs. (10)–(13) into Eq. (9), we get:

$$\begin{aligned} EHSO(B) &= F(2,2)(2h - n_i - r + 4) + F(2,3)2r + F(3,3)(3h - r - 3) \\ &= \frac{2\sqrt{3}}{2}(2h - n_i - r + 4) + \frac{2\sqrt{19}}{2}r + \frac{3\sqrt{3}}{3}(3h - r - 3) \end{aligned}$$

$$= 5\sqrt{3}h + (\sqrt{19} - 2\sqrt{3})r - \sqrt{3}n_i + \sqrt{3}$$
$$= 8.660h + 0.895r - 1.732n_i + 1.732\,.$$

The number of edges of type the (2,2), (2,3), and (3,3) in a phenylene graph PH with h hexagons and r inlets is [24]:

$$m_{2,2} = 2h - r + 4 \tag{14}$$
$$m_{2,3} = 2r \tag{15}$$
$$m_{3,3} = 6h - r - 6\,. \tag{16}$$

By incorporating Eqs. (14), (15), and (16) into Eq. (9), we get:

$$EHSO(PH) = F(2,2)(2h - r + 4) + F(2,3)2r + F(3,3)(6h - r - 6)$$
$$= \frac{2\sqrt{3}}{2}(2h - r + 4) + \frac{2\sqrt{19}}{2}r + \frac{3\sqrt{3}}{3}(6h - r - 6)$$
$$= 8\sqrt{3}h + (\sqrt{19} - 2\sqrt{3})r - 2\sqrt{3}$$
$$= 13.856h + 0.859r - 3.464\,.$$

The above formulas enable straightforward calculation of $EHSO(B)$ and $EHSO(PH)$ of any benzenoid or phenylene species, by counting the simple structural details h, r, n_i. In the same time, these formulas reveal the dependence of $EHSO$ on the structural features of the underlying molecule. Notably, the dependence on r is identical in both cases.

4 Concluding remarks and open problems

In this paper a new Sombor-index-type graph invariant is introduced, named *Euler hyperbolic Sombor index*, $EHSO$, defined via Eq. (5). The basic mathematical properties of $EHSO$ are determined, and some of its possible chemical applications indicated. Comparing $EHSO$ with the Euler Sombor index, Eq. (3) and the hyperbolic Sombor index, Eq. (4), we see that

$$HSO(G) < EHSO(G) < ESO(G)$$

i.e., that the Euler hyperbolic Sombor index is a kind of medium between Euler and hyperbolic Sombor indices. This observation makes the Euler hyperbolic Sombor index a promising tool for further research in both graph theory and mathematical chemistry. Anyway, the relations between $EHSO$ and HSO, as well as between $EHSO$ and ESO, would need to be further investigated. At the first glance, it seems that the applicative properties of $EHSO$ should be similar to those of HSO and ESO, but this also awaits to be examined in due detail.

There remain several additional directions for future research of Euler hyperbolic Sombor index. One such is the study of this index for other classes of graphs, such as unicyclic, bicyclic, or more complex molecular structures. In this paper, we established relations between $EHSO$ and some well-known topological indices, Theorems 3 and 6. However, it is certain that many more such relations exist, and these deserve to be established in the future, see [7, 19, 23, 34]. In this paper, the question which n-vertex graph has maximum $EHSO$-value is also unanswered.

We believe that our results demonstrate that the Euler hyperbolic Sombor index may serve as a useful descriptor in both theoretical graph studies and in the topological analysis of molecular structures, and that it deserves additional investigation. Our results, at least, present a contribution to the emerging theory of Sombor-type graph invariants.

***Acknowledgment*:** Izudin Redžepović acknowledges financial support from the Serbian Ministry of Science, Technological Development, and Innovation (Agreement No. 451-03-34/2026-03/200252), as well as from the State University of Novi Pazar.

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