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Degree-Ratio Sombor Index

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Abstract

The degree-ratio Sombor (DRSO) index is a recently introduced variant of the extensively studied Sombor index. In this paper, we establish an upper bound on the DRSO index in terms of the size, minimum degree, and maximum degree of a graph, which provides the unique graph maximizing the DRSO index among all fixed-order trees. Also, we give an upper bound for the DRSO index in terms of the order, size, and maximum degree of a graph, from which it follows that the star graph uniquely maximizes the DRSO index over the class of all fixed-order connected graphs. In addition, we rectify two results from [MATCH Commun. Math. Comput. Chem.

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97 (2027) 135–164], one related to a lower bound on the DRSO index and the other concerning extremal values of the DRSO index among all fixed-order connected graphs. We characterize the graphs that maximize the DRSO index among all fixed-order (i) unicyclic graphs, (ii) trees with a prescribed number of pendent vertices, and (iii) trees with a fixed matching number. Furthermore, we determine the structure of graphs that minimize the DRSO index among all fixed-order k-cyclic graphs, for $k \geq 1$ and sufficiently large values of the order.

1 Introduction

Topological indices are numerical invariants of graphs that capture structural properties and have found some applications in chemistry, particularly in quantitative structure-property and structure-activity relationships studies. Among these, degree-based indices have attracted considerable attention due to their simplicity and effectiveness. One of the notable additions to the family of degree-based indices is the Sombor index, introduced by Gutman [9] half a decade ago, which is defined for a graph G as

$$\mathcal{SO}(G) = \sum_{vu \in E(G)} \sqrt{d_v^2 + d_u^2},$$

where d_v and d_u denote the degrees of the vertices u and v, respectively, and $E(G)$ represents the edge set of G. Since its introduction, the Sombor index has been extensively investigated (for instance, see the surveys [11, 14, 17]), leading to numerous variants, including the elliptic Sombor index [13], Euler–Sombor index [10, 18], diminished Sombor index [15], augmented Sombor index [5], and hyperbolic Sombor index [4]; a comprehensive list of such Sombor-type indices can be found in the recent survey [11].

This paper is concerned with a recently proposed variant of the Sombor index, namely the degree-ratio Sombor (DRSO) index [16]. The DRSO index of a graph G is defined as

$$\mathcal{DRSO}(G) = \sum_{vu \in E(G)} \sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2}.$$

Due to its novelty, only limited results are currently available, and several fundamental problems regarding its extremal behavior remain open.

Extremal graph theory concerning topological indices focuses on characterizing graphs attaining extremum values of such indices over different classes of graphs. In this context, extremal results concerning the DRSO index are of particular interest. Some preliminary results along these lines were reported in [12] for trees, and in [16] for fixed-order connected graphs; however, those in [16] contains inaccuracies.

In this paper, we contribute to the study of the DRSO index in several directions. First, we establish an upper bound on the DRSO index in terms of the size, minimum degree, and maximum degree of a graph. As a consequence, we identify the unique graph that maximizes the DRSO index among all fixed-order trees. In addition, we give an upper bound for the DRSO index in terms of the order, size, and maximum degree of a graph, which implies that the star graph uniquely maximizes the DRSO index over the class of all fixed-order connected graphs. We also revisit two results from [16], correcting a previously stated lower bound on the DRSO index as well as the characterization of extremal graphs among all fixed-order connected graphs. Furthermore, we provide a complete characterization of graphs that maximize the DRSO index within the class of all fixed-order (i) unicyclic graphs, (ii) trees with a prescribed number of pendent vertices, and (iii) trees with a fixed matching number. Finally, we characterize graphs that attain the minimum DRSO index among all fixed-order k-cyclic graphs, for $k \geq 1$ and sufficiently large order.

2 Results

We first establish an upper bound for the DRSO index in terms of the size, minimum degree, and maximum degree of a graph, which provides the unique graph maximizing the DRSO index among all fixed-order trees.

