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Maximum Sombor Index and Spectral Radius of Hypertrees

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Abstract

In the framework of complex network analysis, hypergraphs provide a natural generalization for modeling higher-order interactions. In this work, we investigate the extremal structural properties of uniform hypertrees with respect to the Sombor index and the Sombor spectral radius, two degree-based measures that capture nonlinear connectivity patterns. For hypertrees of fixed size, we identify the structures that maximize and second-maximize these descriptors. We show that both the Sombor index and the Sombor spectral radius increase strictly under an edge-releasing operation applied to non-pendant hyperedges, revealing a monotonic structural transformation principle. This result enables us to characterize the hyperstar configuration as the unique maximizer of both measures. Furthermore, by systematically employing edge-moving and edge-releasing operations, we determine the hypertree structure that attains the second-highest values of these indices. Our findings contribute to the understanding of how local structural modifications influence global spectral and topological descriptors in higher-order networks, offering insights relevant to the study of nonlinear and complex systems.

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1 Introduction

Degree-based topological indices play a central role in understanding the structural properties of discrete systems, and they have been extensively studied in both graph theory and its applications. Among these, the Sombor index has recently emerged as a notable invariant, attracting considerable attention due to its sensitivity to vertex degrees and its strong correlation with various structural characteristics. Alongside this, spectral parameters associated with such indices, particularly the Sombor spectral radius, provide deeper insight into the interplay between combinatorial structure and eigenvalue behavior.

In parallel, hypergraphs have gained prominence as a natural generalization of graphs for modeling higher-order interactions. This has motivated the extension of numerous graph invariants and spectral notions to the hypergraph setting. However, compared to graphs, the study of extremal problems for degree-based indices and their spectral counterparts (using hypermatrices) in hypergraphs remains relatively underdeveloped. Extremal characterization problems are of particular importance, as they reveal the structural configurations that maximize or minimize a given invariant. While such problems have been widely investigated for graphs, their hypergraph analogues often require new ideas and techniques due to the increased combinatorial complexity.

Motivated by these considerations, the present work focuses on the characterization of extremal hypertrees with respect to the Sombor index and the Sombor spectral radius. We identify the structures that attain extremal values and establish corresponding bounds, thereby contributing to a deeper understanding of degree-based invariants and spectral properties in hypergraphs through hypermatrix.

1.1 Preliminaries: hypermatrix

Hypermatrix of order h (sometimes, it is also called a tensor) is a generalization a matrix is a multi-dimensional array, which uses h indices to represent each element of the array uniquely. The dimension of an order h hypermatrix $\mathcal{M}$ is usually represented as $n_1 \times \ldots \times n_h$, and the ele-

ments of $\mathcal{M}$ are denoted using h indices, that is $\mathcal{M} = [\mathcal{M}_{j_1 \ldots j_h}]$, where $1 \leq j_i \leq n_i, i = 1, \ldots, h$. If $n_1 = \cdots = n_h = n$, then $\mathcal{M}$ is called a hypermatrix of order h and dimension n. An order h, dimension n hypermatrix $\mathcal{M}$ is said to be symmetric if $\mathcal{M}_{j_1 \ldots j_h} = \mathcal{M}_{j_{\sigma(1)} \cdots j_{\sigma(h)}}$, for any permutation σ of the symmetric group S_h. The definitions for the eigenvalues and singular values of a hypermatrix were independently proposed by Qi [18] and Lim [11], respectively. A complex number λ and a complex vector $\mathbf{y}$ of length n is said to form an eigenpair for an order h, dimension n hypermatrix $\mathcal{M}$ if

$$\mathcal{M}\mathbf{y}^{h-1} = \lambda \mathbf{y}^{[h-1]},$$

where $\mathcal{M}\mathbf{y}^{r-1}$ is a column vector of length n whose i-th entry is given by

$$(\mathcal{M}\mathbf{y}^{h-1})_i = \sum_{j_2, \ldots, j_h = 1}^{n} \mathcal{M}_{i j_2 \ldots j_h} y_{j_2} \cdots y_{j_h},$$

and $\mathbf{y}^{[h-1]}$ is also a column vector of length n whose i-th entry is given by y_i^{h-1}. If the eigenvector $\mathbf{y}$ of $\mathcal{M}$ corresponding to the eigenvalue λ is real, then λ is called an H-eigenvalue of $\mathcal{M}$. The spectral radius of a hypermatrix $\mathcal{M}$, denoted by $\rho(\mathcal{M})$ is defined as

$$\rho = \rho(\mathcal{M}) = \max\{|\lambda| \ |\mathcal{M}\mathbf{y}^{h-1} = \lambda \mathbf{y}^{[h-1]} \ \}.$$

For a non-negative hypermatrix $\mathcal{M}$ of order h and dimension n, Yang and Yang [34] have proved that $\rho = \rho(\mathcal{M})$ is an H-eigenvalue of $\mathcal{M}$. Furthermore, for a weakly irreducible hypermatrix $\mathcal{M}$, Friedland, Gaubert and Han have shown [6] that there exists a unique positive eigenvector corresponding to ρ, and for an irreducible hypermatrix $\mathcal{M}$, Chang, Pearson and Zhang have proved [4] that there exists a unique non-negative eigenvector of $\mathcal{M}$ that corresponds to ρ. If $\mathcal{M}$ is a real symmetric hypermatrix of order h and dimension n, then by using the Rayleigh's quotient, we have [18, 19]

$$\rho(\mathcal{M}) = \max\{\mathcal{M}\mathbf{y}^h | \mathbf{y} \text{ is real and non-negative with } ||\mathbf{y}||_h = 1\},$$

where $\mathcal{M}\mathbf{y}^h = \sum_{j_1,\ldots,j_h=1}^{n} \mathcal{M}_{j_1\ldots j_h} y_{j_1}\cdots y_{j_h}$.

1.2 Preliminaries: hypergraph

A hypergraph is a generalization of a graph, where the edges allowed to connect any number of vertices. Formally, a hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$ of sets, where the elements of $\mathcal{V}$ are called the vertices of $\mathcal{H}$, and $\mathcal{E}$ contains the subsets of the vertex set $\mathcal{V}$ that are called the hyperedges of the hypergraph. The cardinality of a hyperedge is the number of vertices it contain. If the cardinality of every hyperedge is same, say h, then the hypergraph is said to be h-uniform hypergraph. The *size* of a hypergraph is the number of hyperedges in the hypergraph and *order*[†] of a hypergraph is the number of vertices in it.

A linear hypergraph is a hypergraph where any two hyperedges can have atmost one vertex in common. A walk in a hypergraph is an alternating sequence of vertices and hyperedges that starts and ends with vertices, where the consecutive elements of the sequence are incident. Length of a walk is defined to be the floor of the length of the sequence divided by two. A walk in a hypergraph is called a path if all the vertices and hyperedges in the sequence are distinct. A hypergraph is said to be connected if there exist at least one path between any pair of vertices in the hypergraph. In a walk of length greater than one if all the vertices and hyperedges are distinct with the exception that the initial and final vertices coincide, then this walk is called a cycle. A connected hypergraph without a cycle is called a hypertree. If the hypergraph is not linear, then the hypergraph contains a cycle of length two. A hypertree is a linear hypergraph. For a vertex w of $\mathcal{H}$, its degree is denoted by $d_{\mathcal{H}}(w)$ or simply d_w, is the number of hyperedges of $\mathcal{H}$ containing w. A vertex w in a hypergraph $\mathcal{H}$ is called a pendant vertex if $d_{\mathcal{H}}(w) = 1$, otherwise it is called non-pendant. A hyperedge is pendant if it has exactly one non-pendant vertex and all other vertices are pendant. A hyperstar is a hyperstar where every hyperedge is a pendant hyperedge. An h-uniform hyperstar with m hyperedges is denoted by $\mathcal{S}_m^{(h)}$. For example, Figure 3 depicts a 4-uniform hyperstar

