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Geometric Approach to Degree–Based Topological Indices: Degree–Ratio Sombor Indices of Graphs

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Abstract

An alternative geometric interpretation of vertex-degree-based topological indices is introduced. Let $(d(u), d(v))$ denote the degree pair of the edge $uv \in E(G)$ and $\left(\frac{d(u)}{d(v)}, \frac{d(v)}{d(u)}\right)$ its corresponding degree-ratio pair. Based on the Euclidean metric, a novel class of graph invariants is considered, of which the simplest member is a Degree-Ratio Sombor index *DRSO*. The *DRSO* index is obtained by summing the terms $\sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2}$ over all edges $uv \in E(G)$. Fundamental mathematical properties of the *DRSO* index are established. Several structural variants of the *DRSO* index, including the modified and reduced versions, are formulated.

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1 Introduction

In this paper we are concerned with simple and connected graphs. Simple graphs are characterized by no loops and no multiple edges, typically representing undirected and unweighted relationships. Connected graphs contain at least one path between each pair of vertices. Let $G = (V(G), E(G))$ be a simple and connected graph with n vertices and m edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex v (or the number of vertices adjacent to v) is denoted by $d(v)$. If u and v are adjacent vertices, then the edge connecting them is denoted by uv.

The path, cycle, tree, star, and complete graphs on n vertices denoted by P_n, C_n, T_n, S_n, and K_n, respectively, are standard graph families considered in this paper. A path graph on $n(\geq 2)$ vertices is the graph with $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(P_n) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\}$. A cycle graph on $n(\geq 3)$ is a connected graph where each vertex has degree 2. A tree is a connected acyclic graph with n vertices and $n-1$ edges. A tree has exactly one path connecting any two distinct vertices. A star graph is the complete bipartite graph, $K_{1,n-1}$, consisting of one vertex of degree $n-1$ and $n-1$ vertices of degree 1. A complete graph is the graph with $V(K_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(K_n) = \{(v_i, v_j) : 1 \leq i < j \leq n\}$.

According to the IUPAC definition [36], a *topological index* of a molecular graph is a numerical value associated with the chemical constitution for correlating structural feature with various physical properties, chemical reactivity, and biological activity. Degree-based topological indices of a graph G, denoted by $TI(G)$, are generally defined as

$$TI(G) = \sum_{uv \in E(G)} \psi(d(u), d(v)) \tag{1}$$

where $\psi(x, y) \geq 0$ is a real-valued kernel function satisfying a symmetric property $\psi(x, y) = \psi(y, x)$.

In the mathematical and chemical graph theory literature, a large number of vertex-degree-based topological indices have been introduced and studied extensively [10, 20, 35]. The classical and prominent degree-based topological indices have been formulated via the algebraic approach such as degree additive, multiplicative, division, and powers. For example Zagreb (M_1 and M_2), symmetric division deg SDD, Albertson Alb, Randić R, atom-bond connectivity ABC, forgotten F, and inverse-sum indeg ISI indices. Some classical degree-based topological indices have been motivated by mean functions such as arithmetic, geometric, and harmonic means. For example, the harmonic H and geometric-arithmetic GA indices. A not necessarily complete list of the classical degree-based topological indices is given in Table 1.

Index Name and Ref.	**Year**	$\psi(d(u), d(v))$
First Zagreb [11]	1972	$d(u) + d(v)$
Second Zagreb [11]	1972	$d(u)d(v)$
Randić [30]	1975	$\frac{1}{\sqrt{d(u)d(v)}}$
Harmonic [8]	1987	$\frac{2}{d(u)+d(v)}$
Albertson [2]	1997	$\lvert d(u) - d(v)\rvert$
Atom-Bond Connectivity [7]	1998	$\sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$
Geometric-Arithmetic [37]	2009	$\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$
Sum-Connectivity [42]	2009	$\frac{1}{\sqrt{d(u)+d(v)}}$
Symmetric Division Deg [39]	2010	$\frac{d(u)}{d(v)} + \frac{d(v)}{d(u)}$
Inverse Sum Indeg [39]	2010	$\frac{d(u)d(v)}{d(u)+d(v)}$
Forgotten [9]	2015	$d(u)^2 + d(v)^2$

Table 1. Kernel functions of some prominent classical degree-based topological indices

