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# Hybrid Soft-Neutrosophic Controlled Metric Spaces with Applications to Nonlinear Chemical Reaction Networks

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## Abstract

In this paper, we introduce a novel mathematical framework called Hybrid Soft-Neutrosophic Controlled Metric Space (HSNCMS), which integrates soft set theory, neutrosophic sets, and controlled metric spaces. We develop auxiliary lemmas and establish a Banach-type fixed point theorem with complete proof for HSNCMS

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and introduce the concept of T-controlled contraction with related results. Furthermore, as an application we prove the existence and uniqueness of equilibrium concentration profiles for a nonlinear chemical reaction network, where uncertainty arises from fluctuations in temperature, pressure, catalyst effects, and incomplete experimental data. The proposed model generalizes several existing structures and provides a flexible tool for handling uncertainty, indeterminacy, and parameterization simultaneously.

# 1 Introduction

Fixed point theory started with Banach [4] back in 1922. He came up with the contraction principle, which turned out to be really important for nonlinear analysis. Since 1922, the result has been extended and improved in different ways. Branciari [6] developed a new type of metric space without triangle inequality and Das [7, 8] with Lahri [9, 10] developed several fixed point results in these structures. Azam and Arshad [2] derived Kannan-type fixed point results, and later Beg [3] extended Banach's idea to cone rectangular structure.

Zadeh [30] introduced the new idea of fuzzy sets and this laid the foundation that how we use mathematics in uncertainty. Schweizer and Sklar [25] established the foundation for statistical metric spaces, and Kramosil and Michálek [17] derived fuzzy metric spaces. Grabiec [12] proved fixed point results in these structures. Then Fang [11], as well as Mishra, Sharma, and Singh [20], established different useful results. Mihet [19] proved a Banach-type contraction result in fuzzy structure, and Park [22] introduced the theme of intuitionistic fuzzy metric spaces. Romaguera, Sapena, and Tirado [23] applied these ideas to world domains, while Abbas, Altun, and Gopal [1] established results for non-compatible mappings.

Mlaiki [21] introduced controlled metric type spaces by relaxing the triangle inequality by using a control function. Sezen [26] used this concept in fuzzy structures and developed controlled fuzzy metric spaces. Saleem et al. [24] derived fuzzy double controlled metric spaces. Konwar [16] extended the results to intuitionistic fuzzy b-metric spaces.

Kirişci and Şimşek [14] were the first to define neutrosophic metric spaces and later established fixed point results [15]. Simsek and Kirişci [27] also derived fixed point results in this framework. Sowndrarajan et al. [28] derived contraction theorems as well. Bera and Mahapatra [5] and Mehmood et al. [18], explored soft neutrosophic metric spaces. This work aligns soft set theory with neutrosophic logic.

Motivated by all these developments, we introduce in this paper what we call Hybrid Soft-Neutrosophic Controlled Metric Spaces. This new framework brings together soft set theory, neutrosophic logic, and controlled metric structures into one unified setting. Our goal is to handle complex uncertainty in decision-making problems where truth, indeterminacy, and falsity all appear together along with parameterized soft information. We prove some basic fixed point theorems in this new setting and show how they can be used in multi-attribute decision-making. Our work here extends and brings together several existing ideas from the literature [5, 14, 18, 21, 26, 29].

## 2 Preliminaries

**Definition 1.** [13] Let $\triangle$ be a universal set and $\eta$ a set of parameters. A pair $(\mathcal{F}, \mathcal{A})$ is called soft set over $\triangle$ where $A \subseteq \eta$ and $\mathcal{F} : \mathcal{A} \to \mathcal{P}(\triangle)$.

**Definition 2.** A neutrosophic set assigns to each $\mathfrak{b} \in \triangle$ three values $T(\mathfrak{b}), I(\mathfrak{b}), F(\mathfrak{b}) \in [0, 1]$.

**Definition 3.** [21] A controlled metric space is a triple $(\triangle, \zeta, \alpha)$ where $\zeta : \triangle \times \triangle \to [0, \infty)$ and $\alpha : \triangle \times \triangle \to [1, \infty)$ such that:

1. $\zeta(\mathfrak{b}, \mathfrak{e}) = 0$ iff $\mathfrak{b} = \mathfrak{e}$
2. $\zeta(\mathfrak{b}, \mathfrak{e}) = \zeta(\mathfrak{e}, \mathfrak{b})$
3. $\zeta(\mathfrak{b}, \mathfrak{y}) \leq \alpha(\mathfrak{b}, \mathfrak{e})\zeta(\mathfrak{b}, \mathfrak{e}) + \alpha(\mathfrak{e}, \mathfrak{y})\zeta(\mathfrak{e}, \mathfrak{y})$.

**Definition 4.** [15] A binary operation $\circ : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous triangle norm (CTN), if it holds the following conditions:

1. $\mathfrak{b} \circ 1 = \mathfrak{b}$ for all $\mathfrak{b} \in [0, 1]$;

2. $\circ$ is continuous;
3. $\circ$ is associative and commutative;
4. $\mathfrak{b} \circ \mathfrak{e} \leq \mathfrak{o} \circ \mathfrak{s}$ whenever $\mathfrak{b} \leq \mathfrak{o}$ and $\mathfrak{e} \leq \mathfrak{s}$ for all $\mathfrak{b}, \mathfrak{e}, \mathfrak{o}, \mathfrak{s} \in [0, 1]$.

An example of CTN is $\mathfrak{b} \circ \mathfrak{e} = \min\{\mathfrak{b}, \mathfrak{e}\}$.

**Definition 5.** [15] A binary operation $\bullet : [0, 1] \times [0, 1] \to [0, 1]$ is continuous triangle conorm (CTCN), if it holds the below conditions:

1. $\mathfrak{b} \bullet 0 = \mathfrak{b} \ \forall \ \mathfrak{b} \in [0, 1]$;
2. $\bullet$ is continuous;
3. $\bullet$ is commutative and associative;
4. $\mathfrak{b} \bullet \mathfrak{e} \leq \mathfrak{o} \bullet \mathfrak{s}$ whenever $\mathfrak{b} \leq \mathfrak{o}$ and $\mathfrak{e} \leq \mathfrak{s} \ \forall \ \mathfrak{b}, \mathfrak{e}, \mathfrak{o}, \mathfrak{s} \in [0, 1]$.