[†]Note the difference between the order of a hypergraph and hypermatrix

with 6 hyperedges. Informally, a h-power hypergraph of a simple graph is obtained by inserting $h-2$ new vertices corresponding to each edge of the graph and the hyperedge of size r is formed by including these $h-2$ vertices along with the end vertices of the edge. Formally, a power (particularly, h-power) hypergraph of a simple graph $G=(V,E)$ (2-uniform hypergraph) is denoted G^h, whose vertex set is given by

$$\mathcal{V}(G^h) = V \cup \{j_1^{(e)}, \ldots, j_{h-2}^{(e)}| \text{ for each } e \in E\},$$

and the edge set is given by

$$\mathcal{E}(G^h) = \{e \cup \{j_1^{(e)}, \ldots, j_{h-2}^{(e)}\} | e \in E\}.$$

In this article, trees are denoted by their order, whereas uniform hypertrees are denoted by their uniformity and size. We also emphasize the distinction between G^h and $H^{(h)}$: the former denotes the h-power hypergraph of a graph G, while the latter denotes an h-uniform hypergraph $\mathcal{H}$. Note that both hypergraphs have the same uniformity, namely h. For all other undefined terminologies in the theory hypergraphs, reader can refer to [3].

Following are the key operations on hypergraphs that will be very useful in our results of characterizing the extremal hypertrees.

EMO [9] (Edge-Moving Operation) Given $t \geq 1$ and $h \geq 2$, let $\mathcal{H}$ be a hypergraph, z be a vertex of $\mathcal{H}$, and $e_1, \ldots, e_t$ be the hyperedges of $\mathcal{H}$ such that $z \notin e_j$, for $j = 1, \ldots, t$. For each $j = 1, \ldots, t$ suppose that $z \neq w_j \in e_j$, and let $e_j^* = (e_j \setminus \{w_j\}) \cup \{z\}$ and, $\mathcal{H}^*$ be the hypergraph on the vertex set $\mathcal{V}(\mathcal{H}^*) = \mathcal{V}(\mathcal{H})$ with $\mathcal{E}(\mathcal{H}^*) = (\mathcal{E}(\mathcal{H}) \setminus \{e_1, \ldots, e_t\}) \cup \{e_1^*, \ldots, e_t^*\}$. Then, we say that $\mathcal{H}^*$ is obtained from $\mathcal{H}$ by moving hyperedges $(e_1, \ldots, e_t)$ from $(w_1, \ldots, w_t)$ to z. As an example the Figure 1 shows the depiction of the moving of hyperedges (e_1, e_2) from w_4 to w_1.

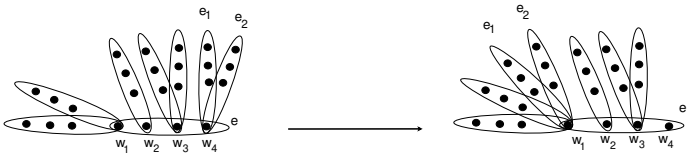

Figure 1. Moving of hyperedges (e_1, e_2) from w_4 to w_1.

ERO [9] (Edge-Releasing Operation) Let $\mathcal{LH}$ be an h-uniform linear hypergraph, and e be a non-pendant hyperedge of $\mathcal{LH}$ and $z \in e$. Suppose that $e_1, \ldots, e_t$ are all the hyperedges of $\mathcal{LH}$ that are adjacent to e but not containing $z \in e$, and also suppose that $e_j \cap e = \{w_j\}$ for $j = 1, \ldots, t$. Let $\mathcal{LH}^\dagger$ be the linear hypergraph obtained from moving the hyperedges $(e_1, \ldots, e_t)$ from $(w_1, \ldots, w_t)$ to z. Then $\mathcal{LH}^\dagger$ is said to be obtained from $\mathcal{LH}$ by edge releasing e at z. Again, as an example, Figure 2 shows the depiction of the edge-releasing operation (ERO) of the hyperedge e at the vertex w_1.

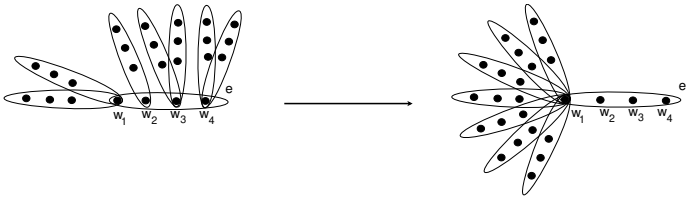

Figure 2. Edge-releasing operation of the non-pendant hyperedge e at w_1.

The normalized adjacency hypermatrix [5] of an h-uniform hypergraph $\mathcal{H}$ on the vertex set $[n] := \{1, \ldots, n\}$ is denoted by $\mathcal{A} = \mathcal{A}(\mathcal{H})$ and its entries are given by

$$\mathcal{A}_{j_1 \ldots j_h} = \begin{cases} \frac{1}{(r-1)!} & \text{if } \{j_1, \ldots, j_h\} \in \mathcal{E}(\mathcal{H}) \\ 0 & \text{otherwise} \end{cases}.$$

The spectral radius of a hypergraph is defined to be the spectral radius of its normalized adjacency hypermatrix. The hypertrees attaining maximum and second maximum spectral radius was characterized by Li, Shao

and Qi [9], and the characterization of hypertrees attaining minimum and second minimum spectral radius was carried out by Zhang, Li and Guo [37]. The ordering of first eight hypertrees attaining maximum spectral radius was characterized by Yuan, Shao and Shan [36]. The connected hypergraphs with spectral radius strictly less than two and equal two was characterized by Lu and Man [15]. Further bounds for the spectral radius of hypergraphs and the extremal hypergraph characterizations can be found in [17, 33, 32, 2, 14, 21, 35, 28, 27].

Given an h-uniform hypergraph $\mathcal{H}$, if f denotes a positive valued symmetric function on h-variables defined on the positive orthant of $\mathbb{R}^h$, then a degree based topological index $\mathcal{F}$ of $\mathcal{H}$ corresponding to the vertex-degree function f can be defined as

$$\mathcal{F}(\mathcal{H}) := \mathcal{F}_f(\mathcal{H}) = \sum_{e\in\mathcal{H}} f(d_e),$$

where $f(d_e) := f(d_{j_1}, \ldots, d_{j_h})$, for a hyperedge $e = \{j_1, \ldots, j_h\}$. Let $\mathcal{A}_{\mathcal{F}}(\mathcal{H})$ denotes an extended adjacency hypermatrix of $\mathcal{H}$ with the associated vertex-degree function f. The hypermatrix $\mathcal{A}_{\mathcal{F}} := \mathcal{A}_{\mathcal{F}}(\mathcal{H})$ is of order h and dimension n, and its entries are given by

$$(\mathcal{A}_{\mathcal{F}})_{j_1\ldots j_h} = \begin{cases} \frac{f(d_e)}{(h-1)!} & \text{if } e = \{j_1, \ldots, j_h\} \in \mathcal{E}(\mathcal{H}), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The Sombor index of a graph, a vertex-degree-based topological index, was introduced in 2021 [8]. Since then, substantial research has been carried out at a rapid pace; for extremal results and bounds, see the survey [13]. In our study, we consider the Sombor index of a hypergraph and its associated degree-based-extended-adjacency hypermatrix of uniform hypertrees. The Sombor index [23] of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is denoted by $\mathcal{SO}(\mathcal{H})$ and is defined as

$$\mathcal{SO}(\mathcal{H}) = \sum_{e\in\mathcal{E}} \sqrt{\sum_{u\in e} d_u^2}.$$

Further studies on vertex-degree-based topological indices of hyper-

graphs [1, 7, 10, 24, 26, 29, 30], directed hypergraphs [22] and their applications [20] have grown rapidly in recent years. Several other generalizations of vertex-degree-based topological indices and the degree-based weighted matrices of hypergraphs have been reported in [16, 31].