Proposition 1. *If G is a graph with size m, minimum degree $\delta \geq 1$ and*

maximum degree Δ*, then*

$$\mathcal{DRSO}(G) \leq m\sqrt{\frac{\delta^2}{\Delta^2} + \frac{\Delta^2}{\delta^2}}. \tag{1}$$

The equality in (1) *holds if and only if one endvertex of every edge of* G *has degree* δ *and the other endvertex has degree* Δ*.*

Proof. Consider a function f defined on the closed interval $[\delta/\Delta, \Delta/\delta]$ as $f(t) = t^2 + t^{-2}$. Then, f attains its maximum value either at $t = \delta/\Delta$ or at $t = \Delta/\delta$. Hence, for any edge $vu \in E(G)$, it holds that

$$f\left(\frac{d_v}{d_u}\right) \leq f\left(\frac{\delta}{\Delta}\right) = f\left(\frac{\Delta}{\delta}\right),$$

which yields

$$\sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2} \leq \sqrt{\frac{\delta^2}{\Delta^2} + \frac{\Delta^2}{\delta^2}} \tag{2}$$

with equality if and only if $\min\{d_v, d_u\} = \delta$ and $\max\{d_v, d_u\} = \Delta$. Therefore, summing (2) over all edges of G yields the desired result. ■

The following result is a notable consequence of Proposition 1.

Corollary 1. *If* G *is a graph of minimum degree at least 1, size* m *and order* n*, then*

$$\mathcal{DRSO}(G) \leq m\sqrt{(n-1)^2 + \frac{1}{(n-1)^2}},$$

where the equality is attained if and only if G *is the star graph* S_n*.*

The set of all distinct elements of the degree sequence of a graph G is known as the degree set of G.

Next, we provide an upper bound on the DRSO index in terms of the order, size, and maximum degree of a graph, which gives the unique graph maximizing the DRSO index among all fixed-order connected graphs.

Proposition 2. *If G is a connected n-order graph with size m and maximum degree $\Delta \geq 2$, then*

$$\mathcal{DRSO}(G) \leq \frac{\sqrt{2}(m\Delta + m - n\Delta)}{\Delta - 1} + \frac{(n\Delta - 2m)\sqrt{\Delta^4 + 1}}{\Delta(\Delta - 1)}, \tag{3}$$

where the equality in (3) *holds if and only if the degree set of G is either* $\{1, \Delta\}$ *or* $\{\Delta\}$.

Proof. Let i, j and Δ be integers satisfying $1 \leq i \leq j \leq \Delta$ and $j \geq 2$ such that $(i, j) \notin \{(1, \Delta), (\Delta, \Delta)\}$. Then, the function $\Psi_{\mathcal{DRSO}}$ defined by

$$\Psi_{\mathcal{DRSO}}(i, j, \Delta) = \sqrt{\frac{i^2}{j^2} + \frac{j^2}{i^2}} + \frac{1}{\Delta - 1}\left(2 - \frac{\Delta}{i} - \frac{\Delta}{j}\right)\sqrt{\Delta^2 + \frac{1}{\Delta^2}} + \frac{\sqrt{2}}{\Delta - 1}\left(\frac{\Delta}{i} + \frac{\Delta}{j} - \Delta - 1\right),$$

is increasing in j, and hence, we have $\Psi_{\mathcal{DRSO}}(i, j, \Delta) \leq \Psi_{\mathcal{DRSO}}(i, \Delta, \Delta)$. However, $\Psi_{\mathcal{DRSO}}(i, \Delta, \Delta)$ is convex with respect to i, and so, for $i \leq \Delta - 1$,

$$\begin{aligned}\Psi_{\mathcal{DRSO}}(i, j, \Delta) &\leq \Psi_{\mathcal{DRSO}}(i, \Delta, \Delta) \\ &\leq \max\{\Psi_{\mathcal{DRSO}}(2, \Delta, \Delta), \Psi_{\mathcal{DRSO}}(\Delta - 1, \Delta, \Delta)\} < 0.\end{aligned}$$

Thus, the required conclusion follows from Theorem 1 of [1]. ■

Since the right-hand expression in (3), say $\Theta(n, m, \Delta)$, strictly decreases with respect to m for $\Delta \geq 3$, we have $\Theta(n, m, \Delta) \leq \Theta(n, n - 1, \Delta)$ whenever $\Delta \geq 3$ as $m \geq n - 1$. Also, the expression $\Theta(n, n - 1, \Delta)$ strictly increases with respect to $\Delta \geq 3$ as $\Delta \leq n - 1$. Consequently, we have $\Theta(n, m, \Delta) \leq \Theta(n, n - 1, n - 1)$ whenever $\Delta \geq 3$. It remains to consider the case where $\Delta = 2$. We recall that a connected n-order graph with maximum degree 2 is either the path graph P_n or the cycle graph C_n. However, it holds that $\mathcal{DRSO}(P_n) < \mathcal{DRSO}(C_n) < \mathcal{DRSO}(S_n)$ for every $n \geq 4$. Based on this discussion, Proposition 2 yields the following result.