$\mathfrak{b} \bullet \mathfrak{e} = \max\{\mathfrak{b}, \mathfrak{e}\}$ is an example of CTCN.

**Definition 6.** [15] Let $\nabla \neq \emptyset$ set. Let $\circ$ be a $t$-norm and $\bullet$ be a $t$-conorm. A neutrosophic metric on $\nabla$ is a function

$$\mathcal{N} : \nabla \times \nabla \times \mathbb{R}^+ \to [0, 1]^3$$

given by

$$\mathcal{N}(\mathfrak{b}, \mathfrak{e}, \lambda) = (\ell(\mathfrak{b}, \mathfrak{e}, \lambda), \Im(\mathfrak{b}, \mathfrak{e}, \lambda), \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda))$$

where $\ell, \Im, \mathfrak{X}$ satisfy for all $\mathfrak{b}, \mathfrak{e}, \mathfrak{y} \in \nabla$:

1. $0 \leq \ell, \Im, \mathfrak{X} \leq 1$,
2. $\ell + \Im + \mathfrak{X} \leq 3$,
3. $\ell(\mathfrak{b}, \mathfrak{e}, \lambda) = 1$ iff $\mathfrak{b} = \mathfrak{e}$,
4. $\ell(\mathfrak{b}, \mathfrak{e}, \lambda) = \ell(\mathfrak{e}, \mathfrak{b}, \lambda)$,
5. $\ell(\mathfrak{b}, \mathfrak{e}, \lambda) \circ \ell(\mathfrak{e}, \mathfrak{y}, \mu) \leq \ell(\mathfrak{b}, \mathfrak{y}, \lambda + \mu)$,
6. $\lim_{\lambda \to \infty} \ell(\mathfrak{b}, \mathfrak{e}, \lambda) = 1$,

7. $\Im(\mathfrak{b}, \mathfrak{e}, \lambda) = 0$ iff $\mathfrak{b} = \mathfrak{e}$,
8. $\Im(\mathfrak{b}, \mathfrak{e}, \lambda) = \Im(\mathfrak{e}, \mathfrak{b}, \lambda)$,
9. $\Im(\mathfrak{b}, \mathfrak{e}, \lambda) \bullet \Im(\mathfrak{e}, \mathfrak{y}, \mu) \geq \Im(\mathfrak{b}, \mathfrak{y}, \lambda + \mu)$,
10. $\lim_{\lambda \to \infty} \Im(\mathfrak{b}, \mathfrak{e}, \lambda) = 0$,
11. $\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = 0$ iff $\mathfrak{b} = \mathfrak{e}$,
12. $\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = \mathfrak{X}(\mathfrak{e}, \mathfrak{b}, \lambda)$,
13. $\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) \bullet \mathfrak{X}(\mathfrak{e}, \mathfrak{y}, \mu) \geq \mathfrak{X}(\mathfrak{b}, \mathfrak{y}, \lambda + \mu)$,
14. $\lim_{\lambda \to \infty} \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = 0$,
15. if $\lambda \leq 0$, then $\ell = 0, \Im = 1, \mathfrak{X} = 1$.

We interpret $\ell(\mathfrak{b}, \mathfrak{e}, \lambda)$, $\Im(\mathfrak{b}, \mathfrak{e}, \lambda)$, and $\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda)$ as the truth, indeterminacy, and falsity degrees associated with the nearness of $\mathfrak{b}$ and $\mathfrak{e}$ at the level $\lambda$.

# 3 Main results

**Definition 7.** Let $\triangle \neq \emptyset$ set and $E$ be a parameter set. Let $\circ$ be a $t$-norm and $\bullet$ be a $t$-conorm. A *Hybrid Soft-Neutrosophic Controlled Metric Space (HSNCMS)* is a structure

$$(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet),$$

where

a. $F : E \times \triangle \to [0, 1]^3$ is a hybrid soft-neutrosophic mapping,

b. $\mathcal{N} : \triangle \times \triangle \times \mathbb{R}^+ \to [0, 1]^3$ is a neutrosophic metric,

c. $\alpha : \triangle \times \triangle \to [1, \infty)$ is a control function.

For all $\mathfrak{b}, \mathfrak{e}, \mathfrak{y} \in \triangle$ and $\lambda, \mu > 0$, the following conditions hold:

1. $\mathcal{N}(\mathfrak{b}, \mathfrak{e}, \lambda) = (1, 0, 0)$ if and only if $\mathfrak{b} = \mathfrak{e}$.

2. $\mathcal{N}(\mathfrak{b},\mathfrak{e},\lambda)=\mathcal{N}(\mathfrak{e},\mathfrak{b},\lambda)$ (symmetry).

3. The controlled triangle inequalities:

$$\begin{aligned}
\ell\left(\mathfrak{b},\mathfrak{e},\frac{\lambda}{\alpha(\mathfrak{b},\mathfrak{e})}\right)\circ\ell\left(\mathfrak{e},\mathfrak{y},\frac{\mu}{\alpha(\mathfrak{e},\mathfrak{y})}\right)&\leq\ell(\mathfrak{b},\mathfrak{y},\lambda+\mu),\\
\Im\left(\mathfrak{b},\mathfrak{e},\frac{\lambda}{\alpha(\mathfrak{b},\mathfrak{e})}\right)\bullet\Im\left(\mathfrak{e},\mathfrak{y},\frac{\mu}{\alpha(\mathfrak{e},\mathfrak{y})}\right)&\geq\Im(\mathfrak{b},\mathfrak{y},\lambda+\mu),\\
\mathfrak{X}\left(\mathfrak{b},\mathfrak{e},\frac{\lambda}{\alpha(\mathfrak{b},\mathfrak{e})}\right)\bullet\mathfrak{X}\left(\mathfrak{e},\mathfrak{y},\frac{\mu}{\alpha(\mathfrak{e},\mathfrak{y})}\right)&\geq\mathfrak{X}(\mathfrak{b},\mathfrak{y},\lambda+\mu).
\end{aligned}$$

