**MATCH**
*Communications in Mathematical and in Computer Chemistry*

*MATCH Commun. Math. Comput. Chem.* **97** (2027) 227–242
**ISSN:** 0340–6253
**doi:** 10.46793/match.97-1.08626

# Sharp Mosaic Dimension of Matching Toggle Graphs, with an Application to Fibonaccenes

**Đặng Võ Phúc**[a,*]

[a] *Department of Mathematics, FPT University, Quy Nhon AI Campus, An Phu Thinh New Urban Area, Vietnam*

`dangphuc150488@gmail.com`

(Received March 9, 2026)

## Abstract

Zigzag hexagonal chains (fibonaccenes) form a classical benzenoid family whose resonance graphs are Fibonacci cubes. Motivated by a question of Németh [16], we study instead the *matching toggle graph* $\mathcal{M}(Z)$ of a zigzag hexagonal chain $Z$ with $n$ hexagons, whose vertices are all matchings of $Z$ and whose edges correspond to adding or deleting a single bond. We prove that

$$\operatorname{md}(\mathcal{M}(Z)) = \left\lceil \frac{5n+1}{2} \right\rceil,$$

thereby confirming Németh's conjectured lattice dimension for zigzag hexagonal chains.

More generally, for a connected graph $G$ with $m = |E(G)| \geq 2$, we show that the degree lower bound $\operatorname{md}(\mathcal{M}(G)) \geq \lceil m/2 \rceil$ is attained whenever either $m$ is even, or $m$ is odd and $G$ contains a non-bridge edge. The proof combines the standard identification $\mathcal{M}(G) \cong \mathcal{I}(L(G))$ with a lattice embedding scheme based on adjacent-edge pairings. An optimal pairing is certified via a perfect matching in a suitable claw-free line graph, guaranteed by the Sumner–Las Vergnas theorem. Thus the chemical case of fibonaccenes appears as a natural and sharp application of a more general graph-theoretic mechanism.

---

[*] Corresponding author.

# 1 Introduction

Zigzag hexagonal chains, or fibonaccenes, are among the standard benzenoid systems studied in chemical graph theory. Their resonance structure is especially well behaved: Klavžar and Žigert proved that the resonance graphs of fibonaccenes are precisely the Fibonacci cubes [12]; see also [18] for related $Z$-transformation graph results and the monograph [6] for broader background on Fibonacci-type cubes. Thus fibonaccenes provide a natural bridge between benzenoid chemistry and the combinatorics of hypercube-like reconfiguration graphs.

More broadly, fibonaccenes have been studied extensively in chemical graph theory from several complementary viewpoints, including enumeration, Kekulé structures, Clar structures, and resonance theory; see the review of Gutman and Klavžar [7]. In particular, the resonance-graph connection mentioned above belongs to a wider classical framework for fibonaccenes. This broader background also suggests looking beyond resonance graphs and perfect matchings to other reconfiguration structures associated with fibonaccenes.

Motivated by the above perspective, Németh [16] proposed a broader reconfiguration viewpoint in which one studies not only Kekulé structures or perfect matchings, but also the graph whose vertices are all matchings and whose edges correspond to adding or deleting a single edge. For a graph $G$, this is the matching toggle graph $\mathcal{M}(G)$. In particular, for a zigzag hexagonal chain $Z$ with $n$ hexagons, Németh computed the first small examples and conjectured that $\mathcal{M}(Z)$ embeds into the integer lattice $\mathbb{Z}^d$ with $d = \lceil (5n+1)/2 \rceil$. From the chemical point of view, this asks for the smallest lattice dimension needed to realize the reconfiguration space of all partial bond matchings of a fibonaccene, not just its perfect-matchings sector.

From a graph-theoretic perspective, the problem fits naturally into the language of reconfiguration graphs and Fibonacci-type cube structures. Fibonacci cubes $\Gamma_n$ were introduced in the context of interconnection networks as sparse, self-similar induced subgraphs of hypercubes [8, 9], and many related variants have since been studied; see the survey of Klavžar [11].

Németh [16] also observed that simple tiling models lead to Fibonacci cubes and to variants such as Lucas cubes [15], Tribonacci cubes [1], Pell graphs [14], and metallic cubes [5]. The present paper shows that, for the chemically important family of fibonaccenes, the matching-toggle analogue also admits a sharp and explicit lattice-dimension formula.