Using Equation (1), the Sombor hypermatrix of an h-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is denoted by $\mathcal{A}_{\mathcal{SO}} = \mathcal{A}_{\mathcal{SO}}(\mathcal{H})$, whose $(j_1, \ldots, j_h)$-th entry is given by

$$(\mathcal{A}_{\mathcal{SO}})_{j_1 \ldots j_h} = \begin{cases} \frac{\sqrt{\sum_{i=1}^{h} d_{\mathcal{H}}(j_i)^2}}{(h-1)!} & \text{if } \{j_1, \ldots, j_h\} \text{ is an element of } \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

Lin and Zhou [12] have defined $\mathcal{ABC}$ hypermatrix of uniform hypergraphs and they studied the variation spectral radius in the class of linear hypergraphs. They have characterized the extremal linear hypergraphs attaining the maximum spectral radius among various classes of linear hypergraphs. The relation between the spectral radius of the hypergraph and its generalized power hypergraph was determined in [25]. In the same article, authors have generalized the idea *determining the spectral radius of the hypergraph using the weighted incidence matrix* [15] to degree-based-weighted-adjacency hypermatrix of the hypergraph.

Remainder of the article is organized as follows: In Section 2, we show that Sombor index and Sombor spectral radius strictly increases by the application of edge-releasing operation on the non-pendant hyperedge of the hypergraph. Using these results, it has been shown that hyperstar is the unique hypertree with maximum Sombor index and Sombor spectral radius among all uniform hypertrees of fixed size. Section 3 is devoted to characterize the hypertree attaing second maximum Sombor index and Sombor spectral radius among all hypertrees of fixed size.

2 Hypertree with maximum Sombor index and spectral radius

In this section, initially we prove some lemmas on the variation of Sombor index and Sombor spectral radius while the edge releasing operation is performed on the non-pendant hyperedge of the hypergraphs and then by using these results, we characterize the hypertree attaining the maximum values Sombor index and Sombor spectral radius. Following observation is due to Li, Shao and Qi [9].

Lemma 1. *[9] If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two hypergraphs obtained from $\mathcal{H}$ by ERO of a non-pendant hyperedge e at $z_1 \in e$ and $z_2 \in e$, respectively, then $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic.*

They have also observed that the hypergraph formed by the application of edge-releasing operation on a non-pendant hyperedge of the hypertree is again a hypertree.

Lemma 2. *Given $t \geq 1, h \geq 2$ and a simple connected h-uniform hypergraph $\mathcal{H}$, let $\mathcal{H}^*$ be the hypergraph obtained from $\mathcal{H}$ by ERO of a non-pendant hyperedge e at $z \in e$, such that $\mathcal{H}^*$ is simple. If $f(d_{e_j}) \leq f(d_{e_j^*})$ for all $e_j \in \mathcal{E}(\mathcal{H})$ with $e_j \cap e \neq \emptyset$, then*

$$\rho_{\mathcal{F}}(\mathcal{H}^*) > \rho_{\mathcal{F}}(\mathcal{H}).$$

Proof. Assume that e is a non-pendant hyperedge of $\mathcal{H}$ and $\mathcal{H}^*$ be the hypergraph obtained from $\mathcal{H}$ by ERO of e at $z \in e$. Let $\mathbf{y}$ be the Perron vector of $\mathcal{A}_{\mathcal{F}}(\mathcal{H})$ associated with spectral radius $\rho_{\mathcal{F}}(\mathcal{H})$. Also, let $z_1 \in e$ be such that $y_{z_1} = \max_{w \in e} y_w$, and $\mathcal{H}_1^*$ be the hypergraph obtained from $\mathcal{H}$ by ERO of e at z_1. Let us assume that, in total there are k hyperedges (including e), say $e_1, \ldots, e_k$ of $\mathcal{H}$ such that $e \cap e_j \neq \emptyset, j \in [k]$, and $e_1^*, \ldots, e_k^*$ be the corresponding hyperedges in $\mathcal{H}_1^*$. Then,

$$\begin{aligned}\rho_{\mathcal{F}}(\mathcal{H}_1^*) - \rho_{\mathcal{F}}(\mathcal{H}) &= \rho_{\mathcal{F}}(\mathcal{H}_1^*) - \mathcal{A}_{\mathcal{F}}(\mathcal{H})\mathbf{y}^h \\ &\geq \mathcal{A}_{\mathcal{F}}(\mathcal{H}_1^*)\mathbf{y}^h - \mathcal{A}_{\mathcal{F}}(\mathcal{H})\mathbf{y}^h\end{aligned}$$

$$= \sum_{e^* \in \mathcal{E}(\mathcal{H}_1^*)} hf(d_{e^*})\mathbf{y}^{e^*} - \sum_{e \in \mathcal{H}} hf(d_e)\mathbf{y}^e$$
$$= h\sum_{j=1}^{k}(f(d_{e_j^*})\mathbf{y}^{e_j^*} - f(d_{e_j})\mathbf{y}^{e_j}) \geq 0.$$

For $t \geq 1$, the equality can not occur as $\mathbf{y}$ can not be an eigenvector of both $\mathcal{F}(\mathcal{H}_1^*)$ and $\mathcal{F}(\mathcal{H})$. Along with that, by Lemma 1 we have, $\mathcal{H}_1^* \cong \mathcal{H}^*$ and hence

$$\rho_{\mathcal{F}}(\mathcal{H}^*) = \rho_{\mathcal{F}}(\mathcal{H}_1^*) > \rho_{\mathcal{F}}(\mathcal{H}).$$

■

Lemma 3. *Let $\mathcal{H}^*$ be a h-uniform hypergraph obtained from $\mathcal{H}$ ($\ncong \mathcal{S}_m^{(h)}$) by ERO on a non-pendant hyperedge e of $\mathcal{H}$ at $z \in e$. Then for all hyperedges $e_j \in \mathcal{E}(\mathcal{H}), j \in [k]$ with $e_j \cap e \neq \emptyset$, and corresponding hyperedges $e_j^* \in \mathcal{E}(\mathcal{H}^*)$,*

$$\sqrt{\sum_{w \in e_j} d_{\mathcal{H}}(w)^2} < \sqrt{\sum_{w \in e_j^*} d_{\mathcal{H}^*}(w)^2}.$$