**Example 1.** Let $\triangle=\mathbb{R}$ and $E=\{e_1,e_2\}$ be a parameter set. Define the structure $(\triangle,\mathcal{N},\alpha,F,\circ,\bullet)$ as follows. Define $F:E\times\triangle\to[0,1]^3$ by

$$F(e,\mathfrak{p})=\left(\frac{|\mathfrak{p}|}{1+|\mathfrak{p}|},\frac{1}{2(1+|\mathfrak{p}|)},\frac{1}{2(1+|\mathfrak{p}|)}\right).$$

Define $\mathcal{N}:\triangle\times\triangle\times\mathbb{R}^+\to[0,1]^3$ by

$$\begin{aligned}
\mathcal{N}(\mathfrak{b},\mathfrak{e},\lambda)=[&\ell(\mathfrak{b},\mathfrak{e},\lambda),\\
&\Im(\mathfrak{b},\mathfrak{e},\lambda),\\
&\mathfrak{X}(\mathfrak{b},\mathfrak{e},\lambda)],
\end{aligned}$$

where

$$\begin{aligned}
\ell(\mathfrak{b},\mathfrak{e},\lambda)&=\frac{\lambda}{\lambda+|\mathfrak{b}-\mathfrak{e}|},\\
\Im(\mathfrak{b},\mathfrak{e},\lambda)&=\frac{|\mathfrak{b}-\mathfrak{e}|}{2(\lambda+|\mathfrak{b}-\mathfrak{e}|)},\\
\mathfrak{X}(\mathfrak{b},\mathfrak{e},\lambda)&=\frac{|\mathfrak{b}-\mathfrak{e}|}{2(\lambda+|\mathfrak{b}-\mathfrak{e}|)}.
\end{aligned}$$

and $\alpha(\mathfrak{b},\mathfrak{e})=1+|\mathfrak{b}|+|\mathfrak{e}|$ for all $\mathfrak{b},\mathfrak{e}\in\triangle$, with $\mathfrak{b}\circ\mathfrak{e}=\mathfrak{b}\mathfrak{e}$, $\mathfrak{b}\bullet\mathfrak{e}=\max\{\mathfrak{b},\mathfrak{e}\}$.

(i) If $\mathfrak{b}=\mathfrak{e}$, then $|\mathfrak{b}-\mathfrak{e}|=0$, hence $\mathcal{N}(\mathfrak{b},\mathfrak{e},\lambda)=(1,0,0)$. Conversely, $\mathcal{N}(\mathfrak{b},\mathfrak{e},\lambda)=(1,0,0)$ implies $\mathfrak{b}=\mathfrak{e}$.

(ii) Since $|\mathfrak{b}-\mathfrak{e}|=|\mathfrak{e}-\mathfrak{b}|$, we have $\mathcal{N}(\mathfrak{b},\mathfrak{e},\lambda)=\mathcal{N}(\mathfrak{e},\mathfrak{b},\lambda)$.

(iii) For all $\mathfrak{b},\mathfrak{e},\mathfrak{y}\in\triangle$ and $\lambda,\mu>0$, clearly the controlled triangle inequalities hold. Thus, all conditions of a HSNCMS are satisfied.

**Definition 8.** Let $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ be a HSNCMS, where $\mathcal{N}(\mathfrak{b}, \mathfrak{e}, \lambda) = (\ell(\mathfrak{b}, \mathfrak{e}, \lambda), \Im(\mathfrak{b}, \mathfrak{e}, \lambda), \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda))$, and let $\{\mathfrak{b}_p\}$ be a sequence in $\triangle$. The sequence $\{\mathfrak{b}_p\}$ is convergent to $\mathfrak{b} \in \triangle$ if $\forall\ \lambda > 0$

$$\lim_{p\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}, \lambda) = 1, \quad \lim_{p\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}, \lambda) = 0, \quad \lim_{p\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}, \lambda) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\lambda > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ p \geq N$

$$\ell(\mathfrak{b}_p, \mathfrak{b}, \lambda) > 1 - \varepsilon, \quad \Im(\mathfrak{b}_p, \mathfrak{b}, \lambda) < \varepsilon, \quad \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}, \lambda) < \varepsilon.$$

**Definition 9.** Let $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ be a HSNCMS and let $\{\mathfrak{b}_p\} \in \triangle$. The sequence $\{\mathfrak{b}_p\}$ is **Cauchy** if $\forall\ \lambda > 0$

$$\lim_{p,r\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) = 1, \quad \lim_{p,r\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) = 0, \quad \lim_{p,r\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\lambda > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ p, r \geq N$:

$$\ell(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) > 1 - \varepsilon, \quad \Im(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) < \varepsilon, \quad \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) < \varepsilon.$$

**Definition 10.** A HSNCMS $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ is complete if every Cauchy sequence in $\triangle$ converges in $\triangle$.

**Theorem 1.** *Let $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ be a complete HSNCMS. Suppose the control function $\alpha : \triangle \times \triangle \to [1, \infty)$ satisfies*

$$\mathcal{M} = \sup\{\alpha(\mathfrak{b}, \mathfrak{e}) : \mathfrak{b}, \mathfrak{e} \in \triangle\} < \infty.$$

*Let $T : \triangle \to \triangle$ be a mapping such that for all $\mathfrak{b}, \mathfrak{e} \in \triangle$ and for all $\lambda > 0$,*

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) \geq \ell\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right), \tag{1}$$

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \Im\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right), \tag{2}$$

$$\mathfrak{X}(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \mathfrak{X}\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right), \tag{3}$$

*where $k \in (0, 1)$. Then $T$ has a unique fixed point $\mathfrak{b}^* \in \triangle$.*

*Proof.* Let $\mathfrak{b}_0 \in \triangle$ and define the iterative sequence $\{\mathfrak{b}_p\}$ by $\mathfrak{b}_{p+1} = T(\mathfrak{b}_p)$ for all $p = 0, 1, 2, \ldots$.

For any $p \in \mathbb{N}$ and $\lambda > 0$, applying the contraction condition repeatedly yields

$$\begin{aligned}\ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) &= \ell(T\mathfrak{b}_{p-1}, T\mathfrak{b}_p, \lambda)\\ &\geq \ell(\mathfrak{b}_{p-1}, \mathfrak{b}_p, \lambda/k)\\ &\geq \cdots\\ &\geq \ell(\mathfrak{b}_0, \mathfrak{b}_1, \lambda/k^p).\end{aligned}$$

Since $k \in (0,1)$, $k^p \to 0$, hence $\lambda/k^p \to \infty$. By the fundamental property $\lim_{t\to\infty} \ell(\mathfrak{b}_0, \mathfrak{b}_1, t) = 1$, we obtain $\lim_{p\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = 1$ for all $\lambda > 0$.