To make the novelty boundary explicit, let us emphasize that several ingredients used below are standard: the correspondence between matchings of $G$ and independent sets of the line graph $L(G)$, the characteristic-vector embedding of independent-set toggle graphs into a hypercube, basic structural facts on line graphs such as claw-freeness, and the perfect-matching theorem of Sumner and Las Vergnas for connected claw-free graphs of even order; see, for example, Diestel [3], Chudnovsky and Seymour [2], and the original papers [13, 17]. We recall these facts—and include short proofs of some of them—only to keep the exposition self-contained. The new point of the present note is that these classical ingredients can be organized through an adjacent-edge pairing argument to yield a *sharp* mosaic-dimension formula for $\mathcal{M}(G)$.

Our main contribution is therefore a sharp mosaic-dimension theorem for matching toggle graphs. For a connected graph $G$ with $m = |E(G)| \geq 2$, we show that the lower bound $\operatorname{md}(\mathcal{M}(G)) \geq \lceil m/2 \rceil$ is tight whenever either $m$ is even, or $m$ is odd and $G$ contains a non-bridge edge. As a consequence, we confirm the conjecture of Németh [16] for zigzag hexagonal chains and obtain the optimal formula $\operatorname{md}(\mathcal{M}(Z)) = \lceil (5n+1)/2 \rceil$ whenever $Z$ has $n$ hexagons.

The key idea is to certify an optimal adjacent-edge pairing of $E(G)$, which yields a lattice embedding via a simple coordinate map. Although the identification $\mathcal{M}(G) \cong \mathcal{I}(L(G))$ is standard, obtaining a sharp lattice dimension requires a structural pairing argument. We use matchings in the claw-free line graph $L(G)$: by the Sumner–Las Vergnas theorem, every connected claw-free graph of even order has a perfect matching, which in our setting produces the required optimal pairing. For zigzag chains, a naive geometric pairing fails because consecutive shared edges may be separated by a “kink” edge (see Figure 1), so the line-graph approach is essential.

The new results of this note are:

- A sharp mosaic-dimension theorem for matching toggle graphs: for connected $G$ with $|E(G)| = m \geq 2$, the lower bound $\mathrm{md}(\mathcal{M}(G)) \geq \lceil m/2 \rceil$ is tight whenever $m$ is even, or $m$ is odd and $G$ contains a non-bridge edge.
- As a corollary, if $Z$ is a zigzag hexagonal chain with $n$ hexagons, then $\mathrm{md}(\mathcal{M}(Z)) = \lceil (5n+1)/2 \rceil$, confirming the conjecture of [16].

For context and notation, Section 3 also records the standard induced embedding $\mathcal{M}(G) \hookrightarrow Q_{|E(G)|}$ via the line graph $L(G)$.

The remainder of this note is organized as follows. In Section 2, we recall necessary graph-theoretic notions and introduce the matching toggle graph. Section 3 records the standard induced embedding $\mathcal{M}(G) \hookrightarrow Q_{|E(G)|}$. In Section 4, we develop the adjacent-edge pairing method, relate pairings to matchings in line graphs, and apply the Sumner–Las Vergnas theorem to obtain sharp mosaic-dimension bounds; we then specialize these bounds to zigzag hexagonal chains and discuss the geometric obstruction illustrated in Figure 1. Finally, Section 5 collects remarks and comparisons with resonance graphs.

# 2 Preliminaries

All graphs are finite, simple, and undirected. We use standard terminology from graph theory; see, e.g., Diestel [3].

## 2.1 Hypercubes and cube mosaics

**Definition 1** (Hypercube)**.** For $m \geq 0$, the *m-dimensional hypercube* $Q_m$ has vertex set $\{0,1\}^m$. Two vertices are adjacent if they differ in exactly one coordinate (cf. [3, Ch. 1, Ex. 2]).

**Definition 2** (Cube mosaic and mosaic dimension)**.** For $d \geq 1$, the *d-dimensional cube mosaic graph* is the integer lattice graph $\mathbb{Z}^d$: its vertex set is $\mathbb{Z}^d$ and $(x_1, \ldots, x_d)$ is adjacent to $(y_1, \ldots, y_d)$ if and only if $|x_i - y_i| = 1$ for exactly one index $i$ and $x_j = y_j$ for all $j \neq i$.