In 2021, Ivan Gutman introduced a novel approach, called the geometric approach, for formulating degree-based topological indices [12]. Based on the Euclidean distance from the origin $(0,0)$ to the degree pair $\big(d(u),d(v)\big)$ for $uv \in E(G)$, a novel degree-based graph invariant, referred to as the *Sombor index SO* was formulated. The index is defined as

$$SO(G) = \sum_{uv\in E(G)} \|(d(u),d(v))\|_2 = \sum_{uv\in E(G)} \sqrt{d(u)^2+d(v)^2}$$

The introduction of the Sombor index marked a new era of geometrically motivated topological indices, see for example [4, 13, 14, 16, 34]. Some remarkable Sombor-type indices include diminished Sombor DSO, Banhatti-Sombor BSO, Euler-Sombor EU, elliptic Sombor ESO, forgotten Sombor FSO, and hyperbolic Sombor HSO indices. A not necessarily complete list of Sombor variants is given in Table 2.

Index Name and Ref.	**Year**	$\psi\,(d(u),d(v))$
Diminished Sombor [29]	2021	$\dfrac{\sqrt{d(u)^2+d(v)^2}}{d(u)+d(v)}$
Banhatti-Sombor [22]	2021	$\dfrac{\sqrt{d(u)^2+d(v)^2}}{d(u)d(v)}$
Euler-Sombor [34]	2024	$\sqrt{d(u)^2+d(v)^2+d(u)d(v)}$
Elliptic Sombor [14]	2024	$(d(u)+d(v))\sqrt{d(u)^2+d(v)^2}$
Forgotten Sombor [23]	2024	$\sqrt{d(u)^4+d(v)^4}$
Hyperbolic Sombor [4]	2025	$\dfrac{\sqrt{d(u)^2+d(v)^2}}{\min\{d(u),d(v)\}}$
Augmented Euler-Sombor [27]	2026	$\sqrt{\dfrac{d(u)^2+d(v)^2+d(u)d(v)}{d(u)+d(v)-2}}$

Table 2. Kernel functions of Sombor-type degree-based topological indices

A common approach to modifying such degree-based topological indices is to take the reciprocal of their edge contributions. For example, see the modified or reciprocal versions of Sombor, diminished Sombor, Banhatti-Sombor, Euler-Sombor, elliptic Sombor, forgotten Sombor, and Augmented Euler-Sombor indices in [15, 17, 24–27, 33].

The Sombor index and its structural variants are fundamentally formulated based on the different geometric approaches: (1) The Euclidean distance from the origin $(0,0)$ to the raw degree pair $(d(u), d(v))$ conceived the original Sombor index SO [12]; (2) The Euclidean distance from the origin $(0,0)$ to the reciprocal degree pair $\left(\frac{1}{d(u)}, \frac{1}{d(v)}\right)$ led to the Banhatti-Sombor index BSO [22]; (3) The Euclidean distance from the origin $(0,0)$ to the squared degree pair $\left(d(u)^2, d(v)^2\right)$ gave rise to the forgotten Sombor index FSO [23]; (4) The area of an ellipse whose focal points correspond to the degree pair $(d(u), d(v))$ for $uv \in E(G)$ motivated the construction of the elliptic Sombor index ESO [14]; (5) The perimeter of an ellipse whose focal points correspond to the degree pair $(d(u), d(v))$ for $uv \in E(G)$ using Euler's formula provided the basis for Euler-Sombor index EU [34]; and (6) The eccentricity of the hyperbola whose length of the semi-major and semi-minor axes correspond to the degree pair $(d(u), d(v))$ inspired the formulation of the hyperbolic Sombor index HSO [4]. In Table 3, we show that it is also possible to interpret some topological indices in terms of the Euclidean distance from the origin $(0,0)$ to certain points in the coordinate system. However, to the best of our knowledge, the Euclidean metric associated with the degree-ratio pair $\left(\frac{d(u)}{d(v)}, \frac{d(v)}{d(u)}\right)$, and other related ratio-based coordinates, has not yet been investigated.

This paper aimed at exploring the recent advancements in the Sombor index and their mathematical properties. The contribution of this paper is threefold:

1. Introduction of a novel geometrically motivated index, referred to as the Degree-Ratio Sombor index $DRSO$, along with its modified and reduced versions.
2. Establishment of basic mathematical properties of $DRSO$ index.
3. Formulation of several associated $DRSO$ variants using geometric approach.

The remainder of this paper is organized as follows. In Section 2, we formulate a novel Degree-Ratio Sombor index *DRSO*, and its modified and reduced versions. Section 3 establishes fundamental mathematical properties of the *DRSO* index. In section 4, we apply the concepts of degree-ratio and the related ratio-based coordinates and the Euclidean metric to formulate the associated *DRSO* structural variants. Finally, Section 5 concludes the paper with summary of the main results and possible open problems.