Similarly, $\lim_{p\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = 0$ and $\lim_{p\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = 0$ for all $\lambda > 0$.

To show $\{\mathfrak{b}_p\}$ is Cauchy, take $p > q$. Using the controlled triangle inequality iteratively with the $t$-norm $\circ$,

$$\ell(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) \geq \bigcirc_{i=q}^{p-1} \ell\left(\mathfrak{b}_i, \mathfrak{b}_{i+1}, \frac{\lambda}{\mathcal{M}}\right),$$

where $\mathcal{M} = \sup\{\alpha(\mathfrak{b}, \mathfrak{e}) : \mathfrak{b}, \mathfrak{e} \in \triangle\} < \infty$.

By the contraction condition,

$$\ell\left(\mathfrak{b}_i, \mathfrak{b}_{i+1}, \frac{\lambda}{\mathcal{M}}\right) \geq \ell\left(\mathfrak{b}_0, \mathfrak{b}_1, \frac{\lambda}{k^i\mathcal{M}}\right).$$

Thus,

$$\ell(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) \geq \bigcirc_{i=q}^{p-1} \ell\left(\mathfrak{b}_0, \mathfrak{b}_1, \frac{\lambda}{k^i\mathcal{M}}\right).$$

Since $k \in (0,1)$, $k^i \to 0$ as $i \to \infty$, so $\lambda/(k^i\mathcal{M}) \to \infty$. Hence $\ell(\mathfrak{b}_0, \mathfrak{b}_1, \lambda/(k^i\mathcal{M})) \to 1$. By continuity of the $t$-norm $\circ$, the iterated $t$-norm tends to 1 as $q, p \to \infty$. Therefore,

$$\lim_{q,p\to\infty} \ell(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) = 1 \quad \forall\lambda > 0.$$

For the falsity component, using the $t$-conorm $\bullet$,

$$\mathfrak{X}(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) \leq \bigotimes_{i=q}^{p-1} \mathfrak{X}\left(\mathfrak{b}_i, \mathfrak{b}_{i+1}, \frac{\lambda}{\mathcal{M}}\right).$$

By the contraction condition,

$$\mathfrak{X}\left(\mathfrak{b}_i, \mathfrak{b}_{i+1}, \frac{\lambda}{\mathcal{M}}\right) \leq \mathfrak{X}\left(\mathfrak{b}_0, \mathfrak{b}_1, \frac{\lambda}{k^i \mathcal{M}}\right) \to 0.$$

Thus $\lim_{q,p\to\infty} \mathfrak{X}(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) = 0$ and similarly $\lim_{q,p\to\infty} \Im(\mathfrak{b}_q, \mathfrak{b}_p, \lambda) = 0$ for all $\lambda > 0$. Hence $\{\mathfrak{b}_p\}$ is Cauchy. Since the space is complete, there exists $\mathfrak{b}^* \in \triangle$ such that $\mathfrak{b}_p \to \mathfrak{b}^*$; i.e., for all $\lambda > 0$,

$$\lim_{p\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}^*, \lambda) = 1, \quad \lim_{p\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}^*, \lambda) = 0, \quad \lim_{p\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}^*, \lambda) = 0.$$

Now we prove $T(\mathfrak{b}^*) = \mathfrak{b}^*$. Using the controlled triangle inequality,

$$\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) \geq \ell\left(T\mathfrak{b}^*, T\mathfrak{b}_p, \frac{\lambda}{2\alpha(T\mathfrak{b}^*, \mathfrak{b}_{p+1})}\right) \circ \ell\left(\mathfrak{b}_{p+1}, \mathfrak{b}^*, \frac{\lambda}{2\alpha(\mathfrak{b}_{p+1}, \mathfrak{b}^*)}\right).$$

By the contraction condition,

$$\ell\left(T\mathfrak{b}^*, T\mathfrak{b}_p, \frac{\lambda}{2\alpha(T\mathfrak{b}^*, \mathfrak{b}_{p+1})}\right) \geq \ell\left(\mathfrak{b}^*, \mathfrak{b}_p, \frac{\lambda}{2k \cdot \alpha(T\mathfrak{b}^*, \mathfrak{b}_{p+1})}\right).$$

As $p \to \infty$, $\mathfrak{b}_p \to \mathfrak{b}^*$, so

$$\ell\left(\mathfrak{b}^*, \mathfrak{b}_p, \frac{\lambda}{2k \cdot \alpha(T\mathfrak{b}^*, \mathfrak{b}_{p+1})}\right) \to 1, \qquad \ell\left(\mathfrak{b}_{p+1}, \mathfrak{b}^*, \frac{\lambda}{2\alpha(\mathfrak{b}_{p+1}, \mathfrak{b}^*)}\right) \to 1.$$

Thus,

$$\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) \geq 1 \circ 1 = 1 \quad \forall \lambda > 0.$$

Since $\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) \leq 1$ always holds, we obtain $\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 1$ for all $\lambda > 0$. By property (iii) of neutrosophic metric spaces, $T\mathfrak{b}^* = \mathfrak{b}^*$. A similar argument using the falsity and indeterminacy components yields $\Im(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 0$ and $\mathfrak{X}(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 0$ for all $\lambda > 0$, confirming $T\mathfrak{b}^* = \mathfrak{b}^*$.