Corollary 2. *If G is a connected n-order graph such that $n \geq 4$ then*

$$\mathcal{DRSO}(G) \leq \sqrt{(n-1)^4 + 1} \tag{4}$$

where the equality in (4) *holds if and only if G is the star graph S_n.*

The following erroneous inequality was inadvertently established in [16]:

$$\mathcal{DRSO}(G) \geq \frac{1}{\delta\Delta}\sqrt{\frac{\delta^2 + \Delta^2}{2}} \cdot \mathcal{SO}(G). \tag{5}$$

For instance, the n-order path graph P_n serves as a counterexample to (5) whenever $n \geq 4$. The next result provides a chain of inequalities, which gives a lower bound on the DRSO index in terms of the Sombor index and maximum degree.

Remark 1. The above error was independently noticed also by Dr. Aleksandar Petojević.

Proposition 3. *If G is a connected graph with maximum degree $\Delta \geq 1$, then*

$$\mathcal{DRSO}(G) \geq \frac{1}{\sqrt{2}} \cdot \mathcal{SDD}(G) \geq \frac{1}{\Delta} \cdot \mathcal{SO}(G) \geq \frac{1}{\sqrt{2}\Delta^2} \cdot \mathcal{F}(G), \tag{6}$$

where either of the equalities in (6) *holds if and only if G is regular. Here, $\mathcal{F}(G)$ is the forgotten index [8] and $\mathcal{SDD}(G)$ is the symmetric division deg index [19], which are defined as*

$$\mathcal{F}(G) = \sum_{uv \in E(G)} (d_v^2 + d_u^2) \quad \textit{and} \quad \mathcal{SDD}(G) = \sum_{uv \in E(G)} \frac{d_v^2 + d_u^2}{d_v d_u}.$$

Proof. The chain of inequalities (6) holds because for any edge $vu \in E(G)$, we have

$$\frac{1}{\sqrt{2}\Delta^2} \cdot (d_v^2 + d_u^2) \leq \frac{1}{\Delta} \cdot \sqrt{d_v^2 + d_u^2} \leq \frac{\sqrt{d_v^2 + d_u^2}}{\sqrt{2} d_v d_u} \cdot \sqrt{d_v^2 + d_u^2} \leq \frac{\sqrt{d_v^4 + d_u^4}}{d_v d_u}$$

where either of the first two equalities holds if and only if $d_v = d_u = \Delta$, whereas the last one holds if and only if $d_v = d_u$. ∎

Next, we provide two lemmas that are needed to prove the corrected form of Theorem 6 in [16]. The following result is due to Deng, Huang, and Jiang [6]; here, we provide its shorter proof.

Lemma 1. *Let G be an n-order graph containing no vertex of degree zero such that $n \geq 4$. Consider a graph invariant of the form $\sum_{vu \in E(G)} \phi(d_v, d_u)$, where ϕ is a symmetric non-negative function depending on d_v and d_u. Let $E^*(G) = \{uv \in E(G) : \{d_v, d_u\} \neq \{n-1, 1\}\}$. For any edge $vu \in E^*(G)$, define*

$$\Theta(d_v, d_u) := \phi(d_v, d_u) - \frac{(n-1)\,\phi(n-1,1)}{n}\left(\frac{1}{d_v} + \frac{1}{d_u}\right). \tag{7}$$

Then,

$$\sum_{vu \in E(G)} \phi(d_v, d_u) \begin{cases} \leq (n-1)\,\phi(n-1,1) & \text{if } \Theta(d_v, d_u) < 0 \ \forall\ vu \in E^*(G), \\ \geq (n-1)\,\phi(n-1,1) & \text{if } \Theta(d_v, d_u) > 0 \ \forall\ vu \in E^*(G), \end{cases}$$

where the equality in either case holds if and only if $E^*(G) = \emptyset$; that is, G is the star graph S_n.