2 Geometric interpretation of topological indices via the Euclidean metric

The notation $\sum_{uv\in E(G)}$ in Eq. (1) is conventionally interpreted as summation over all edges of the graph G. The edge contribution ψ would depend on the degree pair $(x, y) = (d(u), d(v))$. To ensure consistency, we shall always assume that $0 < x \leq y$. Table 3 presents a geometric interpretation of some topological indices via the Euclidean norm.

From Table 3, the abbreviations ABC, ABS, and SDD stand for atom-bond-connectivity, atom-bond sum-connectivity, and symmetric division degree indices, respectively. In the geometric interpretation of the second Zagreb and Harmonic indices, $(1, 1)$ is an isolated-edge degree pair, and $\omega_2(x, y) = \frac{xy}{\sqrt{2}}$ and $\omega_1(x, y) = \frac{\sqrt{2}}{x+y}$ are the respective weighting functions. The notation $\|(\cdot,\cdot)\|_2^2$ means the square of an Euclidean norm of an ordered degree pair.

The formal definitions of degree-coordinate (d-coordinate) or degree point (d-point), dual degree-point (dd-point), and degree-radius (d-radius) are provided in [12]. For clarity and consistency, we also define the following relevant and key terms used throughout this paper.

Definition 1. Let the ordered raw degree point (d-point) be (x, y). The corresponding degree-ratio point (dr-point) of the (x, y)-edge $e_{ij} \in E(G)$ is defined as $\left(\frac{x}{y}, \frac{y}{x}\right)$.

Definition 2. The point with coordinates $\left(\frac{y}{x}, \frac{x}{y}\right)$ is the dual-degree-ratio point (ddr-point) of the (x, y)-edge $e_{ij} \in E(G)$.

Definition 3. The distance between the dr-point $\left(\frac{x}{y}, \frac{y}{x}\right)$ and the origin of

Euclidean Metric	$\psi(x,y)$	Index Name and Ref.
$\left\Vert \frac{1}{x+y}(x,y)\right\Vert_2$	$\frac{\sqrt{x^2+y^2}}{x+y}$	Diminished Sombor [29]
$\left\Vert \left(x+\frac{1}{2}y, \frac{\sqrt{3}}{2}y\right)\right\Vert_2$	$\sqrt{x^2+y^2+xy}$	Euler-Sombor [34]
$\left\Vert \left(\sqrt{\frac{x-1}{xy}}, \sqrt{\frac{y-1}{xy}}\right)\right\Vert_2$	$\sqrt{\frac{x+y-2}{xy}}$	ABC [7]
$\left\Vert \left(\sqrt{\frac{x-1}{x+y}}, \sqrt{\frac{y-1}{x+y}}\right)\right\Vert_2$	$\sqrt{\frac{x+y-2}{x+y}}$	ABS [3]
$\left\Vert \left(1, \frac{y}{x}\right)\right\Vert_2$	$\sqrt{1+\left(\frac{y}{x}\right)^2}$	Hyperbolic Sombor [4]
$\left\Vert (x^{\frac{1}{2}}, y^{\frac{1}{2}})\right\Vert_2$	$\sqrt{x+y}$	Nirmala [21]
$\left\Vert \frac{1}{x+y}\left(x^{\frac{1}{2}}, y^{\frac{1}{2}}\right)\right\Vert_2$	$\frac{1}{\sqrt{x+y}}$	Sum-Connectivity [42]
$\left\Vert (x^2, y^2)\right\Vert_2$	$\sqrt{x^4+y^4}$	Forgotten Sombor [23]
$\left\Vert \left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)\right\Vert_2$	$\sqrt{x^3+y^3}$	Dharwad [18]
$\left\Vert \left(\sqrt{\frac{x}{y}}, \sqrt{\frac{y}{x}}\right)\right\Vert_2^2$	$\frac{x}{y}+\frac{y}{x}$	SDD [38, 39]
$\left\Vert (x^{\frac{1}{2}}, y^{\frac{1}{2}})\right\Vert_2^2$	$x+y$	First Zagreb [11]
$\Vert (x,y)\Vert_2^2$	x^2+y^2	Forgotten [9]
$\Vert (x+y)(x,y)\Vert_2$	$(x+y)\sqrt{x^2+y^2}$	Elliptic Sombor [14]
$\left\Vert \left(x^{-\frac{1}{2}}, y^{-\frac{1}{2}}\right)\right\Vert_2$	$\sqrt{x^{-1}+y^{-1}}$	First Inverse Nirmala [19]
$\left\Vert (x^{-1}, y^{-1})\right\Vert_2$	$\sqrt{x^{-2}+y^{-2}}$	First Banhatti-Sombor [22]
$\left\Vert \frac{xy}{\sqrt{2}}(1,1)\right\Vert_2$	xy	Second Zagreb [11]
$\left\Vert \frac{\sqrt{2}}{x+y}(1,1)\right\Vert_2$	$\frac{2}{x+y}$	Harmonic [8]