To establish uniqueness, suppose $\mathfrak{b}^*$ and $\tau^*$ are two fixed points with

$\mathfrak{b}^* \neq \tau^*$. Then for any $\lambda > 0$,

$$\begin{aligned}\ell(\mathfrak{b}^*, \tau^*, \lambda) &= \ell(T\mathfrak{b}^*, T\tau^*, \lambda)\\ &\geq \ell(\mathfrak{b}^*, \tau^*, \lambda/k)\\ &\geq \ell(\mathfrak{b}^*, \tau^*, \lambda/k^2)\\ &\geq \cdots\\ &\geq \ell(\mathfrak{b}^*, \tau^*, \lambda/k^p).\end{aligned}$$

As $p \to \infty$, $\lambda/k^p \to \infty$, so

$$\ell(\mathfrak{b}^*, \tau^*, \lambda) \geq \lim_{t\to\infty} \ell(\mathfrak{b}^*, \tau^*, t) = 1.$$

Since $\ell(\mathfrak{b}^*, \tau^*, \lambda) \leq 1$, we have $\ell(\mathfrak{b}^*, \tau^*, \lambda) = 1$ for all $\lambda > 0$. By property (iii), $\mathfrak{b}^* = \tau^*$. Alternatively, using the falsity function,

$$\begin{aligned}X(\mathfrak{b}^*, \tau^*, \lambda) &= X(T\mathfrak{b}^*, T\tau^*, \lambda)\\ &\leq X(\mathfrak{b}^*, \tau^*, \lambda/k) \leq \cdots \leq X(\mathfrak{b}^*, \tau^*, \lambda/k^p) \to 0\end{aligned}$$

as $p \to \infty$, giving $X(\mathfrak{b}^*, \tau^*, \lambda) = 0$ for all $\lambda > 0$, which also confirms $\mathfrak{b}^* = \tau^*$. Therefore, $T$ has a unique fixed point $\mathfrak{b}^* \in \triangle$. ■

**Example 2.** Let $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ be a complete HSNCMS, where $\triangle = [0, 1]$ and the control function $\alpha : \triangle \times \triangle \to [1, \infty)$ is bounded, i.e.

$$\mathcal{M} = \sup\{\alpha(\mathfrak{b}, \mathfrak{e}) : \mathfrak{b}, \mathfrak{e} \in \triangle\} < \infty.$$

Define the mapping $T : \triangle \to \triangle$ by

$$T(\vartheta) = \frac{\vartheta}{2}, \quad \forall \vartheta \in \triangle.$$

Assume the neutrosophic metric components are defined as:

$$\ell(\mathfrak{b}, \mathfrak{e}, \lambda) = \exp\left(-\frac{|\mathfrak{b} - \mathfrak{e}|}{\lambda}\right), \quad \Im(\mathfrak{b}, \mathfrak{e}, \lambda) = 1 - \exp\left(-\frac{|\mathfrak{b} - \mathfrak{e}|}{\lambda}\right),$$

$$\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = \exp\left(-\frac{2\lambda}{1 + |\mathfrak{b} - \mathfrak{e}|}\right),$$

where $\circ$ is the product $t$-norm and $\bullet$ is the probabilistic sum. For $T(\mathfrak{b}) = \mathfrak{b}/2$, we have $|T(\mathfrak{b}) - T(\mathfrak{e})| = |\mathfrak{b} - \mathfrak{e}|/2$. Thus,

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) = \exp\left(-\frac{|\mathfrak{b} - \mathfrak{e}|}{2\lambda}\right) = \ell(\mathfrak{b}, \mathfrak{e}, 2\lambda).$$

Hence,

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) \geq \ell(\mathfrak{b}, \mathfrak{e}, \lambda/k), \quad k = 1/2.$$

Similarly,

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \Im(\mathfrak{b}, \mathfrak{e}, \lambda/k), \quad \mathfrak{X}(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda/k).$$

Thus $T$ satisfies the neutrosophic contraction conditions. Let $\delta_0 \in \triangle$ and define

$$\mathfrak{b}_{p+1} = T(\mathfrak{b}_p) = \frac{\mathfrak{b}_p}{2}.$$

Then

$$\mathfrak{b}_p = \frac{\mathfrak{b}_0}{2^p}, \quad \lim_{p\to\infty} \mathfrak{b}_p = 0.$$

$$\lim_{p\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = \lim_{p\to\infty} \exp\left(-\frac{|\mathfrak{b}_0|}{2^{p+1}\lambda}\right) = 1,$$

$$\lim_{p\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = 0, \quad \lim_{p\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) = 0.$$

For $p > q$,

$$|\mathfrak{b}_p - \mathfrak{b}_q| = |\mathfrak{b}_0| \left(\frac{1}{2^p} - \frac{1}{2^q}\right) \to 0.$$

Hence,

$$\lim_{p,q\to\infty} \ell(\mathfrak{b}_p, \mathfrak{b}_q, \lambda) = 1, \quad \lim_{p,q\to\infty} \Im(\mathfrak{b}_p, \mathfrak{b}_q, \lambda) = 0, \quad \lim_{p,q\to\infty} \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_q, \lambda) = 0.$$

This means $\{\mathfrak{b}_p\}$ is Cauchy. Since $\triangle$ is complete, $\exists\ \mathfrak{b}^* \in \triangle$ such that $\mathfrak{b}_p \to \mathfrak{b}^*$ and

$$\mathfrak{b}_p = \frac{\mathfrak{b}_0}{2^n} \to 0 \Rightarrow \mathfrak{b}^* = 0.$$

$$T(\mathfrak{b}^*) = T(0) = 0 = \mathfrak{b}^*.$$

Thus $\mathfrak{b}^*$ is a fixed point.

**Corollary.** *Let* $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ *be a complete HSNCMS with bounded control function* $\alpha$*. If* $\alpha(\mathfrak{b}_1, \mathfrak{b}_2) = 1$ *for all* $\mathfrak{b}_1, \mathfrak{b}_2 \in \triangle$ *and* $T : \triangle \to \triangle$ *satisfies (1)-(3) for all* $\lambda > 0$ *and some* $k \in (0,1)$*, then* $T$ *has a unique fixed point in* $\triangle$*.*

**Corollary.** *Let* $(\triangle, \mathcal{N}, \circ, \bullet)$ *be a complete neutrosophic metric space. If* $T : \triangle \to \triangle$ *satisfies (1)-(3) for all* $\mathfrak{b}_1, \mathfrak{b}_2 \in \triangle$*,* $\lambda > 0$*, and* $k \in (0,1)$*, then* $T$ *has a unique fixed point.*

**Definition 11.** Let $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ be a HSNCMS, where

$$\mathcal{N}(\mathfrak{b}, \mathfrak{e}, \lambda) = (\ell(\mathfrak{b}, \mathfrak{e}, \lambda), \Im(\mathfrak{b}, \mathfrak{e}, \lambda), \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda)).$$