Proof. It is a well-known fact that

$$n = \sum_{vu \in E(G)} \left(\frac{1}{d_v} + \frac{1}{d_u}\right),$$

(for example, see [7]) which yields

$$|E(G) \setminus E^*(G)| = n - 1 - \frac{n-1}{n} \sum_{vu \in E^*(G)} \left(\frac{1}{d_v} + \frac{1}{d_u}\right).$$

Hence, we have

$$\sum_{vu \in E(G)} \phi(d_v, d_u) = \sum_{vu \in E(G) \setminus E^*(G)} \phi(d_v, d_u) + \sum_{vu \in E^*(G)} \phi(d_v, d_u)$$

$$= \phi(n-1,1)\left[n - 1 - \frac{n-1}{n} \sum_{vu \in E^*(G)} \left(\frac{1}{d_v} + \frac{1}{d_u}\right)\right] + \sum_{vu \in E^*(G)} \phi(d_v, d_u)$$

$$= (n-1)\,\phi(n-1,1)$$

$$+ \sum_{vu \in E^*(G)} \left[\phi(d_v, d_u) - \frac{(n-1)\,\phi(n-1,1)}{n} \left(\frac{1}{d_v} + \frac{1}{d_u} \right) \right]. \tag{8}$$

Now, the required conclusion follows from (7) and (8). ■

Lemma 2. [16] *If G is a connected graph, then $\mathcal{DRSO}(G) \geq \sqrt{2}\ |E(G)|$, where the equality holds if and only if G is regular.*

Now, we are in a position to prove the corrected version of Theorem 6 in [16].

Theorem 4. *Among all n-order connected graphs only the star graph S_n (respectively, the path graph P_n) maximizes (respectively, minimizes) the DRSO index for every $n \geq 4$.*

Remark 2. Using computer-aided search, the maximality property of the star S_n was independently discovered, but not mathematically proven, by Dr. Izudin Redžepović.

Proof. In (7), substituting

$$\phi(d_v, d_u) = \sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2},$$

we obtain

$$\Theta(d_v, d_u) := \frac{n\sqrt{d_v^4 + d_v^4} - (d_v + d_v)\sqrt{(n-1)^4 + 1}}{n d_v d_u}.$$

We claim that the inequality $\Theta(d_v, d_u) < 0$ holds for every edge belonging to the set $E^*(G) = \{uv \in E(G) : \{d_v, d_u\} \neq \{n-1, 1\}\}$. We note that the inequality $\Theta(d_v, d_u) < 0$ is equivalent to

$$\frac{\sqrt{d_v^4 + d_u^4}}{d_v + d_u} < \frac{\sqrt{(n-1)^4 + 1}}{n},$$

which is evidently true for every $vu \in E^*(G)$ and $n \geq 4$. Hence, by

Lemma 1, we have

$$\mathcal{DRSO}(G) \leq \sqrt{(n-1)^4+1}$$

with equality if and only if G is the star graph S_n. (We remark here that the conclusion made in the previous sentence also follows from Corollary 2.)

It remains to prove that $\mathcal{DRSO}(G) \geq \mathcal{DRSO}(P_n)$ with equality if and only if G is P_n. If $|E(G)| \geq n$ then from Lemma 2 it follows that $\mathcal{DRSO}(G) \geq \sqrt{2}n$, with equality if and only if $|E(G)| = n$ and G is regular; that is, G is the cycle graph C_n. However, $\mathcal{DRSO}(C_n) > \mathcal{DRSO}(P_n)$ for every $n \geq 3$. Therefore, $\mathcal{DRSO}(G) > \mathcal{DRSO}(P_n)$ whenever $|E(G)| \geq n$.

In the rest of the proof, we suppose that $|E(G)| = n-1$. Then, G is a tree. Let p be the number of pendent vertices of G. For $\min\{d_v, d_u\} \geq 2$, we note that

$$\sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2} \geq \sqrt{2} \tag{9}$$

and

$$\sqrt{d_v^2 + \frac{1}{d_v^2}} \geq \frac{\sqrt{17}}{2}, \tag{10}$$

where the equality in (9) (respectively, in (10)) holds if and only if $d_v = d_u$ (respectively, $d_v = 2$). Therefore, we have

$$\begin{aligned}\mathcal{DRSO}(G) = &\sum_{vu \in E(G);\ \min\{d_v,d_u\} \geq 2} \sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2} \\ &+ \sum_{vu \in E(G);\ d_u = 1} \sqrt{d_v^2 + \frac{1}{d_v^2}} \\ &\geq \sqrt{2}(n-p-1) + \frac{\sqrt{17}}{2}p \geq \sqrt{2}(n-3) + \sqrt{17}.\end{aligned}$$

We observe that $\mathcal{DRSO}(G) = \sqrt{2}(n-3) + \sqrt{17}$ holds if and only if G is the path graph P_n.