Proof. Suppose that e_j is a non-pendant hyperedge of $\mathcal{H}$ ($\ncong \mathcal{S}_m^{(h)}$) and e_j^* be the corresponding hyperedge in $\mathcal{H}^*$, which is obtained from $\mathcal{H}$ by edge releasing e at z. If $e = \{z_1, z_2, \ldots, z_h\}$, then $d_e = \{d_{z_1}, d_{z_2}, \ldots, d_{z_h}\}$. Also, $e^* = e$ and $d_{e^*} = \{d_{\mathcal{H}^*}(z), 1, \ldots, 1\}$, where $d_{\mathcal{H}^*}(z) = 1 - h + \sum_{j=1}^{h} d_{\mathcal{H}}(z_j)$. For $a > b \geq 1$, we have

$$a^2 + b^2 < (a+1)^2 + (b-1)^2. \tag{2}$$

By using inequality (2) repeatedly, it can be shown that

$$d_{\mathcal{H}^*}(z)^2 + (h-1) \geq \sum_{j=1}^{h} d_{\mathcal{H}}(z_j)^2.$$

For a hyperedge $e_j \neq e$, let w_j be the unique (since the hypergraph is linear) vertex in $e \cap e_j$. Since, $d_{\mathcal{H}^*}(z_1) > d_{\mathcal{H}}(w_j)$, we have

$$\sqrt{\sum_{w\in e^*} d_{\mathcal{H}^*}(w)^2} - \sqrt{\sum_{w\in e} d_{\mathcal{H}}(w)^2} = \left(\sqrt{d_{\mathcal{H}^*}(z)^2 + (h-1)} - \sqrt{\sum_{j=1}^{h} d_{\mathcal{H}}(z_j)^2}\right)$$
$$+ \sum_{\substack{e_j^*\in\mathcal{E}(\mathcal{H}^*)\\ e_j\in\mathcal{E}(\mathcal{H})}} \sqrt{d_{\mathcal{H}^*}(z_1)^2 + \sum_{\substack{w\in e_j^*\\ w\neq z_1}} d_{\mathcal{H}^*}(w)^2} - \sqrt{d_{\mathcal{H}}(w_j)^2 + \sum_{\substack{w\in e_j\\ w\neq w_j}} d_{\mathcal{H}}(w)^2}$$
$$\geq \left(\sqrt{d_{\mathcal{H}^*}(z)^2 + (h-1)} - \sqrt{\sum_{j=1}^{h} d_{\mathcal{H}}(z_j)^2}\right) \geq 0.$$

■

Theorem 1. *Let $\mathcal{T}$ be a h-uniform hypertree with m hyperedges. Then*

$$\mathcal{SO}(\mathcal{T}) \leq m\sqrt{m^2 + h - 1},$$

Equality is attained if and only if $\mathcal{T} \cong \mathcal{S}_m^{(h)}$.

Proof. By applying ERO on a non-pendant hyperedge e of a hypertree $\mathcal{T}$ at $z \in e$, e becomes a pendant hyperedge. For any hypertree $\mathcal{T}$, by repeatedly applying ERO on the non-pendant hyperedges of $\mathcal{T}$, we finally get a h-uniform hyperstar, where all the hyperedges in the hypergraph are pendant. Now, by using Lemma 3 we can see that the h-uniform hyperstar, $\mathcal{S}_m^{(h)}$ attains the maximum value of Sombor index among all hypertrees of size m. Since, each hyperedge of $\mathcal{S}_m^{(h)}$ contains exactly $h-1$ pendant vertices and a vertex of degree m, we have the result. ■

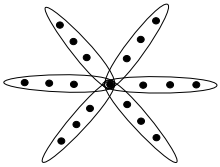

Figure 3. The 4-uniform hyperstar with 6 hyperedges, $\mathcal{S}_6^{(4)}$

Theorem 2. *If $\mathcal{T}$ is a h-uniform hypertree of size m, then*

$$\rho_{\mathcal{SO}}(\mathcal{T}) \leq m^{\frac{1}{h}}\sqrt{m^2 + h - 1},$$

with equality if and only if $\mathcal{T} \cong \mathcal{S}_m^{(h)}$.

Proof. On contrary, suppose that $\mathcal{T}_1 \ncong \mathcal{S}_m^{(h)}$ is a hypertree that attains the maximum value of the spectral radius among all hypertrees of sixed size. Then, $\mathcal{T}_1$ must have at least one non-pendant hyperedge, say e_1. Let $\mathcal{T}_1^*$ be the hypergraph obtained from $\mathcal{T}_1$ by applying ERO on the hyperedge e_1 at some $z \in e_1$. By using Lemma 2 and Lemma 3, we have that $\rho_{\mathcal{SO}}(\mathcal{T}_1) < \rho_{\mathcal{SO}}(\mathcal{T}_1^*)$, a contradiction. Hence, if $\mathcal{T}_1$ is a hypertree that attains the maximum spectral radius among all hypertrees, then $\mathcal{T}_1$ does not contain any non-pendant hyperedges.

The spectral radius of the Sombor hypermatrix of a h-uniform hyperstar can be determined easily, since $\mathcal{A}_{\mathcal{SO}}(\mathcal{S}_m^{(h)}) = (\sqrt{m^2+h-1})\mathcal{A}(\mathcal{S}_m^{(h)})$. The equality is clear, as there exists a unique hypertree without any non-pendant hyperedge. ■

3 Hypertree with second maximum Sombor index and Sombor spectral radius

In this section, we characterize the hypertree attaining the second maximum Sombor index among all uniform hypertrees of fixed size. First we show that the hypertree $\mathcal{S}_{1,m-2} \cong \mathcal{D}_m^{(h)}$ is the unique hypertree attaining the maximum Sombor index and Sombor spectral radius among all h-uniform double hyperstars of size m, and later we show that the hypertree attaining second maximum Sombor index must belong to the class of double hyperstars.

Definition 1. Let $\mathcal{S}_{t_1+1}$ and $\mathcal{S}_{t_2+1}$ be the stars (2-uniform hyperstars) on t_1+1 and t_2+1 vertices, respectively. A double star $\mathcal{S}_{t_1,t_2}$ is obtained by attaching the centers of the two stars $\mathcal{S}_{t_1+1}$ and $\mathcal{S}_{t_2+1}$, to the distinct end vertices of K_2.

By definition, note that degrees of the two non-pendant vertices of the double star $\mathcal{S}_{t_1,t_2}$ are equal to t_1+1 and t_2+1. We call the h-power hypergraph of the double star $\mathcal{S}_{t_1,t_2}$ as an h-uniform double hyperstar, and is denoted by $\mathcal{S}_{t_1,t_2}^h$. For example, Figure 4 pictorially depicts the double

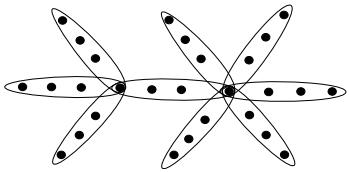

Figure 4. The 4-uniform double hyperstar, $\mathcal{S}^4_{3,5}$

hyperstar $\mathcal{S}^4_{3,5}$, which contain two non-pendant vertices (in the unique non-pendant hyperedge) of degrees 4 and 6.

Lemma 4. *Let $\mathcal{S}^h_{t_1,t_2}$ be the h-power hypergraph of the double star $\mathcal{S}_{t_1,t_2}$. If $t_1 \geq t_2$, then*

$$\mathcal{SO}(\mathcal{S}^h_{t_1,t_2}) < \mathcal{SO}(\mathcal{S}^h_{t_1+t,t_2-t}).$$

Proof. For $a_1 > a_2 \geq 2$, and for $\alpha \geq 1, 1 \leq t \leq a_2$, we have

$$\sqrt{(a_1+t)^2+\alpha} + \sqrt{(a_2-t)^2+\alpha} > \sqrt{a_1^2+\alpha} + \sqrt{a_2^2+\alpha}. \tag{3}$$