A mapping $T : \triangle \to \triangle$ is called a $T$**-controlled contraction** if $\exists\ \theta \in (0,1)$ such that for all $\mathfrak{b}, \mathfrak{e} \in \triangle$, $\mathfrak{b} \neq \mathfrak{e}$, and for all $\lambda > 0$, the following conditions hold:

$$\begin{gathered}\frac{1}{\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda)} - 1 \leq \theta \left( \frac{1}{\ell(\mathfrak{b}, \mathfrak{e}, \lambda/\alpha(\mathfrak{b}, \mathfrak{e}))} - 1 \right), \\ \Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \theta\, \Im(\mathfrak{b}, \mathfrak{e}, \lambda/\alpha(\mathfrak{b}, \mathfrak{e})), \\ \mathfrak{X}(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \theta\, \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda/\alpha(\mathfrak{b}, \mathfrak{e})).\end{gathered}$$

**Theorem 2.** *Let* $(\triangle, \mathcal{N}, \alpha, F, \circ, \bullet)$ *be a complete HSNCMS, assume that the control function* $\alpha : \triangle \times \triangle \to [1, \infty)$ *is bounded, i.e.,* $\exists\ \mathcal{M} \geq 1$ *such that*

$$\alpha(\mathfrak{b}, \mathfrak{e}) \leq \mathcal{M} \quad \forall \mathfrak{b}, \mathfrak{e} \in \triangle.$$

*Let* $T : \triangle \to \triangle$ *be a* $T$*-controlled contraction. Then* $\vartheta \in \triangle$ *is a unique fixed point of* $T$*.*

*Proof.* Let $\mathfrak{b}_0 \in \triangle$ and define a sequence $\{\mathfrak{b}_p\}$ by

$$\mathfrak{b}_{p+1} = T\mathfrak{b}_p, \quad p \geq 0.$$

By $T$-controlled contraction, we obtain

$$\frac{1}{\ell(\mathfrak{b}_{p+1}, \mathfrak{b}_{p+2}, \lambda)} - 1 \leq \theta \left( \frac{1}{\ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda/\alpha(\mathfrak{b}_p, \mathfrak{b}_{p+1}))} - 1 \right).$$

Let

$$a_p(\lambda) = \frac{1}{\ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda)} - 1.$$

Then

$$a_{p+1}(\lambda) \leq \theta\, a_p \left( \frac{\lambda}{\alpha(\mathfrak{b}_p, \mathfrak{b}_{p+1})} \right).$$

Since $\alpha(\mathfrak{b}, \mathfrak{e}) \leq \mathcal{M}$, we have

$$a_{p+1}(\lambda) \leq \theta\, a_p \left( \frac{\lambda}{\mathcal{M}} \right).$$

By induction,

$$a_p(\lambda) \leq \theta^p a_0 \left( \frac{\lambda}{\mathcal{M}^p} \right).$$

Since $0 < \theta < 1$, we have $\theta^p \to 0$ as $p \to \infty$. Hence,

$$a_p(\lambda) \to 0 \Rightarrow \ell(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) \to 1.$$

From contraction,

$$\Im(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) \leq \theta\, \Im(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda/\mathcal{M}),$$

so

$$\Im(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) \leq \theta^n \Im(\mathfrak{b}_0, \mathfrak{b}_1, \lambda/\mathcal{M}^p) \to 0.$$

Similarly,

$$\mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_{p+1}, \lambda) \to 0.$$

Using controlled triangle inequality repeatedly, we obtain

$$\ell(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) \to 1, \quad \Im(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) \to 0, \quad \mathfrak{X}(\mathfrak{b}_p, \mathfrak{b}_r, \lambda) \to 0$$

as $p, r \to \infty$. Hence $\{\mathfrak{b}_p\}$ is Cauchy. Since the space is complete, there exists $\mathfrak{b}^* \in \triangle$ such that $\mathfrak{b}_p \to \mathfrak{b}^*$. Using controlled triangle inequality with $\lambda = \lambda_1 + \lambda_2$:

$$\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) \geq \ell(T\mathfrak{b}^*, T\mathfrak{b}_p, \lambda_1) \circ \ell(T\mathfrak{b}_p, \mathfrak{b}^*, \lambda_2).$$

As $p \to \infty$,

$$\ell(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 1.$$

Similarly,

$$\Im(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 0, \quad \mathfrak{X}(T\mathfrak{b}^*, \mathfrak{b}^*, \lambda) = 0.$$

Thus $T\mathfrak{b}^* = \mathfrak{b}^*$. Let $\mathfrak{b}^*, \mathfrak{g}^*$ be fixed points. Then

$$\frac{1}{\ell(\mathfrak{b}^*, \mathfrak{g}^*, \lambda)} - 1 \leq \theta \left( \frac{1}{\ell(\mathfrak{b}^*, \mathfrak{g}^*, \lambda/\mathcal{M})} - 1 \right).$$

Letting $\lambda \to \infty$, we obtain

$$\ell(\mathfrak{b}^*, \mathfrak{g}^*, \lambda) = 1.$$

Similarly,

$$\Im(\mathfrak{b}^*, \mathfrak{g}^*, \lambda) = 0, \quad \mathfrak{X}(\mathfrak{b}^*, \mathfrak{g}^*, \lambda) = 0.$$

Hence $\mathfrak{b}^* = \mathfrak{g}^*$. Therefore,

$$\lim_{p \to \infty} \mathcal{N}(\mathfrak{b}_p^*, \mathfrak{b}^*, \lambda) = (1, 0, 0).$$

Thus, $T$ has a unique fixed point $\mathfrak{b}^* \in \triangle$. ■

**Example 3.** Let $\triangle = (0, 1]$ and let $E = \{e_1, e_2\}$. Define a hybrid soft-neutrosophic mapping $F : E \times \triangle \to [0, 1]^3$ by:

$$T_{e_1}(\mathfrak{b}) = \frac{\mathfrak{b}}{1+\mathfrak{b}}, \quad I_{e_1}(\mathfrak{b}) = \frac{1-\mathfrak{b}}{2}, \quad F_{e_1}(\mathfrak{b}) = \frac{1}{1+\mathfrak{b}},$$
$$T_{e_2}(\mathfrak{b}) = \frac{\mathfrak{b}^2}{1+\mathfrak{b}^2}, \quad I_{e_2}(\mathfrak{b}) = \frac{|\mathfrak{b}-0.5|}{2}, \quad F_{e_2}(\mathfrak{b}) = \frac{1}{1+\mathfrak{b}^2}.$$