■

Remark 3. Theorem 4 is straightforwardly extended to the case of general n-order graphs (both connected and disconnected): the star S_n (respectively, the edge-less graph $\overline{K_n}$) maximizes (respectively, minimizes) the DRSO index for every $n \geq 4$.

The next three theorems are direct consequences of existing results.

Theorem 5. *Among all n-order trees with p pendent vertices, the graph deduced from the $(n-p+1)$-order path graph P_{n-p+1} by attaching $p-1$ pendent vertices to exactly one pendent vertex of P_{n-p+1}, uniquely maximizes the DRSO index for every $n \geq 4$.*

Proof. The result follows from Theorem 4.1 of [3]. ■

Theorem 6. *Among all n-order trees with matching number ℓ, the graph deduced from the $(n-\ell+1)$-order star graph $S_{n-\ell+1}$ by subdividing its $\ell-1$ edges uniquely maximizes the DRSO index for every $n \geq 4$.*

Proof. The result follows from Theorem 3.3 of [3]. ■

A connected graph containing exactly one cycle is called a unicylic graph. Let S_n^+ denote the n-order unicyclic graph of maximum degree $n-1$.

Theorem 7. *Among all n-order unicyclic graphs, only the graph S_n^+ (respectively, the cycle graph C_n) maximizes (respectively, minimizes) the DRSO index for every $n \geq 4$.*

Proof. The conclusion concerning C_n follows from Lemma 2. The claim regarding S_n^+ for $n \geq 6$ follows from Theorem 2.2 of [2]. For $n = 4$, there exist only two non-isomorphic n-order unicyclic graphs, namely S_n^+ and C_n. Also, for $n = 5$, all distinct (up to isomorphism) n-order unicyclic graphs, together with their associated DRSO indices, are shown in Figure 1. ■

A connected n-order graph of size $n+1$ is known as an n-order bicyclic graph. In order to establish a result concerning the minimum DRSO index of fixed-order bicyclic graphs, we require the following proposition.

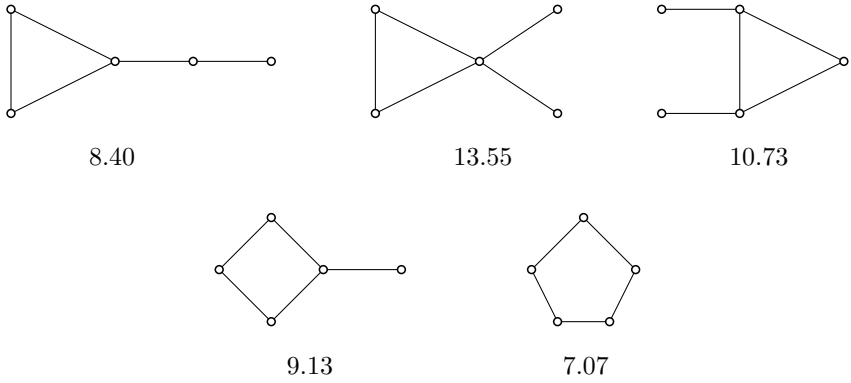

Figure 1. All distinct (up to isomorphism) unicyclic graphs of order 5, together with their associated DRSO indices, accurate to two decimal places.

Proposition 8. *Consider a connected graph G having $m \geq 2$ edges, among which exactly p are pendent edges. Then the following inequality holds:*

$$\mathcal{DRSO}(G) \geq \sqrt{2}\, m + \left(\frac{\sqrt{17}}{2} - \sqrt{2} \right) p. \tag{11}$$

The equality in (11) *is attained if and only if each pendent edge (if any) is incident to a vertex of degree* 2*, and every non-pendent edge joins two vertices of equal degree.*

Proof. Using (9) and (10), we have

$$\begin{aligned} \mathcal{DRSO}(G) &= \sum_{vu \in E(G);\ \min\{d_v, d_u\} \geq 2} \sqrt{\left(\frac{d_v}{d_u}\right)^2 + \left(\frac{d_u}{d_v}\right)^2} \\ &\quad + \sum_{vu \in E(G);\ d_u = 1} \sqrt{d_v^2 + \frac{1}{d_v^2}} \\ &\geq \sqrt{2}(m-p) + \frac{\sqrt{17}}{2}\, p. \end{aligned}$$

where the equality holds if and only if G satisfies the constraints mentioned in the statement of the proposition. ∎

For a graph G, let $m_{i,j}(G)$ (which is the same as $m_{j,i}(G)$) denote the number of elements of the set $\{vu \in E(G) : d_v = i, d_u = j\}$.