We define the *mosaic dimension* of a finite graph $H$ by

$$\operatorname{md}(H) \;=\; \min\Bigl\{d \geq 1 : \ H \text{ is isomorphic to a (not necessarily induced) subgraph of } \mathbb{Z}^d\Bigr\}.$$

**Lemma 1** (Degree obstruction)**.** *For any finite graph $H$,* $\operatorname{md}(H) \geq \lceil \Delta(H)/2 \rceil$.

*Proof.* Every vertex of $\mathbb{Z}^d$ has degree at most $2d$, hence any subgraph has maximum degree at most $2d$. So if $H \subseteq \mathbb{Z}^d$ then $\Delta(H) \leq 2d$. ■

## 2.2 Matchings, line graphs, and toggle graphs

**Definition 3** (Matchings)**.** Let $G = (V, E)$. A *matching* is a set $M \subseteq E$ such that no two edges of $M$ share an endpoint (see, e.g., [3, Ch. 2]).

**Definition 4** (Line graph)**.** The *line graph* $L(G)$ has vertex set $E(G)$, with two vertices adjacent in $L(G)$ if and only if the corresponding edges of $G$ share a vertex (see, e.g., [3, Sec. 1.1]).

We analyze the structure of matchings using the standard Token Addition/Removal (TAR for short) reconfiguration model; see Ito et al. [10]. Under this model, we formalize the adjacency of matchings and independent sets as follows.

**Definition 5** (Toggle graphs)**.**(a) The *independent-set toggle graph* $\mathcal{I}(H)$ has as vertices all independent sets $I \subseteq V(H)$. Two independent sets are adjacent if their symmetric difference has size 1, i.e., $I' = I \triangle \{v\}$ for some $v \in V(H)$.

(b) The *matching toggle graph* $\mathcal{M}(G)$ has as vertices all matchings $M \subseteq E(G)$. Two matchings are adjacent if their symmetric difference has size 1, i.e., $M' = M \triangle \{e\}$ for some $e \in E(G)$.

**Lemma 2** (Standard identification of matchings with independent sets)**.** *For every graph $G$, there is a natural graph isomorphism*

$$\mathcal{M}(G) \;\cong\; \mathcal{I}(L(G)),$$

*sending a matching $M \subseteq E(G)$ to the independent set $I_M \subseteq V(L(G))$ consisting of the vertices corresponding to edges in $M$.*

*Proof.* An edge set $M$ is a matching in $G$ if and only if its corresponding vertices in $L(G)$ are pairwise non-adjacent, i.e. form an independent set. Moreover, toggling an edge $e$ in a matching corresponds exactly to toggling the corresponding vertex in $L(G)$. ■

*Remark.* The correspondence in Lemma 2 is standard; it is the familiar equivalence between matchings in $G$ and independent sets in its line graph, written here in TAR language.

**Lemma 3** (Elementary characteristic-vector embedding of $\mathcal{I}(H)$)**.** *Let $H$ be a graph with $N = |V(H)|$. Then $\mathcal{I}(H)$ is an induced subgraph of $Q_N$.*

*Proof.* Identify each independent set $I \subseteq V(H)$ with its characteristic vector $\chi_I \in \{0,1\}^N$. Two independent sets differ by toggling one vertex if and only if their characteristic vectors differ in exactly one bit, hence adjacency in $\mathcal{I}(H)$ agrees with adjacency in $Q_N$ restricted to these vectors. ■

**Lemma 4** (Maximum degree of $\mathcal{M}(G)$)**.** *If $G$ has $m = |E(G)|$ edges, then $\Delta(\mathcal{M}(G)) = m$.*

*Proof.* The empty matching $\varnothing$ is adjacent to $\{e\}$ for every $e \in E(G)$, so $\deg(\varnothing) = m$ and hence $\Delta(\mathcal{M}(G)) \geq m$. On the other hand, every vertex of $\mathcal{M}(G)$ is a matching $M$, and toggling must choose some edge $e \in E(G)$ (either remove it if $e \in M$ or add it if $e \notin M$). Thus $\deg(M) \leq m$ for all $M$, so $\Delta(\mathcal{M}(G)) \leq m$. ■

## 2.3 Zigzag hexagonal chains

We follow standard terminology in chemical graph theory; see, e.g. [4,12]. A *hexagonal chain* is obtained by gluing hexagons edge-to-edge so that each hexagon shares at most two edges with other hexagons, and the inner dual graph is a path. A *zigzag hexagonal chain* (also called a planar fibonaccene) is a hexagonal chain in which no three consecutive hexagons lie in a straight line. In general, there exist non-isomorphic zigzag hexagonal chains with the same number $n$ of hexagons. Accordingly, throughout the sequel, $Z$ denotes an arbitrary zigzag hexagonal chain with $n$ hexagons.