Table 3. Interpretation of topological indices via the Euclidean metric

the coordinate system is the degree-ratio radius (dr-radius) of the (x, y)-edge $e_{ij} \in E(G)$, denoted by $r\left(\frac{x}{y}, \frac{y}{x}\right)$.

Based on the classical geometry, the Euclidean distance from the origin of the coordinated system to a dr-point (or the Euclidean norm of a dr-point), we have

$$r\left(\frac{x}{y}, \frac{y}{x}\right) = \sqrt{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2} = \frac{\sqrt{x^4 + y^4}}{xy} \qquad (2)$$

It is notable that the kernel function $\psi(x, y) = \sqrt{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2} = \frac{\sqrt{x^4+y^4}}{xy}$ has not been used in the theory of vertex-degree-based topological indices.

Motivated by the Euclidean norm of the dr-point in Eq. (2), the definition of the original Sombor index [12], and geometric interpretation of the *SDD* and *HSO* indices in Table 3, we introduce a novel index, for which we propose the name *Degree-Ratio Sombor index DRSO*, defined as

$$DRSO(G) = \sum_{uv \in E(G)} \sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2} \qquad (3)$$

or equivalently

$$DRSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u)d(v)} \qquad (4)$$

Motivated by the works on modified Sombor index [15, 17, 33], we propose a *modified Degree-Ratio Sombor index* ${}^{m}DRSO$, defined as

$${}^{m}DRSO(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2}} \qquad (5)$$

or equivalently

$${}^{m}DRSO(G) = \sum_{uv \in E(G)} \frac{d(u)d(v)}{\sqrt{d(u)^4 + d(v)^4}} \qquad (6)$$

Motivated by the works on reduced Sombor index [6, 12, 28, 40], we propose a *reduced Degree-Ratio Sombor index* $DRSO_{red}$, defined as

$$DRSO_{red}(G) = \sum_{uv \in E(G)} \sqrt{\left(\frac{d_r(u)}{d_r(v)}\right)^2 + \left(\frac{d_r(v)}{d_r(u)}\right)^2} \tag{7}$$

or equivalently

$$DRSO_{red}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_r(u)^4 + d_r(v)^4}}{d_r(u)d_r(v)} \tag{8}$$

where $d_r(u) = d(u) - 1$ and $d_r(v) = d(v) - 1$. The $DRSO_{red}$ index is well-defined only for graphs with minimum degree $\delta(G) \geq 2$.

In the subsequent section, we establish some fundamental mathematical properties of the $DRSO$ index.

3 Fundamental mathematical properties of the degree-ratio Sombor index

In this section, we establish and prove some propositions and theorems related to the Degree-Ratio Sombor index $DRSO$, Symmetric Division Deg index SDD, Sombor index SO, F-Sombor index FSO, and first Zagreb index M_1. The theorems enable rigorous comparison across graph families. Bounds and extremal values of the $DRSO$ index provide fundamental insights about its range and structural relevance.

Theorem 1. *Let G be a simple connected graph of size m. Then*

$$DRSO(G) \geq \sqrt{2}m$$

Moreover, the equality holds if and only if G is regular.

Proof. Let $t = \frac{d(u)}{d(v)}$. Then the edge contribution of the $DRSO$ index become

$$\psi(t) = \sqrt{t^2 + t^{-2}}$$

By considering the AM-GM inequality $\frac{1}{2}(a+b) \geq \sqrt{ab}$ and letting $a = t^2$

and $b = t^{-2}$, we have

$$\begin{aligned} t^2 + t^{-2} &\geq 2 \\ \implies \sqrt{t^2 + t^{-2}} &\geq \sqrt{2} \end{aligned}$$

By substituting $t = \frac{d(u)}{d(v)}$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned} \sum_{uv \in E(G)} \sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2} &\geq \sum_{uv \in E(G)} \sqrt{2} = \sqrt{2}m \\ \implies DRSO(G) &\geq \sqrt{2}m \end{aligned}$$