The following theorem provides the graphs attaining the minimum DRSO index among all fixed-order bicyclic graphs.

Theorem 9. *Let G be a bicyclic n-order graph such that $n \geq 5$. Then,*

$$\mathcal{DRSO}(G) \geq \sqrt{2}\, n + \frac{2\sqrt{97}}{3} - 3\sqrt{2}. \tag{12}$$

The equality in (12) *holds if and only if G is isomorphic either to a graph formed by inserting an edge in the cycle C_n or to a graph constructed by joining two vertex-disjoint cycles with a single edge.*

Proof. Let p be the number of pendent edge(s) of G.

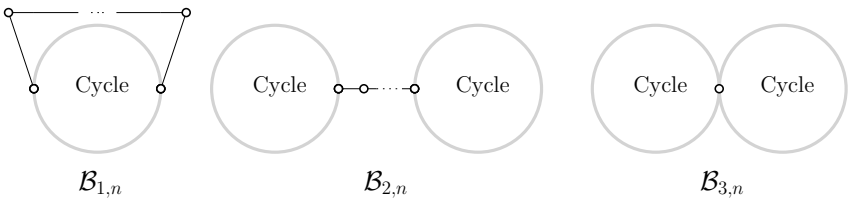

Figure 2. The structural representations of the graphs in the classes $\mathcal{B}_{1,n}$, $\mathcal{B}_{2,n}$ and $\mathcal{B}_{3,n}$ used in the proof of Theorem 9.

First, we consider the case wehere $p = 0$. Then, $G \in \cup_{i=1}^{3} \mathcal{B}_{i,n}$, where $\mathcal{B}_{1,n}$ (respectively, $\mathcal{B}_{2,n}$ (respectively, $\mathcal{B}_{3,n}$)) is the class of all n-order bicyclic graphs formed by inserting a path between two vertices of the cycle C_n (respectively, by joining two vertex-disjoint cycles with a path (respectively, by joining two vertex-disjoint cycles through a common vertex)); see Figure 2. We note that $m_{2,3}(G) + m_{2,4}(G) \geq 4$, and hence, we have

$$\mathcal{DRSO}(G) \geq \sqrt{2}\,\big(|E(G)| - 4\big) + 4 \cdot \frac{\sqrt{97}}{6},$$

with equality if and only if $m_{2,3}(G) = 4$, as required.

Next, we consider the case where $p = 1$. Then, G is obtained from one of the structures depicted in Figure 2 by attaching a pendent path, and hence, the maximum degree of G is not more than 5. Consequently, we

have $m_{1,i}(G) = 1$ and $m_{2,j}(G) \geq 2$ for some i and j satisfying $2 \leq i \leq 5$ and $3 \leq j \leq 5$, and thus, the desired inequality follows from

$$\mathcal{DRSO}(G) \geq \sqrt{2}\left(|E(G)| - 3\right) + 2 \cdot \frac{\sqrt{97}}{6} + \frac{\sqrt{17}}{2} > \sqrt{2}\, n + \frac{2\sqrt{97}}{3} - 3\sqrt{2}.$$

Finally, we consider the case where $p \geq 2$. Then, Proposition 8 yields

$$\mathcal{DRSO}(G) \geq \sqrt{2}\, n + \sqrt{17} - \sqrt{2} > \sqrt{2}\, n + \frac{2\sqrt{97}}{3} - 3\sqrt{2},$$

as required. ■

For any non-negative integer k, a connected graph G satisfying $|E(G)| = |V(G)| + k - 1$ is called a k-cyclic graph. In order to characterize graphs that attain the minimum DRSO index among all n-order k-cyclic graphs, for $n \geq 2(k-1) \geq 4$, we need the following lemma.

Lemma 3. *Let G be a graph with minimum DRSO index among all k-cyclic n-order graphs, provided that $n \geq 2(k-1) \geq 4$. Then, G contains no pendent edge.*

Proof. First, we consider the case where $n = 2(k-1) \geq 4$. In this situation, there exists at least one 3-regular k-cyclic graph of order $2(k-1)$ for every $k \geq 3$. Also, there is no ℓ-regular k-cyclic graph of order $2(k-1)$ for any $k \geq 3$ and $\ell \neq 3$. Hence, by Proposition 8, the graphs that minimize the DRSO index over the class of all k-cyclic graphs of order $2(k-1)$ are precisely the 3-regular ones. Consequently, G has no pendent edge.