Define the neutrosophic metric components $\mathcal{N} = (\ell, \Im, \mathfrak{X})$ by:

$$\ell(\mathfrak{b}, \mathfrak{e}, \lambda) = \exp\left(-\frac{|\mathfrak{b}-\mathfrak{e}|}{\lambda}\right),$$
$$\Im(\mathfrak{b}, \mathfrak{e}, \lambda) = 1 - \exp\left(-\frac{|\mathfrak{b}-\mathfrak{e}|}{\lambda}\right),$$
$$\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = \frac{|\mathfrak{b}-\mathfrak{e}|}{1+\lambda}.$$

Let the control function $\alpha : \triangle \times \triangle \to [1, \infty)$ be defined as: $\alpha(\mathfrak{b}, \mathfrak{e}) = 1 + \frac{1}{1+\mathfrak{b}} + \frac{1}{1+\mathfrak{e}}$. Then $\alpha(\mathfrak{b}, \mathfrak{e}) \leq 3$ for all $\mathfrak{b}, \mathfrak{e} \in \triangle$, so $\mathcal{M} = 3$. Let $\circ$ be the product $t$-norm and $\bullet$ be the probabilistic sum:

$$\mathfrak{b} \circ \mathfrak{e} = \mathfrak{b}\mathfrak{e}, \quad \mathfrak{b} \bullet \mathfrak{e} = \mathfrak{b} + \mathfrak{e} - \mathfrak{b}\mathfrak{e}.$$

Define $T : \triangle \to \triangle$ by:

$$T(\mathfrak{b}) = \frac{\mathfrak{b}}{1 + \mathfrak{b}}.$$

We compute:

$$|T(\mathfrak{b}) - T(\mathfrak{e})| = \left| \frac{\mathfrak{b}}{1 + \mathfrak{b}} - \frac{\mathfrak{e}}{1 + \mathfrak{e}} \right| = \frac{|\mathfrak{b} - \mathfrak{e}|}{(1 + \mathfrak{b})(1 + \mathfrak{e})}.$$

Thus,

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) = \exp\left( -\frac{|\mathfrak{b} - \mathfrak{e}|}{\lambda(1 + \mathfrak{b})(1 + \mathfrak{e})} \right).$$

Hence,

$$\frac{1}{\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda)} - 1 = \exp\left( \frac{|\mathfrak{b} - \mathfrak{e}|}{\lambda(1 + \mathfrak{b})(1 + \mathfrak{e})} \right) - 1.$$

Since $(1 + \mathfrak{b})(1 + \mathfrak{e}) \geq 1$, we obtain

$$\frac{1}{\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda)} - 1 \leq \frac{1}{2} \left( \exp\left( \frac{|\mathfrak{b} - \mathfrak{e}|}{\lambda} \right) - 1 \right).$$

Thus,

$$\frac{1}{\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda)} - 1 \leq \theta \left( \frac{1}{\ell(\mathfrak{b}, \mathfrak{e}, \lambda)} - 1 \right),$$

where $\theta = 1/2 \in (0, 1)$. Similarly, one can verify:

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \theta \, \Im(\mathfrak{b}, \mathfrak{e}, \lambda), \quad \mathfrak{X}(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \theta \, \mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda).$$

Hence $T$ is a $T$-controlled contraction. Let $\mathfrak{b}_1 \in \triangle$ and define $\mathfrak{b}_{p+1} = T(\mathfrak{b}_p)$. Then

$$\mathfrak{b}_{p+1} = \frac{\mathfrak{b}_p}{1 + \mathfrak{b}_p}.$$

This sequence is decreasing and bounded below by 0, hence convergent.

Let $\lim_{p\to\infty} \mathfrak{b}_p = \mathfrak{b}^*$:

$$\mathfrak{b}^* = \frac{\mathfrak{b}^*}{1+\mathfrak{b}^*} \Rightarrow \mathfrak{b}^*(1+\mathfrak{b}^*) = \mathfrak{b}^* \Rightarrow (\mathfrak{b}^*)^2 = 0.$$

Thus $\mathfrak{b}^* = 0$. $T(0) = \frac{0}{1+0} = 0$, so $\mathfrak{b}^* = 0$ is a fixed point.

# 4 Application to nonlinear chemical reaction network in HSNCMS

Chemical reactions are important in mathematical chemistry. They help us study how chemical concentrations change over time. But in real life, things are never perfect. Temperature changes, pressure shifts, and other effects create uncertainty. Sometimes we also have missing data from experiments. The HSNCMS is a good tool to handle this kind of uncertainty and confusion. In this section, we use our main fixed point result to show that a nonlinear chemical reaction system has one and only one balanced concentration profile.

**Theorem 3.** *Let*

$$\Delta = C([0,1], \mathbb{R}^n)$$

*be the Banach space of all continuous vector-valued functions representing the concentration profiles of* $n$ *chemical species, endowed with the norm*

$$\|\mathfrak{b}\| = \sup_{t\in[0,1]} \max_{1\le i\le n} |\mathfrak{b}_i(t)|.$$

*Define a Hybrid Soft-Neutrosophic Controlled Metric Space*

$$(\Delta, \mathcal{N}, \alpha, F, \circ, \bullet)$$

*by*

$$\ell(\mathfrak{b}, \mathfrak{e}, \lambda) = \frac{\lambda}{\lambda + \|\mathfrak{b} - \mathfrak{e}\|},$$

$$\Im(\mathfrak{b}, \mathfrak{e}, \lambda) = \frac{\|\mathfrak{b} - \mathfrak{e}\|}{2(\lambda + \|\mathfrak{b} - \mathfrak{e}\|)},$$

$$\mathfrak{X}(\mathfrak{b}, \mathfrak{e}, \lambda) = \frac{\|\mathfrak{b} - \mathfrak{e}\|}{2(\lambda + \|\mathfrak{b} - \mathfrak{e}\|)},$$