Let e be the non-pendant hyperedge of $\mathcal{H} := \mathcal{S}^h_{t_1,t_2}$, and u_1 be the non-pendant vertex of $\mathcal{H}$ that is incident to t_1 pendant hyperedges and u_2 be another non-pendant vertex of $\mathcal{H}$ that is incident to t_2 pendant hyperedges. Also, let $\mathcal{H}^* := \mathcal{S}^h_{t_1+t,t_2-t}$ be the hypergraph obtained from $\mathcal{H}$ by moving t hyperedges from u_2 to u_1, and e^* be the non-pendant hyperedge of $\mathcal{H}^*$. Then

$$\begin{aligned}
&\sqrt{\sum_{w\in e^*} d_{\mathcal{H}^*}(w)^2} - \sqrt{\sum_{w\in e} d_{\mathcal{H}}(w)^2} \\
&= \sqrt{(t_1+t+1)^2+(t_2-t+1)^2+h-2} - \sqrt{(t_1+1)^2+(t_2+1)^2+h-2} \geq 0.
\end{aligned}$$

Let $e_1, \ldots, e_t$ be the hyperedges of $\mathcal{H}$ that are moved from u_2 to u_1, and $e_j^* = (e_j \setminus \{u_2\}) \cup \{u_1\}, j \in [t]$. For all $j \in [t]$, it is obvious that

$$\begin{aligned}
&\sqrt{\sum_{w\in {e_j}^*} d_{\mathcal{H}^*}(w)^2} - \sqrt{\sum_{w\in e_j} d_{\mathcal{H}}(w)^2} \\
&\qquad = \sqrt{(t_1+t+1)^2+h-1} - \sqrt{(t_2+1)^2+h-1} \geq 0.
\end{aligned}$$

Suppose that $e_{t+1}, \ldots, e_{t_2}$ are the hyperedges that are incident to u_2 both

in $\mathcal{H}$ and $\mathcal{H}^*$. Then for $t+1 \leq i \leq t_2$,

$$\sqrt{\sum_{w\in e_i} d_{\mathcal{H}}(w)^2} > \sqrt{\sum_{w\in e_i^*} d_{\mathcal{H}}(w)^2}.$$

But, on choosing $t_2 - t$ hyperedges, say $f_1, \ldots, f_{t_2-t}$ among t_1 pendant hyperedges that are incident to u_1 both in $\mathcal{H}$ and $\mathcal{H}^*$, we have by using inequality (3),

$$\begin{aligned}\sqrt{\sum_{w\in f_i^*} d_{\mathcal{H}^*}(w)^2} + \sqrt{\sum_{w\in e_i^*} d_{\mathcal{H}^*}(w)^2} - \left(\sqrt{\sum_{w\in f_i} d_{\mathcal{H}}(w)^2} + \sqrt{\sum_{w\in e_i} d_{\mathcal{H}}(w)^2}\right)\\ = \sqrt{(t_1+t+1)^2+h-1} + \sqrt{(t_2-t+1)^2+h-1}\\ - (\sqrt{(t_1+1)^2+h-1} + \sqrt{(t_2+1)^2+h-1}) \geq 0.\end{aligned}$$

For each of the remaining hyperedges, say $f_{t_2-t+1}, \ldots, f_{t_1}$, it is obvious that

$$\begin{aligned}\sqrt{\sum_{w\in f_j^*} d_{\mathcal{H}^*}(w)^2} - \sqrt{\sum_{w\in f_j} d_{\mathcal{H}}(w)^2}\\ = \sqrt{(t_1+t+1)^2+h-1} - \sqrt{(t_1+1)^2+h-1} \geq 0.\end{aligned}$$

Hence, we have $\mathcal{SO}(\mathcal{H}) < \mathcal{SO}(\mathcal{H}^*)$. ■

The extremal hypertree $D_9^{(4)}$, which is also isomorphic to the double hyperstar $\mathcal{S}_{7,1}^h$ that attains the second maximum Sombor index and Sombor spectral radius (which will be discussed in the later part) among all hypertrees of fixed size and uniformity is pictorially depicted in Figure 5.

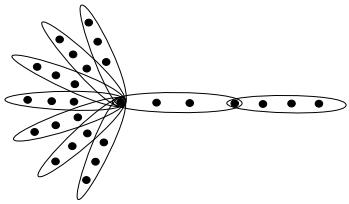

Figure 5. The hypertree $\mathcal{D}_9^{(4)}$

Theorem 3. *Let $\mathcal{T} \ncong \mathcal{S}_m^{(h)}$ be a h-uniform hypertree with m hyperedges. Then,*

$$\mathcal{SO}(\mathcal{T}) \leq (m-2)\sqrt{(m-1)^2+h-1} + \sqrt{(m-1)^2+h+2} + \sqrt{h+3}.$$

Equality is attained if and only if $\mathcal{T} \cong \mathcal{D}_m^{(h)}$, which is obtained from $\mathcal{S}_{m-1}^{(h)}$ by attaching a pendant hyperedge to any one of the pendant vertices.

Proof. Assume that $\mathcal{T}'$ be the hypergraph that attains the maximum Sombor index among all hypertrees (except $\mathcal{S}_m^{(h)}$) of size m. On contrary, suppose that $\mathcal{T}'$ contains more than one non-pendant hyperedges. Then, $\mathcal{T}_1'$ be the hypergraph obtained from $\mathcal{T}'$ by applying ERO on a non-pendant hyperedge of the hypergraph, until it has exactly one non-pendant hyperedge. Then, by using Lemma 3, we have $\mathcal{SO}(\mathcal{T}_1') > \mathcal{SO}(\mathcal{T}')$, a contradiction. So $\mathcal{T}'$ contains exactly one non-pendant hyperedge.

Again on contrary, suppose that $\mathcal{T}'$ contains more than two non pendant vertices. Let us assume that e be the non-pendant hyperedge of $\mathcal{T}'$, and $w_1, \ldots, w_t$, $3 \leq t \leq h$ be the non-pendant vertices in e. Without loss of generality, assume that $d(w_1) \geq d(w_2) \geq \cdots \geq d(w_t)$ be the degrees of the vertices in $\mathcal{T}'$. Note that, as $\mathcal{T}'$ contains exactly one non-pendant hyperedge, the hyperedges (except e) incident at $w_j, 1 \leq j \leq t$ are all pendant. Let $\mathcal{T}_2'$ be the hypergraph obtained from $\mathcal{T}'$ by moving all the pendant hyperedges from $w_t, t \geq 3$ to w_1. Now by using the inequality (2), we acquire that $\mathcal{SO}(\mathcal{T}_2') > \mathcal{SO}(\mathcal{T}')$, again a contradiction. Therefore we have that $\mathcal{T}'$ contains exactly one non-pendant hyperedge, and two non-pendant vertices, that is, $\mathcal{T}'$ is isomorphic to a power hypergraph of a double star. Hence, by using Lemma 4, we have that $\mathcal{T}' \cong \mathcal{D}_m^{(h)} \cong \mathcal{S}_{m-2,1}^h$.

The Sombor index of $\mathcal{D}_m^{(h)}$ can be computed directly, as

• it contain exactly $m-2$ pendant hyperedges with degree of the non-pendant edge equal to $m-2$, hence its contribution is $(m-2)\sqrt{(m-1)^2+h-1}$;

• it contains a hyperedge with exactly two non-pendant vertices, hence its contribution is $\sqrt{(m-1)^2+4+h-2} = \sqrt{(m-1)^2+h+2}$;

• it contains a pendant hyperedge with the degree of the non-pendant vertex equal to 2. ∎

Lemma 5. *Let $\mathcal{S}^h_{t_1,t_2}$ be the h-power hypergraph of the double star $\mathcal{S}_{t_1,t_2}$. Then*

$$\rho_{\mathcal{SO}}(\mathcal{S}^h_{t_1,t_2}) < \rho_{\mathcal{SO}}(\mathcal{S}^h_{t_1+t_2-1,1}).$$

Proof. Let e be a non-pendant hyperedge of $\mathcal{H} := \mathcal{S}^h_{t_1,t_2}$ and u_1 and u_2 be the non-pendant vertices of $\mathcal{H}$ with t_1 pendant vertices incident to u_1 and t_2 pendant vertices incident to u_2. Without loss of generality, assume that $t_1 \geq t_2$. Let $\mathbf{y}$ be the positive eigenvector of $\mathcal{A}_{\mathcal{SO}}(\mathcal{H})$ corresponding to the largest H-eigenvalue $\rho_{\mathcal{SO}}(\mathcal{H})$.