Equality holds for any regular graph because every edge contributes $\sqrt{2}$. Conversely, at the edge level, we have

$$\begin{aligned} \sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2} &= \sqrt{2}, \\ \left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2 &= 2 \end{aligned}$$

Let $t = \frac{d(u)}{d(v)} > 0$ and $s = t^{-1} > 0$, then we have to solve the equations, $t^2 + s^2 = 2$ (circle) and $ts = 1$ (hyperbola), simultaneously. Then, $(t, s) = (1, 1), (-1, -1)$. The only valid solution is $(t, s) = (1, 1)$.

$$\begin{aligned} \implies \frac{d(u)}{d(v)} &= 1, \\ \implies d(u) &= d(v). \end{aligned}$$

Hence, the equality holds if and only if G is regular. ∎

Theorem 2. *Let G be a simple and connected graph. Then*

$$\frac{1}{\sqrt{2}} \cdot SDD(G) \leq DRSO(G) < SDD(G)$$

Moreover, the left equality holds when G is regular.

Proof. Under the conditions $p, q > 0$ with $pq = 1$, we have $p^2 + q^2 <$

$p^2+q^2+2pq=(p+q)^2$. Then,

$$\sqrt{p^2+q^2} < p+q$$

From the Cauchy-Schwartz inequality $(p+q)^2 \le 2(p^2+q^2)$, we have

$$\begin{aligned} &\frac{1}{\sqrt{2}}\cdot(p+q) \le \sqrt{p^2+q^2} \\ \implies &\frac{1}{\sqrt{2}}\cdot(p+q) \le \sqrt{p^2+q^2} < (p+q) \end{aligned}$$

By letting $p=\frac{d(u)}{d(v)}$ and $q=\frac{d(v)}{d(u)}$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned} \frac{1}{\sqrt{2}}\cdot\sum_{uv\in E(G)}\left(\frac{d(u)}{d(v)}+\frac{d(v)}{d(u)}\right) &\le \sum_{uv\in E(G)}\sqrt{\left(\frac{d(u)}{d(v)}\right)^2+\left(\frac{d(v)}{d(u)}\right)^2} \\ &< \sum_{uv\in E(G)}\left(\frac{d(u)}{d(v)}+\frac{d(v)}{d(u)}\right) \\ \implies \frac{1}{\sqrt{2}}\cdot SDD(G) \le DRSO(G) &< SDD(G) \end{aligned}$$

When G is regular, we have $d(u)=d(v) \implies \frac{d(u)}{d(v)}=1$. Thus

$$\begin{aligned} \frac{1}{\sqrt{2}}\cdot\sum_{uv\in E(G)}\left(\frac{d(u)}{d(v)}+\frac{d(v)}{d(u)}\right) &\le \sum_{uv\in E(G)}\sqrt{\left(\frac{d(u)}{d(v)}\right)^2+\left(\frac{d(v)}{d(u)}\right)^2}, \\ \frac{1}{\sqrt{2}}\cdot\sum_{uv\in E(G)}(1+1) &\le \sum_{uv\in E(G)}\sqrt{1^2+1^2}, \\ \implies \sum_{uv\in E(G)}\sqrt{2} &\le \sum_{uv\in E(G)}\sqrt{2}. \end{aligned}$$

Thus, the left equality holds when G is regular. ■

Theorem 3. *Let G be a simple connected graph and Δ and δ be the maximum and minimum degrees of G, respectively. Then*

$$\frac{1}{\delta\Delta}\sqrt{\frac{\delta^2+\Delta^2}{2}}\cdot SO(G) \le DRSO(G) \le \frac{1}{\delta}\cdot SO(G)$$

Moreover, equality holds on both sides when G is regular.

Proof. Consider the Cauchy-Schwartz inequality $(a+b)^2 \leq 2(a^2+b^2)$. By letting $a = x^2$ and $b = y^2$, we have

$$\begin{aligned}
(x^2+y^2)^2 &\leq 2(x^4+y^4) \\
\frac{x^2+y^2}{\sqrt{2}} &\leq \sqrt{x^4+y^4} \\
\frac{x^2+y^2}{\sqrt{2}xy} &\leq \frac{\sqrt{x^4+y^4}}{xy} \\
\frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{2}xy} &\leq \frac{\sqrt{x^4+y^4}}{xy} \\
\implies \left(\frac{\sqrt{x^2+y^2}}{\sqrt{2}xy}\right)\cdot\sqrt{x^2+y^2} &\leq \frac{\sqrt{x^4+y^4}}{xy}
\end{aligned}$$