*for all* $\mathfrak{b}, \mathfrak{e} \in \Delta$ *and* $\lambda > 0$*, with control function*

$$\alpha(\mathfrak{b}, \mathfrak{e}) = 1 + \|\mathfrak{b}\| + \|\mathfrak{e}\|.$$

*Consider the nonlinear chemical reaction network system*

$$\mathfrak{b}_i(t) = \mathfrak{b}_i^0 + \int_0^t \Phi_i\left(s, \mathfrak{b}_1(s), \mathfrak{b}_2(s), \ldots, \mathfrak{b}_n(s)\right) ds, \quad i = 1, 2, \ldots, n,$$

*where* $\mathfrak{b}_i^0$ *denotes the initial concentration and* $\Phi_i$ *represents the nonlinear production-consumption rate of the* $i$*-th species.*

*Assume that:*

1. *each function*

   $$\Phi_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$$

   *is continuous;*

2. *there exists a constant* $L > 0$ *such that*

   $$|\Phi_i(t, x) - \Phi_i(t, y)| \leq L\|x - y\|,$$

   *for all* $t \in [0,1]$*,* $x, y \in \mathbb{R}^n$*, and* $i = 1, 2, \ldots, n$*;*

3. $$k = L < 1.$$

*Then the nonlinear chemical reaction system admits a unique equilibrium concentration profile in* $\Delta$*.*

*Proof.* Define the operator

$$T : \Delta \to \Delta$$

by

$$(T\mathfrak{b})_i(t) = \mathfrak{b}_i^0 + \int_0^t \Phi_i\left(s, \mathfrak{b}_1(s), \mathfrak{b}_2(s), \ldots, \mathfrak{b}_n(s)\right) ds, \quad i = 1, 2, \ldots, n.$$

Then the fixed points of $T$ are exactly the equilibrium concentration profiles of the nonlinear chemical reaction network system. Since each

$$\Phi_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$$

is continuous, the operator $T$ is well-defined and maps $\Delta$ into itself. Moreover, since

$$\Delta = C([0,1], \mathbb{R}^n)$$

is a Banach space and the Hybrid Soft-Neutrosophic Controlled Metric Space

$$(\Delta, \mathcal{N}, \alpha, F, \circ, \bullet)$$

is constructed from the supremum norm, it follows that this space is complete. Now for any

$$\mathfrak{b}, \mathfrak{e} \in \Delta,$$

we estimate

$$\|(T\mathfrak{b}) - (T\mathfrak{e})\|.$$

Using the Lipschitz condition, we obtain

$$\begin{aligned}
|(T\mathfrak{b})_i(t) - (T\mathfrak{e})_i(t)| &= \left| \int_0^t [\Phi_i(s, \mathfrak{b}(s)) - \Phi_i(s, \mathfrak{e}(s))] \, ds \right| \\
&\leq \int_0^t |\Phi_i(s, \mathfrak{b}(s)) - \Phi_i(s, \mathfrak{e}(s))| \, ds \\
&\leq \int_0^t L\|\mathfrak{b}(s) - \mathfrak{e}(s)\| ds \\
&\leq L\|\mathfrak{b} - \mathfrak{e}\|.
\end{aligned}$$

Hence

$$\|T\mathfrak{b} - T\mathfrak{e}\| \leq L\|\mathfrak{b} - \mathfrak{e}\|.$$

Let

$$k = L < 1.$$

Now we verify condition (1) of the main theorem:

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) = \frac{\lambda}{\lambda + \|T\mathfrak{b} - T\mathfrak{e}\|}.$$

Using the above estimate,

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) \geq \frac{\lambda}{\lambda + k\|\mathfrak{b} - \mathfrak{e}\|}.$$

Also,

$$\ell\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right) = \frac{\lambda/k}{\lambda/k + \|\mathfrak{b} - \mathfrak{e}\|} = \frac{\lambda}{\lambda + k\|\mathfrak{b} - \mathfrak{e}\|}.$$

Therefore,

$$\ell(T\mathfrak{b}, T\mathfrak{e}, \lambda) \geq \ell\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right).$$

Next,

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) = \frac{\|T\mathfrak{b} - T\mathfrak{e}\|}{2(\lambda + \|T\mathfrak{b} - T\mathfrak{e}\|)}.$$

Thus

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \frac{k\|\mathfrak{b} - \mathfrak{e}\|}{2(\lambda + k\|\mathfrak{b} - \mathfrak{e}\|)}.$$

Also,

$$\Im\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right) = \frac{\|\mathfrak{b} - \mathfrak{e}\|}{2\left(\frac{\lambda}{k} + \|\mathfrak{b} - \mathfrak{e}\|\right)} = \frac{k\|\mathfrak{b} - \mathfrak{e}\|}{2(\lambda + k\|\mathfrak{b} - \mathfrak{e}\|)}.$$

Hence

$$\Im(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \Im\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right).$$

Similarly,

$$\mathfrak{X}(T\mathfrak{b}, T\mathfrak{e}, \lambda) \leq \mathfrak{X}\left(\mathfrak{b}, \mathfrak{e}, \frac{\lambda}{k}\right).$$

Thus all conditions (1), (2), and (3) of the main fixed point theorem are satisfied. Since

$$(\Delta, \mathcal{N}, \alpha, F, \circ, \bullet)$$

is complete and $k \in (0,1)$, by the main theorem, the operator $T$ has a unique fixed point

$$\mathfrak{b}^* \in \Delta.$$

Therefore, the nonlinear chemical reaction network system admits a unique equilibrium concentration profile. ■

# 5 Conclusions

In this paper, we introduced the Hybrid Soft-Neutrosophic Controlled Metric Space (HSNCMS), a unified framework combining soft set theory, neutrosophic logic, and controlled metric structures, and established Banach-type and $T$-controlled contraction fixed point theorems with rigorous proofs, supported by illustrative examples and successfully applied to Nonlinear Chemical Reaction Networks. Future work includes extending to generalized contraction mappings (Kannan, Chatterjea, Reich, and Ciric-type), common fixed point theorems, topological properties, fractional HSNCMS, integration with machine learning clustering and neural networks, and applications in image processing, multi-agent consensus systems, economic equilibrium modeling, and computational algorithms for real-world decision-making under uncertainty.

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