Case 1. $y_{u_1} \geq y_{u_2}$.

Let $\mathcal{H}^*_1 := \mathcal{S}_{t_1+t_2-1,1}$ be the hypergraph obtained from $\mathcal{H}$ by moving the hyperedges $e_1, \ldots, e_{t_2-1}$ from u_2 to u_1, and $e^*_1, \ldots, e^*_{t_2-1}$ be the corresponding hyperedges in $\mathcal{H}^*_1$. Let e^* be the non-pendant hyperedge of $\mathcal{H}^*_1$. Also, let e_{t_2} be the pendant hyperedge incident to u_2 both in $\mathcal{H}$ and $\mathcal{H}^*_1$, and let $g_1, \ldots, g_{t_1}$ be the hyperedges incident to u_1 both in $\mathcal{H}$ and $\mathcal{H}^*_1$. Then

$$\begin{aligned}
\rho_{\mathcal{SO}}(\mathcal{H}^*_1) - \rho_{\mathcal{SO}}(\mathcal{H}) &= \rho_{\mathcal{SO}}(\mathcal{H}^*_1) - \mathcal{A}_{\mathcal{SO}}(\mathcal{H})\mathbf{y}^h \\
&\geq \mathcal{A}_{\mathcal{SO}}(\mathcal{H}^*_1)\mathbf{y}^h - \mathcal{A}_{\mathcal{SO}}(\mathcal{H})\mathbf{y}^h \\
&= h \sum_{j=1}^{t_2-1} \left(\sqrt{\sum_{w\in e^*_j} d_{\mathcal{H}^*_1}(w)^2}\ \mathbf{y}^{e^*_j} - \sqrt{\sum_{w\in e_j} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_j} \right) \\
&\quad + h \left(\sqrt{\sum_{w\in e^*} d_{\mathcal{H}^*_1}(w)^2}\ \mathbf{y}^{e^*} - \sqrt{\sum_{w\in e} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e} \right) \\
&\quad + h \left(\sqrt{\sum_{w\in e^*_{t_2}} d_{\mathcal{H}^*_1}(w)^2}\ \mathbf{y}^{e^*_{t_2}} + \sqrt{\sum_{w\in g^*_{t_1}} d_{\mathcal{H}^*_1}(w)^2}\ \mathbf{y}^{g^*_{t_1}} \right) \\
&\quad - h \left(\sqrt{\sum_{w\in e_{t_2}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_{t_2}} + \sqrt{\sum_{w\in g_{t_1}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_{t_1}} \right) \\
&\quad + h \sum_{i=1}^{t_1-1} \left(\sqrt{\sum_{w\in g^*_i} d_{\mathcal{H}^*_1}(w)^2}\ \mathbf{y}^{g^*_i} - \sqrt{\sum_{w\in g_i} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_i} \right).
\end{aligned}$$

Since $y_{u_1} \geq y_{u_2} > 0$, we have for each $j, 1 \leq j \leq t_2 - 1$,

$$\sqrt{\sum_{w \in e_j^*} d_{\mathcal{H}_1^*}(w)^2}\ \mathbf{y}^{e_j^*} - \sqrt{\sum_{w \in e_j} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_j}$$
$$= \sqrt{(t_1+t_2)^2+h-1}\ \mathbf{y}^{e_j^*} - \sqrt{(t_2+1)^2+h-1}\mathbf{y}^{e_j}$$

$$\geq \left(\sqrt{(t_1+t_2)^2+h-1}\ - \sqrt{(t_2+1)^2+h-1}\right)\mathbf{y}^{e_j} \geq 0.$$

For the non-pendant hyperedge e of $\mathcal{H}$ and corresponding hyperedge e^* of $\mathcal{H}_1^*$, we have $\sqrt{\sum_{w \in e^*} d_{\mathcal{H}_1^*}(w)^2}\mathbf{y}^{e^*} - \sqrt{\sum_{w \in e} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e}$ is equal to

$$\sqrt{\left(\sqrt{(t_1+t_2)^2+h+2} - \sqrt{(t_1+1)^2+(t_2+1)^2+h-2}\right)}\mathbf{y}^{e} \geq 0.$$

Clearly, for each $i, 1 \leq i \leq t_1 - 1$,

$$\sqrt{\sum_{w \in g_i^*} d_{\mathcal{H}_1^*}(w)^2}\ \mathbf{y}^{g_i^*} - \sqrt{\sum_{w \in g_i} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_i}$$
$$= \left(\sqrt{(t_1+t_2)^2+h-1} - \sqrt{(t_1+1)^2+h-1}\right)\mathbf{y}^{g_i}$$
$$\geq 0.$$

Finally, consider the hyperedges e_{t_2} and g_{t_1} of the hypergraph $\mathcal{H}$, and the corresponding hyperedges $e_{t_2}^*$ and $g_{t_1}^*$ of the hypergraph $\mathcal{H}_1^*$. Since $y_{u_1} \geq y_{u_2}$, and by using inequality (3), we have

$$\sqrt{\sum_{w \in e_{t_2}^*} d_{\mathcal{H}_1^*}(w)^2}\,\mathbf{y}^{e_{t_2}^*} + \sqrt{\sum_{w \in g_{t_1}^*} d_{\mathcal{H}_1^*}(w)^2}\,\mathbf{y}^{g_{t_1}^*}$$
$$- \left(\sqrt{\sum_{w \in e_{t_2}} d_{\mathcal{H}}(w)^2}\,\mathbf{y}^{e_{t_2}} + \sqrt{\sum_{w \in g_{t_1}} d_{\mathcal{H}}(w)^2}\,\mathbf{y}^{g_{t_1}}\right)$$

$$
\begin{aligned}
&= \sqrt{h+3}\,\mathbf{y}^{e^*_{t_2}} + \sqrt{(t_1+t_2)^2+h-1}\,\mathbf{y}^{g^*_{t_1}} \\
&\quad - \sqrt{(t_2+1)^2+h-1}\,\mathbf{y}^{e_{t_2}} - \sqrt{(t_1+1)^2+h-1}\,\mathbf{y}^{g_{t_1}} \\
&\geq \Big(\sqrt{h+3} + \sqrt{(t_1+t_2)^2+h-1} \\
&\qquad - \sqrt{(t_2+1)^2+h-1} - \sqrt{(t_1+1)^2+h-1}\Big)\,\mathbf{y}^{e_{t_2}} \geq 0.
\end{aligned}
$$

Case 2. $y_{u_1} \leq y_{u_2}$.