In what follows, we assume that $n > 2(k-1)$. Let $G^{\dagger}$ be the k-cyclic n-order graph deduced from a 3-regular k-cyclic graph with $2(k-1)$ vertices by inserting $n - 2(k-1)$ vertices of degree 2 along a single edge. Then,

$$m_{2,2}(G^{\dagger}) = n - 2k + 1, \quad m_{3,3}(G^{\dagger}) = 3k - 4, \quad m_{2,3}(G^{\dagger}) = 2.$$

Suppose, for contradiction, that G contains $p \geq 1$ pendent edge(s). Then, by Proposition 8, we have

$$\mathcal{DRSO}(G) \geq \sqrt{2}(n+k-2) + \frac{\sqrt{17}}{2} > \sqrt{2}(n+k-3) + \frac{\sqrt{97}}{3} = \mathcal{DRSO}(G^{\dagger}),$$

which contradicts the definition of G. Therefore, we must have $p = 0$, completing the proof. ■

The following result characterizes the graphs that attain the minimum DRSO index among all n-order k-cyclic graphs, for $n \geq 2(k-1) \geq 4$.

Theorem 10. *For any k-cyclic n-order graph G with $n \geq 2(k-1) \geq 4$,*

$$\mathcal{DRSO}(G) \geq \begin{cases} 3\sqrt{2}(k-1) & \text{when } n = 2k-2, \\ \sqrt{2}(n+k) + \frac{\sqrt{97}}{3} - 3\sqrt{2} & \text{when } n > 2k-2. \end{cases} \tag{13}$$

The equality in (13) *holds if and only if G is isomorphic either to a* 3*-regular graph (in the case $n = 2k-2$) or to the graph $G^{\dagger}$ (when $n > 2k-2$) as constructed in the proof of Lemma* 3*.*

Proof. Let G^* be a k-cyclic n-order graph that minimizes the DRSO index among all such graphs, where $n \geq 2(k-1) \geq 4$. Then

$$\mathcal{DRSO}(G) \geq \mathcal{DRSO}(G^*). \tag{14}$$

Also, by Lemma 3, we have $m_{1,j}(G^*) = 0$ for every $j \in \{2, 3, \ldots, n-1\}$. For simplicity, we omit the explicit reference to G^* and denote $m_{i,j}(G^*)$ by $m_{i,j}$ in the remainder of the proof. Moreover, we define

$$A := \big\{(i,j) : 2 \leq i \leq j \leq n-1 \text{ and } i, j \in \mathbb{Z}_+\big\},$$

where $\mathbb{Z}_+$ is the set of positive integers. Here, $n = \sum_{(i,j)\in A} m_{i,j} \left(\frac{1}{i} + \frac{1}{j}\right)$, which yields

$$\sum_{(i,j)\in A^*} m_{i,j} \left(\frac{1}{i} + \frac{1}{j}\right) = n - \frac{2}{3}m_{3,3} - \frac{5}{6}m_{2,3} - m_{2,2}, \tag{15}$$

where $A^* := A \setminus \{(2,2), (3,3), (2,3)\}$. Also, $n + k - 1 = \sum_{(i,j)\in A} m_{i,j}$ implies

$$\sum_{(i,j)\in A^*} m_{i,j} = n + k - m_{2,3} - m_{3,3} - m_{2,2} - 1. \tag{16}$$

Upon solving (15) and (16) for the variables $m_{3,3}$ and $m_{2,3}$, we obtain

$$m_{2,3} = 2(n - m_{2,2} - 2k + 2) + \sum_{(i,j)\in A^*} \left(4 - \frac{6}{i} - \frac{6}{j}\right) m_{i,j}. \tag{17}$$

and

$$m_{3,3} = m_{2,2} - n + 5(k-1) + \sum_{(i,j)\in A^*} \left(\frac{6}{i} + \frac{6}{j} - 5\right) m_{i,j} \tag{18}$$