**Lemma 5** (Edge count of a zigzag hexagonal chain). *Let $Z$ be a zigzag hexagonal chain with $n$ hexagons. Then $|E(Z)| = 5n + 1$.*

*Proof.* For $n = 1$, the graph $Z$ is a 6-cycle and has $6 = 5 \cdot 1 + 1$ edges. Each time we attach a new hexagon along a shared edge, we add 5 new edges (the sixth edge is identified with the shared edge). Hence the number of edges increases by 5 at each step, and the claim follows by induction on $n$. ■

# 3 Standard hypercube embedding for matching toggle graphs

**Theorem 1** (Canonical hypercube embedding). *Let $G$ be a graph with $m = |E(G)|$ edges. Then $\mathcal{M}(G)$ is an induced subgraph of $Q_m$. Equivalently, there is an injective map $\Phi : \mathsf{V}(\mathcal{M}(G)) \to \{0,1\}^m$ such that $M \sim M'$ in $\mathcal{M}(G)$ if and only if $\Phi(M) \sim \Phi(M')$ in $Q_m$.*

*Proof.* By Lemma 2, $\mathcal{M}(G) \cong \mathcal{I}(L(G))$. Now apply Lemma 3 to $H = L(G)$, noting that $|V(L(G))| = |E(G)| = m$. ■

*Remark.* Theorem 1 is included for completeness and notation. It is the immediate standard consequence of Lemmas 2 and 3 and is not part of the novelty claimed here.

**Corollary 2** (Canonical embedding for zigzag chains). *If $Z$ is a zigzag hexagonal chain with $n$ hexagons, then the matching toggle graph $\mathcal{M}(Z)$ is an induced subgraph of $Q_{5n+1}$.*

*Proof.* Immediate from Theorem 1 and Lemma 5. ■

# 4 Mosaic embeddings and sharp dimension bounds

## 4.1 An edge-pairing embedding lemma

Let $P_2$ denote the path on vertex set $\{0,1\}$, and $P_3$ the path on vertex set $\{-1,0,1\}$. Their Cartesian products $P_3^{\square k} \square P_2^{\square \ell}$ are finite induced subgraphs of $\mathbb{Z}^{k+\ell}$.

**Definition 6** (Adjacent edge pairing)**.** Let $G$ be a graph. An *adjacent edge pairing* of $E(G)$ is a partition

$$E(G) = \{e_1, f_1\} \,\dot\cup\, \cdots \,\dot\cup\, \{e_k, f_k\} \,\dot\cup\, \{g_1\} \,\dot\cup\, \cdots \,\dot\cup\, \{g_\ell\},$$

where each pair $\{e_i, f_i\}$ consists of two adjacent edges of $G$ (sharing a vertex), and the $g_j$ are single edges.

**Theorem 3** (Embedding via adjacent pairings)**.** *If $G$ admits an adjacent edge pairing with parameters $(k,\ell)$ as in Definition 6, then $\mathcal{M}(G)$ embeds as a subgraph of $P_3^{\square k} \square P_2^{\square \ell}$ and hence of $\mathbb{Z}^{k+\ell}$. Consequently,* $\operatorname{md}(\mathcal{M}(G)) \le k + \ell$*.*

*Proof.* Let $M$ be a matching of $G$. For $1 \le i \le k$ and $1 \le j \le \ell$, define coordinates

$$x_i(M) = \begin{cases} 1, & e_i \in M, \\ -1, & f_i \in M, \\ 0, & \text{otherwise}, \end{cases} \qquad y_j(M) = \begin{cases} 1, & g_j \in M, \\ 0, & \text{otherwise}. \end{cases}$$

Because $e_i$ and $f_i$ are adjacent, they cannot both lie in a matching, so $x_i(M)$ is well-defined. Set $\Psi(M) = (x_1(M), \ldots, x_k(M), y_1(M), \ldots, y_\ell(M)) \in \{-1,0,1\}^k \times \{0,1\}^\ell$.

If $M'$ is obtained from $M$ by toggling a single edge:

- toggling $e_i$ changes $x_i$ by ±1 and leaves all other coordinates unchanged;
- toggling $f_i$ changes $x_i$ by ±1 and leaves all other coordinates unchanged;

- toggling $g_j$ changes $y_j$ by $\pm 1$ and leaves all other coordinates unchanged.