Let $\mathcal{H}^*_2 := \mathcal{S}_{1,t_1+t_2-1}$ be the hypergraph obtained from $\mathcal{H}$ by moving the hyperedges $g_1, \ldots, g_{t_1-1}$ from u_1 to u_2, and $g^*_1, \ldots, g^*_{t_1-1}$ be the corresponding hyperedges in $\mathcal{H}^*_2$. Let e^* be the non-pendant hyperedge of $\mathcal{H}^*_2$. Also, let g_{t_1} be the pendant hyperedge incident to u_1 both in $\mathcal{H}$ and $\mathcal{H}^*_2$, and let $e_1, \ldots, e_{t_2}$ be the hyperedges incident to u_2 both in $\mathcal{H}$ and $\mathcal{H}^*_2$. Then

$$
\begin{aligned}
\rho_{\mathcal{SO}}(\mathcal{H}^*_2) - \rho_{\mathcal{SO}}(\mathcal{H}) &= \rho_{\mathcal{SO}}(\mathcal{H}^*_2) - \mathcal{A}_{\mathcal{SO}}(\mathcal{H})\mathbf{y}^h \\
&\geq \mathcal{A}_{\mathcal{SO}}(\mathcal{H}^*_2)\mathbf{y}^h - \mathcal{A}_{\mathcal{SO}}(\mathcal{H})\mathbf{y}^h \\
&= h \sum_{j=1}^{t_1-1} \left(\sqrt{\sum_{w \in g^*_j} d_{\mathcal{H}^*_2}(w)^2}\ \mathbf{y}^{g^*_j} - \sqrt{\sum_{w \in g_j} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_j} \right) \\
&\quad + h \left(\sqrt{\sum_{w \in e^*} d_{\mathcal{H}^*_2}(w)^2}\ \mathbf{y}^{e^*} - \sqrt{\sum_{w \in e} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e} \right) \\
&\quad + h \left(\sqrt{\sum_{w \in g^*_{t_1}} d_{\mathcal{H}^*_2}(w)^2}\ \mathbf{y}^{g^*_{t_1}} + \sqrt{\sum_{w \in e^*_{t_2}} d_{\mathcal{H}^*_2}(w)^2}\ \mathbf{y}^{e^*_{t_2}} \right. \\
&\quad \left. - \sqrt{\sum_{w \in g_{t_1}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_{t_1}} - \sqrt{\sum_{w \in e_{t_2}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_{t_2}} \right) \\
&\quad + h \sum_{i=1}^{t_2-1} \left(\sqrt{\sum_{w \in e^*_i} d_{\mathcal{H}^*_2}(w)^2}\ \mathbf{y}^{e^*_i} - \sqrt{\sum_{w \in e_i} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_i} \right).
\end{aligned}
$$

Because, $0 < y_{u_1} \leq y_{u_2}$, we have for each $j, 1 \leq j \leq t_1 - 1$,

$$\begin{aligned}
&\sqrt{\sum_{w \in g_j^*} d_{\mathcal{H}_2^*}(w)^2}\ \mathbf{y}^{g_j^*} - \sqrt{\sum_{w \in g_j} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_j} \\
&\qquad = \sqrt{(t_1+t_2)^2 + h - 1}\ \mathbf{y}^{g_j^*} - \sqrt{(t_1+1)^2 + h - 1}\ \mathbf{y}^{g_j} \\
&\qquad \geq \left(\sqrt{(t_1+t_2)^2 + h - 1}\ - \sqrt{(t_2+1)^2 + h - 1}\right) \mathbf{y}^{g_j} \geq 0.
\end{aligned}$$

For the non-pendant hyperedge e of $\mathcal{H}$ and corresponding hyperedge e^* of $\mathcal{H}_2^*$, we have

$$\begin{aligned}
&\sqrt{\sum_{w \in e^*} d_{\mathcal{H}_2^*}(w)^2}\ \mathbf{y}^{e^*} - \sqrt{\sum_{w \in e} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e} \\
&\qquad = \left(\sqrt{(t_1+t_2)^2 + h + 2} - \sqrt{(t_1+1)^2 + (t_2+1)^2 + h - 2}\right) \mathbf{y}^{e},
\end{aligned}$$

which is greater than zero. Clearly, for each $i, 1 \leq i \leq t_2 - 1$,

$$\begin{aligned}
&\sqrt{\sum_{w \in e_i^*} d_{\mathcal{H}_2^*}(w)^2}\ \mathbf{y}^{e_i^*} - \sqrt{\sum_{w \in e_i} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_i} \\
&\qquad = \left(\sqrt{(t_1+t_2)^2 + h - 1} - \sqrt{(t_2+1)^2 + h - 1}\right) \mathbf{y}^{e_i} \geq 0.
\end{aligned}$$

Now, consider the hyperedges e_{t_2} and g_{t_1} of the hypergraph $\mathcal{H}$, and the corresponding hyperedges $e_{t_2}^*$ and $g_{t_1}^*$ of the hypergraph $\mathcal{H}_2^*$. Because $y_{u_1} \leq y_{u_2}$ and by using inequality (3), we have

$$\begin{aligned}
&\sqrt{\sum_{w \in e_{t_2}^*} d_{\mathcal{H}_1^*}(w)^2}\ \mathbf{y}^{e_{t_2}^*} + \sqrt{\sum_{w \in g_{t_1}^*} d_{\mathcal{H}_1^*}(w)^2}\ \mathbf{y}^{g_{t_1}^*} \\
&\qquad - \sqrt{\sum_{w \in e_{t_2}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{e_{t_2}} - \sqrt{\sum_{w \in g_{t_1}} d_{\mathcal{H}}(w)^2}\ \mathbf{y}^{g_{t_1}} \\
&\qquad = \sqrt{(t_1+t_2)^2 + h - 1}\ \mathbf{y}^{e_{t_2}^*} + \sqrt{h+3}\ \mathbf{y}^{g_{t_1}^*} \\
&\qquad - \sqrt{(t_2+1)^2 + h - 1}\ \mathbf{y}^{e_{t_2}} - \sqrt{(t_1+1)^2 + h - 1}\ \mathbf{y}^{g_{t_1}} \\
&\qquad \geq \left(\sqrt{h+3} + \sqrt{(t_1+t_2)^2 + h - 1}\right. \\
&\qquad \left. - \sqrt{(t_2+1)^2 + h - 1} - \sqrt{(t_1+1)^2 + h - 1}\right) \mathbf{y}^{g_{t_1}} \geq 0.
\end{aligned}$$

Since $\mathcal{S}_{1,t_1+t_2-1} \cong \mathcal{S}_{t_1+t_2-1,1}$, we have the desired result. ■

Theorem 4. *Let $\mathcal{T} \ncong \mathcal{S}_m^h$ be a h-uniform hypertree with m hyperedges. Then*

$$\rho_{\mathcal{SO}}(\mathcal{T}) \leq \rho_{\mathcal{SO}}(\mathcal{D}_m^{(h)}).$$

Equality is attained if and only if $\mathcal{T} \cong \mathcal{D}_m^{(h)}$.

Proof. Let $\mathcal{T}_1$ be the hypertree that attains the maximum value of Sombor spectral radius among all h-uniform hypertrees of size m except the hyperstar $\mathcal{S}_m^{(h)}$. We show that $\mathcal{T}_1$ contains exactly one non-pendant hyperedge. On contrary, assume that $\mathcal{T}_1$ contains $p, p \geq 2$ non-pendant hyperedges, say $e_1, \ldots, e_p$. Let $\mathcal{T}^*$ be the hypertree obtained from T_1 by performing ERO on the hyperedges $e_2, \ldots, e_t$. But, by using Lemma 2 and Lemma 3, we conclude that $\rho_{\mathcal{SO}}(\mathcal{T}^*) > \rho_{\mathcal{SO}}(\mathcal{T}_1)$, a contradiction. Now, suppose that $\mathcal{T}_1$ contains more than two non-pendant vertices. Let $w_1, \ldots, w_t$, $3 \leq t \leq h$ be the non-pendant vertices in the non-pendant hyperedge e_1 of $\mathcal{T}_1$. Without loss of generality, assume that $y_{w_1} \geq y_{w_2} \geq \cdots \geq y_{w_r}$ be the components of the Perron vector $\mathbf{y}$ of $\mathcal{A}_{\mathcal{SO}}(\mathcal{T}_1)$ corresponding to the vertices $w_1, \ldots, w_r$, respectively. In the hypertree $\mathcal{T}_1$, let $d_1 + 1, \ldots, d_r + 1$ be the degrees of the vertices $w_1, \ldots, w_r$, respectively. For $d_i > 0$, let $e_{ij}, 1 \leq j \leq d_i$ be the hyperedges of $\mathcal{T}_1$ incident to the vertex w_i. As $\mathcal{T}_1$ contains exactly one non-pendant hyperedge, the hyperedges incident at $w_j, 1 \leq j \leq t$ are all pendant (except the one corresponding to e_1). Since our assumption is that e_1 contains more than 2 non-pendant vertices, and $y_{w_1} \geq y_{w_2} \geq \cdots \geq y_{w_h}$, let $\mathcal{T}_1'$ be the hypergraph obtained from $\mathcal{T}_1$ by EMO on the pendant hyperedges incident to the vertices of e_1 through the following cases depending the degree of the vertex w_2:

Case 1. $d_2 \geq 1$:

If $d_2 \geq 1$, then $\mathcal{T}_1'$ be the hypergraph obtained from $\mathcal{T}_1$ by moving the pendant hyperedges $e_{ij}, 3 \leq i \leq h, 1 \leq j \leq d_i$ (with $d_i \geq 1$) from w_i to w_1.

Case 2. $d_2 = 0$:

If $d_2 = 0$, then there exist at least two vertices in the set $\{w_3, \ldots, w_h\}$ with vertex degree strictly greater than 1. Without loss of generality, assume that w_3 is one such vertex with $d_3 \geq 1$. Now the hypertree $\mathcal{T}_1'$ is obtained from $\mathcal{T}_1$ by

a) moving all the pendant hyperedges $e_{3j}, 1 \le j \le d_3$ from w_3 to w_2; and

b) moving all the remaining pendant hyperedges $e_{ij}, 4 \le i \le h, 1 \le j \le d_i$ from w_i to w_1.

Now by using Rayleigh's quotient, we have

$$\begin{aligned}
\rho_{\mathcal{SO}}(\mathcal{T}_1') - \rho_{\mathcal{SO}}(\mathcal{T}_1) &= \rho_{\mathcal{SO}}(\mathcal{T}_1') - \mathcal{A}_{\mathcal{SO}}(\mathcal{T}_1)\mathbf{y}^h \\
&\ge \mathcal{A}_{\mathcal{SO}}(\mathcal{T}_1')\mathbf{y}^h - \mathcal{A}_{\mathcal{SO}}(\mathcal{T}_1)\mathbf{y}^h \\
&= h\left(\sum_{e\in\mathcal{E}(\mathcal{T}_1')}\sqrt{\sum_{w\in e} d_{\mathcal{T}_1'}(w)^2}\,\mathbf{y}^e - \sum_{e\in\mathcal{E}(\mathcal{T})}\sqrt{\sum_{w\in e} d_{\mathcal{T}_1'}(w)^2}\,\mathbf{y}^e\right) \\
&= h\left(\sqrt{\sum_{w\in e_1} d_{\mathcal{T}^*}(w)^2} - \sqrt{\sum_{w\in e_1'} d_{\mathcal{T}_1'}(w)^2}\right)\mathbf{y}^{e_1} \\
&+ h\sum_{i=1}^{h}\sum_{j=1}^{d_i}\left(\sqrt{\sum_{w\in e_{ij}'} d_{\mathcal{T}_1'}(w)^2} - \sqrt{\sum_{w\in e_{ij}} d_{\mathcal{T}_1}(w)^2}\right)\mathbf{y}^{e_{ij}} \\
&= h\left(\sqrt{h-2+(d_2+1)^2+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} - \sqrt{\sum_{i=1}^{h}(d_i+1)^2}\right)\mathbf{y}^{e_1}
\end{aligned}$$

$$+h\sum_{\substack{i=1,\\ i\neq 2}}^{h} d_i\left(\frac{\mathbf{y}^{e_{ij}}}{y_{w_i}}y_{w_1}\sqrt{h-1+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} - \mathbf{y}^{e_{ij}}\sqrt{h-1+(d_i+1)^2}\right).$$

Let us consider the case where $d_2 \ge 1$, that is Case 1. By the repeated application of inequality (2), we have

$$\sqrt{h-2+(d_2+1)^2+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} \ge \sqrt{\sum_{i=1}^{h}(d_i+1)^2}.$$

For any $3 \leq i \leq h$ with $d_i \geq 1$,

$$\frac{\mathbf{y}^{e_{ij}}}{y_{w_i}} y_{w_1} \sqrt{h-1+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} - \mathbf{y}^{e_{ij}} \sqrt{h-1+(d_i+1)^2}$$
$$\geq \left(\sqrt{h-1+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} - \sqrt{h-1+(d_i+1)^2}\right) \mathbf{y}^{e_{ij}} > 0.$$

Now consider the Case 2, that is when $d_2 = 0$. Since there exist at least one vertex w_i in the set $\{w_4, \ldots, w_r\}$ with $d_i \geq 1$, we have

$$\sqrt{h-2+(d_3+1)^2+\left(1+\sum_{\substack{t=1\\t\neq 2}}^{h} d_i\right)^2} - \sqrt{\sum_{i=1}^{h}(d_i+1)^2} \geq 0.$$

For any hyperedge e_{3j} with $1 \leq j \leq d_3$ of $\mathcal{T}_1$ and corresponding hyperedge in $\mathcal{T}_1'$,

$$\mathbf{y}^{e_{3j}} \frac{y_2}{y_3} \sqrt{h-1+(d_3+1)^2} - \mathbf{y}^{e_{3j}} \sqrt{h-1+(d_3+1)^2} \geq 0.$$

Also, for any hyperedge $e_{ij}, 4 \leq i \leq h, 1 \leq j \leq d_i$ where $d_i \geq 1$ (of $\mathcal{T}_1$) and for the corresponding hyperedge in $\mathcal{T}_1'$, we have

$$\mathbf{y}^{e_{ij}} \frac{y_1}{y_i} \sqrt{h-1+\left(1+d_1+\sum_{i=4}^{r} d_i\right)^2} - \mathbf{y}^{e_{ij}} \sqrt{h-1+(d_i+1)^2} \geq 0.$$

By combining the above two cases, we can conclude that $\rho_{\mathcal{SO}}(\mathcal{T}_1') > \rho_{\mathcal{SO}}(\mathcal{T}_1)$, a contradiction. Therefore, the hypertree $\mathcal{T}_1$ contains exactly one non-pendant hyperedge, and it contains exactly two non-pendant vertices, that is $\mathcal{T}_1$ is isomorphic to a double hyperstar. Finally by using Lemma 5, we conclude that $\mathcal{T}_1$ is isomorphic to $\mathcal{S}_{t_1+t_2-1,1}$. ■

4 Conclusion

In this work, we investigated the extremal behavior of the Sombor index and the Sombor spectral radius within the class of hypertrees. By developing structural insights and employing analytical techniques, we characterized the hypertrees that attain the maximum and second maximum values of both the Sombor index and the Sombor spectral radius. These findings not only extend the theory of Sombor-type invariants from graphs to hypertrees but also provide a deeper understanding of how hypergraph structure influences spectral and degree-based descriptors. The characterizations obtained in this paper may serve as a foundation for further studies on extremal problems in hypergraphs and could be useful in applications where such indices are used to model complex systems.

Future work may focus on generalizing these results to broader classes of hypergraphs, exploring additional variants of vertex-degree-based topological indices, or investigating relationships between spectral invariants and its structural properties.

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