Substituting (17) and (18) into the expression for $\mathcal{DRSO}(G^*)$, we have

$$\begin{aligned}
\mathcal{DRSO}(G^*) &= \sum_{(i,j)\in A} m_{i,j} \sqrt{\frac{i^2}{j^2} + \frac{j^2}{i^2}} \\
&= \sqrt{2}(m_{3,3} + m_{2,2}) + \frac{\sqrt{97}}{6} m_{2,3} + \sum_{(i,j)\in A^*} m_{i,j} \sqrt{\frac{i^2}{j^2} + \frac{j^2}{i^2}} \\
&= \sqrt{2} m_{2,2} + \sqrt{2} \left[m_{2,2} - n + 5(k-1) + \sum_{(i,j)\in A^*} \left(\frac{6}{i} + \frac{6}{j} - 5\right) m_{i,j} \right] \\
&+ \frac{\sqrt{97}}{6} \left[2(n - m_{2,2} - 2k + 2) + \sum_{(i,j)\in A^*} \left(4 - \frac{6}{i} - \frac{6}{j}\right) m_{i,j} \right] \\
&+ \sum_{(i,j)\in A^*} m_{i,j} \sqrt{\frac{i^2}{j^2} + \frac{j^2}{i^2}} \\
&= \left(2\sqrt{2} - \frac{\sqrt{97}}{3}\right) m_{2,2} + \left(\frac{\sqrt{97}}{3} - \sqrt{2}\right) n + \left(5\sqrt{2} - \frac{2\sqrt{97}}{3}\right)(k-1) \\
&+ \sum_{(i,j)\in A^*} \underbrace{\left[\sqrt{\frac{i^2}{j^2} + \frac{j^2}{i^2}} + \left(6\sqrt{2} - \sqrt{97}\right)\left(\frac{1}{i} + \frac{1}{j}\right) + \frac{2\sqrt{97}}{3} - 5\sqrt{2} \right]}_{\Psi(i,j)} m_{i,j}
\end{aligned}$$

$$= \left(\underbrace{2\sqrt{2} - \frac{\sqrt{97}}{3}}_{\text{negative}} \right) m_{2,2} + \left(\frac{\sqrt{97}}{3} - \sqrt{2} \right) n + \left(5\sqrt{2} - \frac{2\sqrt{97}}{3} \right) (k-1). \tag{19}$$

Here, we note that equality in (19) holds by the definition of G^*, by Table 1, and by the following chain of inequalities; this chain holds for all i and j satisfying $j \geq i \geq 3$ with $j > 3$, and also for $i = 2$ with $j > 6$:

$$\begin{aligned} \Psi(i,j) &\geq \sqrt{2} + \left(6\sqrt{2} - \sqrt{97}\right)\left(\frac{1}{i} + \frac{1}{j}\right) + \frac{2\sqrt{97}}{3} - 5\sqrt{2} \\ &= \frac{1}{3ij}\left[\left(\sqrt{97} - 6\sqrt{2}\right)\big(j(i-3) + i(j-3)\big)\right] > 0. \end{aligned}$$

Table 1. Some approximate values of function $\Psi(i,j)$, defined in (19).

(i,j)	$(2,4)$	$(2,5)$	$(2,6)$
$\Psi(i,j)$	0.5337	1.0721	1.6043

Consequently, $m_{i,j} = 0$ for each $(i,j) \in A^*$. Therefore, it follows that $(\delta(G^*), \Delta(G^*)) \in \{(2,3),(3,3)\}$, where $\delta(G^*)$ and $\Delta(G^*)$ denote the minimum degree and maximum degree of G^*, respectively.

For $n = 2(k-1)$, we have $(\delta(G^*), \Delta(G^*)) = (3,3)$, and hence, we are done. Next, in what follows, we suppose that $n > 2(k-1)$. Then, $(\delta(G^*), \Delta(G^*)) = (2,3)$. Since G^* is connected, $m_{2,3}$ is necessarily positive. From the relation

$$2m_{2,2} + m_{2,3} = 2n_2,$$

it follows that $m_{2,3}$ must be a positive even integer. In particular, this implies

$$2m_{2,2} + 2 \leq 2m_{2,2} + m_{2,3} = 2n_2 = 2\big(n - 2(k-1)\big),$$

which yields the bound

$$m_{2,2} \leq n - 2k + 1,$$

with equality if and only if $m_{2,3} = 2$. Furthermore, taking into account the definition of G^* and using (19), we deduce that $m_{2,2} = n - 2k + 1$. Consequently, $m_{2,3} = 2$, and hence it follows that $m_{3,3} = 3k-4$. Therefore, we have

$$\mathcal{DRSO}(G^*) = \sqrt{2}(n + k) + \frac{\sqrt{97}}{3} - 3\sqrt{2},$$

which, together with (14), yields the required conclusion. ■

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