Hence adjacency in $\mathcal{M}(G)$ maps to adjacency in the Cartesian product $P_3^{\square k} \square P_2^{\square \ell}$, which is a subgraph of $\mathbb{Z}^{k+\ell}$. Injectivity follows because $\Psi(M)$ records exactly which edge is chosen from each pair/singleton. ■

## 4.2 Pairings via matchings in the line graph

**Lemma 6** (Adjacent edge pairings and matchings of $L(G)$)**.** *Let $G$ be a graph with $m = |E(G)|$. Giving an adjacent edge pairing of $E(G)$ with parameters $(k, \ell)$ is equivalent to giving a matching $F$ in the line graph $L(G)$ of size $k$; the unmatched vertices of $L(G)$ correspond to the $\ell$ singletons. In particular, if $L(G)$ has a matching of size $\lfloor m/2 \rfloor$, then $E(G)$ admits an adjacent edge pairing with*

$$k + \ell = \left\lceil \frac{m}{2} \right\rceil .$$

*Proof.* Vertices of $L(G)$ are edges of $G$, and an edge of $L(G)$ joins two vertices precisely when the corresponding edges of $G$ are adjacent. Thus a matching $F$ in $L(G)$ is a set of disjoint adjacent pairs of edges of $G$. Taking these pairs as $\{e_i, f_i\}$ and leaving all remaining edges as singletons gives an adjacent edge pairing. Conversely, any adjacent edge pairing determines a matching in $L(G)$ by selecting the line-graph edges joining each pair. The final statement follows because $m - 2\lfloor m/2 \rfloor \in \{0, 1\}$. ■

**Lemma 7** (Line graphs are claw-free; cf. [2])**.** *For every graph $G$, the line graph $L(G)$ is claw-free (has no induced $K_{1,3}$).*

*Proof.* Suppose a vertex $v_e$ of $L(G)$ (corresponding to an edge $e = ab$ of $G$) has three neighbors $v_{e_1}, v_{e_2}, v_{e_3}$ that are pairwise non-adjacent in $L(G)$. Each $e_i$ shares an endpoint with $e$, so each $e_i$ is incident to $a$ or $b$. By pigeonhole, at least two of $e_1, e_2, e_3$ are incident to the same endpoint (say $a$), hence those two edges share $a$ and are adjacent in $L(G)$—a contradiction. ■

**Lemma 8** (Connectivity of line graphs)**.** *If $G$ is connected and has at least two edges, then $L(G)$ is connected.*

*Proof.* Let $e$ and $f$ be edges of $G$. Because $G$ is connected, there is a path in $G$ joining an endpoint of $e$ to an endpoint of $f$. Walking along that path yields a sequence of edges in $G$ where consecutive edges share a vertex; thus the corresponding vertices in $L(G)$ form a path from $v_e$ to $v_f$. ■

**Theorem 4** (Sumner [17], Las Vergnas [13]). *Every connected claw-free graph of even order has a perfect matching.*

*Remark.* This classical theorem was proved by Sumner [17] and independently by Las Vergnas [13]. We will use it only as a black box to guarantee large matchings in line graphs.

**Theorem 5** (A general sharp formula for $\mathrm{md}(\mathcal{M}(G))$). *Let $G$ be a connected graph with $m = |E(G)| \geq 2$ edges.*

(a) *If $m$ is even, then*

$$\mathrm{md}(\mathcal{M}(G)) = \frac{m}{2}.$$

(b) *If $m$ is odd and $G$ has a non-bridge edge $e$ (equivalently, $G - e$ is connected), then*

$$\mathrm{md}(\mathcal{M}(G)) = \frac{m+1}{2}.$$

*Proof.* We first record the universal lower bound coming from the degree obstruction. By Lemma 4, $\Delta(\mathcal{M}(G)) = m$. Hence by Lemma 1,

$$\mathrm{md}(\mathcal{M}(G)) \ \geq \ \left\lceil \frac{m}{2} \right\rceil. \tag{1}$$

*Case (a): $m$ is even.* The line graph $L(G)$ is claw-free by Lemma 7. Since $G$ is connected and has at least two edges, $L(G)$ is connected by Lemma 8. Moreover, $|V(L(G))| = |E(G)| = m$ is even. Therefore, by Theorem 4, $L(G)$ has a perfect matching. In particular, $L(G)$ has a matching of size $m/2 = \lfloor m/2 \rfloor$.

By Lemma 6, this yields an adjacent edge pairing of $E(G)$ with parameters $(k, \ell) = (m/2, 0)$, i.e., $E(G)$ decomposes into $m/2$ adjacent pairs and no singletons. Applying Theorem 3 gives an embedding

$$\mathcal{M}(G) \ \hookrightarrow \ \mathbb{Z}^{m/2},$$

and hence

$$\mathrm{md}(\mathcal{M}(G)) \;\leq\; \frac{m}{2}. \tag{2}$$

Combining (1) with (2), and noting that $\lceil m/2 \rceil = m/2$ here, we obtain $\mathrm{md}(\mathcal{M}(G)) = m/2$.

*Case (b): $m$ is odd and $G$ has a non-bridge edge $e$.* Let $G' := G - e$. Since $e$ is not a bridge, $G'$ is connected. Also $|E(G')| = m - 1$ is even. By Case (a) applied to $G'$, we know that $\mathcal{M}(G')$ admits an embedding into $\mathbb{Z}^{(m-1)/2}$. We now upgrade this to an embedding of $\mathcal{M}(G)$ into $\mathbb{Z}^{(m+1)/2}$ by keeping $e$ as a singleton coordinate.

More explicitly, by Case (a) for $G'$ and the proof above, the line graph $L(G')$ has a perfect matching, hence by Lemma 6 the set $E(G')$ admits an adjacent edge pairing with parameters $(k, \ell) = ((m-1)/2, 0)$. Add $\{e\}$ as a singleton to obtain an adjacent edge pairing of $E(G)$ with parameters

$$(k, \ell) = \left( \frac{m-1}{2}, 1 \right).$$

Applying Theorem 3 yields an embedding

$$\mathcal{M}(G) \;\hookrightarrow\; \mathbb{Z}^{k+\ell} \;=\; \mathbb{Z}^{(m+1)/2},$$

so

$$\mathrm{md}(\mathcal{M}(G)) \;\leq\; \frac{m+1}{2}. \tag{3}$$

On the other hand, the lower bound (1) gives $\mathrm{md}(\mathcal{M}(G)) \geq \lceil m/2 \rceil = (m+1)/2$ because $m$ is odd. Together with (3), we conclude $\mathrm{md}(\mathcal{M}(G)) = (m+1)/2$. ■

*Note.* For completeness, if $G$ is connected with $|E(G)| \leq 1$, then

$$\mathrm{md}(\mathcal{M}(G)) = 1.$$

Indeed, if $|E(G)| = 0$, then $\mathcal{M}(G)$ consists of a single vertex, while if $|E(G)| = 1$, then $\mathcal{M}(G) \cong K_2$. Since in Definition 2 we require $d \geq 1$, both graphs have mosaic dimension 1.

*Remark.* For a connected graph $G$, an edge $e$ is a non-bridge if and only if

$G - e$ remains connected. In particular, every connected graph containing a cycle satisfies the hypothesis of Theorem 5*(b)* (take any edge on a cycle). Moreover, the hypothesis in Theorem 5*(b)* is sufficient but not necessary. For example, let $G = P_4$ be the path on four vertices. Then $m = 3$ and every edge is a bridge, yet $E(P_4)$ admits an adjacent edge pairing with parameters $(k, \ell) = (1, 1)$ (pair the two adjacent edges at the middle vertex and leave the remaining edge as a singleton). Hence Theorem 3 yields $\mathrm{md}(\mathcal{M}(P_4)) \leq 2$, and the degree obstruction Lemma 1 together with Lemma 4 gives $\mathrm{md}(\mathcal{M}(P_4)) \geq \lceil 3/2 \rceil = 2$. Therefore $\mathrm{md}(\mathcal{M}(P_4)) = 2 = \lceil m/2 \rceil$ although $G$ has no non-bridge edge.

**Corollary 6** (Mosaic dimension of zigzag chains)**.** *Let $Z$ be a zigzag hexagonal chain with $n$ hexagons. Then*

$$\mathrm{md}(\mathcal{M}(Z)) = \left\lceil \frac{5n+1}{2} \right\rceil .$$

*Proof.* By Lemma 5, $|E(Z)| = 5n + 1$. The graph $Z$ is connected. If $n$ is odd then $5n + 1$ is even, so Theorem 5*(a)* gives $\mathrm{md}(\mathcal{M}(Z)) = (5n + 1)/2$.

If $n$ is even then $5n + 1$ is odd. Since $Z$ contains cycles, it has a non-bridge edge $e$ (for instance, any edge on a hexagon), so $Z - e$ remains connected. Thus Theorem 5*(b)* applies and yields $\mathrm{md}(\mathcal{M}(Z)) = (5n + 2)/2 = \lceil (5n + 1)/2 \rceil$. ■

## 4.3 Application to zigzag hexagonal chains

*Remark* (A geometric obstruction to naive pairings). In a standard zigzag hexagonal chain (fibonaccene), the two edges shared by consecutive hexagons need not be adjacent. In phenanthrene, the two shared edges on the middle hexagon are separated by one boundary edge; see Figure 1. Hence a naive pairing of shared edges may fail, motivating the line-graph argument used in Theorem 5.

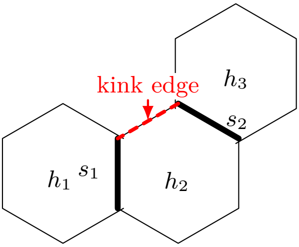

**Figure 1.** Phenanthrene, a zigzag hexagonal chain with three hexagons, drawn in a benzenoid style without duplicated fused edges. On the middle hexagon $h_2$, the two shared edges $s_1$ and $s_2$ are separated by one boundary edge (red dashed), so $s_1$ and $s_2$ are not adjacent edges. This illustrates why pairing consecutive shared edges $(s_{i-1}, s_i)$ may fail geometrically.

We now summarize the dimension argument for $\mathcal{M}(Z)$. The lower bound comes from the degree obstruction (Lemma 1), since $\Delta(\mathcal{M}(Z)) = |E(Z)| = 5n+1$ by Lemma 4 and Lemma 5. The matching upper bound is achieved by Theorem 5, which produces an optimal adjacent-edge pairing (via Lemma 6) and hence an embedding through Theorem 3. Therefore, Corollary 6 follows.

**Example 1** (The naphthalene graph, $n = 2$)**.** For $n = 2$, let $Z$ be the chain consisting of two fused hexagons (the naphthalene graph). By Lemma 5, $|E(Z)| = 11$, so Corollary 2 yields a canonical induced embedding $\mathcal{M}(Z) \hookrightarrow Q_{11}$. Moreover, $\Delta(\mathcal{M}(Z)) = 11$ by Lemma 4, so Lemma 1 gives $\mathrm{md}(\mathcal{M}(Z)) \geq \lceil 11/2 \rceil = 6$.

Since $Z$ contains a cycle, it has a non-bridge edge $e$, and hence Theorem 5*(b)* implies $\mathrm{md}(\mathcal{M}(Z)) = (11+1)/2 = 6$. This agrees with the general formula in Corollary 6.

# 5 Conclusions

From the perspective of chemical graph theory, the main outcome of this note is that the reconfiguration graph of all matchings of a fibonaccene still admits a sharp and explicit lattice realization. If $Z$ is a zigzag hexagonal chain with $n$ hexagons, then Corollary 6 confirms Németh's prediction and gives

$$\mathrm{md}(\mathcal{M}(Z)) = \left\lceil \frac{5n+1}{2} \right\rceil .$$

Thus, alongside the classical fact that the resonance graph of a fibonaccene is a Fibonacci cube, we obtain a complementary statement for the larger reconfiguration space formed by all partial matchings. In this sense, the paper extends the familiar hypercube-type picture from Kekulé-structure reconfiguration to a broader matching-based setting that is still tightly controlled geometrically.

From the graph-theoretic side, this chemical application is a consequence of a more general sharp mosaic-dimension theorem. While Theorem 1 only records the standard induced embedding $\mathcal{M}(G) \hookrightarrow Q_{|E(G)|}$ obtained from line graphs, the main new point is that Theorem 5 turns the degree lower bound $\mathrm{md}(\mathcal{M}(G)) \geq \lceil |E(G)|/2 \rceil$ into an equality under mild connectivity hypotheses. The proof proceeds by converting a large matching in the claw-free line graph into an adjacent-edge pairing (Lemma 6) and then applying the pairing embedding construction (Theorem 3). Figure 1 highlights a geometric subtlety behind the fibonaccene case: although the final pairing is certified abstractly via matchings in a line graph, the zigzag geometry prevents a naive "pair consecutive shared edges" construction.

It is instructive to compare this behavior with resonance graphs $R(Z)$, where adjacency is defined by single-hexagon flips on perfect matchings. While $R(Z)$ is isomorphic to the Fibonacci cube $\Gamma_n$ and hence behaves like a subgraph of $Q_n$ [12], the matching toggle graph $\mathcal{M}(Z)$ includes all matchings and thus has maximum degree $|E(Z)| = 5n + 1$. Consequently, the minimal lattice dimension needed to realize $\mathcal{M}(Z)$ grows on the order of $|E(Z)|/2$, i.e. approximately $2.5n$, reflecting a substantially richer reconfiguration landscape for partial matchings than for Kekulé structures.

Several directions remain open. First, Theorem 5*(b)* requires the existence of a non-bridge edge when $|E(G)|$ is odd; it would be interesting to understand the exact mosaic dimension of $\mathcal{M}(G)$ in families where many edges are bridges, and to determine when the degree obstruction is still tight. Second, one may ask for finer geometric information about $\mathcal{M}(G)$ inside the hypercube $Q_{|E(G)|}$: for instance, which induced subcubes are realized, and how these relate to classical invariants studied for resonance graphs (e.g. cube polynomials and Clar-type parameters; see [6]). Finally, it would be natural to extend these methods to other benzenoid families (linear acenes, armchair chains, or more general systems) and to other matching reconfiguration models beyond TAR, that is, beyond moves that add or delete a single edge at a time.

***Acknowledgment*:** The author is grateful to the two anonymous referees for their careful reading of the manuscript and for their helpful and encouraging comments. In particular, the detailed suggestions of one referee led to several improvements in the presentation, notation, and references, while the positive assessment of the other referee is also sincerely appreciated.

# References

[1] H. Belbachir, R. Ould-Mohamed, Enumerative properties and cube polynomials of Tribonacci cubes, *Discr. Math.* **343** (2020) 111922.

[2] M. Chudnovsky, P. Seymour, The structure of claw-free graphs, in: B. S. Webb (Ed.), *Surveys in Combinatorics 2005*, Cambridge Univ. Press, Cambridge, 2005, pp. 153–171.

[3] R. Diestel, *Graph Theory*, Springer, Berlin, 2017.

[4] T. Došlić, Counting conjugated circuits in benzenoid chains, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 775–784.

[5] T. Došlić, L. Podrug, Metallic cubes, *Discr. Math.* **347** (2024) 113851.

[6] Ö. Eğecioğlu, S. Klavžar, M. Mollard, *Fibonacci Cubes with Applications and Variations*, World Scientific, Singapore, 2023.

[7] I. Gutman, S. Klavžar, Chemical graph theory of fibonacenes, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 39–54.

[8] W. J. Hsu, Fibonacci cubes: a new interconnection topology, *IEEE Trans. Parallel Distrib. Syst.* **4** (1993) 3–12.

[9] W. J. Hsu, C. V. Page, J.S. Liu, Fibonacci cubes – a class of self-similar graphs, *Fibonacci Quart.* **31** (1993) 65–72.

[10] T. Ito, E. D. Demaine, N. J. A. Harvey, C. H. Papadimitriou, M. Sideri, R. Uehara, Y. Uno, On the complexity of reconfiguration problems, *Theoret. Comput. Sci.* **412** (2011) 1054–1065.

[11] S. Klavžar, Structure of Fibonacci cubes: A survey, *J. Comb. Optim.* **25** (2013) 505–522.

[12] S. Klavžar, P. Žigert, Fibonacci cubes are the resonance graphs of fibonaccenes, *Fibonacci Quart.* **43** (2005) 269–276.

[13] M. Las Vergnas, A note on matchings in graphs, *Cahiers Centre Études Rech. Opér.* **17** (1975) 257–260.

[14] E. Munarini, Pell graphs, *Discr. Math.* **342** (2019) 2415–2428.

[15] E. Munarini, C. P. Cippo, N. Zagaglia Salvi, On the Lucas cubes, *Fibonacci Quart.* **39** (2001) 12–21.

[16] L. Németh, From tiling to matching: exploring Fibonacci cube structures, *Dimenziók – Matematikai Közlemények (Math. Notes)* **13** (2025) 55–62.

[17] D. P. Sumner, Graphs with 1-factors, *Proc. Am. Math. Soc.* **42** (1974) 8–12.

[18] H. Zhang, L. Ou, H. Yao, Fibonacci-like cubes as Z-transformation graphs, *Discr. Math.* **309** (2009) 1284–1293.