Since $0 < \delta \leq \min\{x,y\} \leq \max\{x,y\} \leq \Delta \implies 0 < \delta \leq x \leq y \leq \Delta$, then $x \geq \delta, y \leq \Delta$ and it follows that

$$\begin{aligned}
\frac{\sqrt{x^2+y^2}}{\sqrt{2}xy} &\leq \frac{\sqrt{\delta^2+\Delta^2}}{\sqrt{2}\delta\Delta} \\
\implies \left(\frac{\sqrt{\delta^2+\Delta^2}}{\sqrt{2}\delta\Delta}\right)\cdot\sqrt{x^2+y^2} &\leq \frac{\sqrt{x^4+y^4}}{xy}
\end{aligned}$$

By letting $x = d(u)$ and $y = d(v)$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned}
\left(\frac{\sqrt{\delta^2+\Delta^2}}{\sqrt{2}\delta\Delta}\right)\sum_{uv\in E(G)}\sqrt{d(u)^2+d(v)^2} &\leq \sum_{uv\in E(G)}\frac{\sqrt{d(u)^4+d(v)^4}}{d(u)d(v)} \\
\implies \frac{1}{\delta\Delta}\sqrt{\frac{\delta^2+\Delta^2}{2}}\cdot SO(G) &\leq DRSO(G)
\end{aligned}$$

Consider the elementary inequality

$$\begin{aligned}
\frac{x^4+y^4}{x^2+y^2} &\leq \max\{x^2,y^2\} \\
\sqrt{\frac{x^4+y^4}{x^2+y^2}} &\leq \max\{x,y\}
\end{aligned}$$

$$\frac{\sqrt{x^4+y^4}}{xy} = \sqrt{\frac{x^4+y^4}{x^2+y^2}} \cdot \frac{\sqrt{x^2+y^2}}{xy} \leq \max\{x,y\} \cdot \frac{\sqrt{x^2+y^2}}{xy}$$
$$\Longrightarrow \frac{\sqrt{x^4+y^4}}{xy} \leq \frac{\max\{x,y\}}{xy} \cdot \sqrt{x^2+y^2}$$

Since $\min\{x,y\} \geq \delta$, we have

$$xy = \min\{x,y\} \cdot \max\{x,y\} \geq \max\{x,y\} \cdot \delta$$
$$\Longrightarrow \frac{\max\{x,y\}}{xy} \leq \frac{1}{\delta}$$

Then, we have

$$\frac{\sqrt{x^4+y^4}}{xy} \leq \frac{1}{\delta} \cdot \sqrt{x^2+y^2}$$

By letting $x = d(u)$ and $y = d(v)$ and summing over all edges $uv \in E(G)$, we have

$$\sum_{uv\in E(G)} \frac{\sqrt{d(u)^4+d(v)^4}}{d(u)d(v)} \leq \frac{1}{\delta} \cdot \sum_{uv\in E(G)} \sqrt{d(u)^2+d(v)^2}$$
$$\Longrightarrow DRSO(G) \leq \frac{1}{\delta} \cdot SO(G)$$

Therefore

$$\frac{1}{\delta\Delta}\sqrt{\frac{\delta^2+\Delta^2}{2}} \cdot SO(G) \leq DRSO(G) \leq \frac{1}{\delta} \cdot SO(G)$$

If G is regular, we have $d(u) = d(v) = \delta = \Delta = k$, then

$$\frac{1}{k^2}\sqrt{\frac{2k^2}{2}} \sum_{uv\in E(G)} \sqrt{2k^2} \leq \sum_{uv\in E(G)} \sqrt{1^2+1^2} \leq \frac{1}{k} \sum_{uv\in E(G)} \sqrt{2k^2}$$
$$\Longrightarrow \sum_{uv\in E(G)} \sqrt{2} \leq \sum_{uv\in E(G)} \sqrt{2} \leq \sum_{uv\in E(G)} \sqrt{2}$$

which implies that the equality holds when G is a regular graph. ■

Theorem 4. *Let G be a simple connected graph and Δ and δ be the* *maximum and minimum degrees of G, respectively. Then*

$$\frac{1}{\Delta^2} \cdot FSO(G) \le DRSO(G) \le \frac{1}{\delta^2} \cdot FSO(G)$$

Moreover, equality holds on both sides when G is regular.

Proof. Let $x = d(u)$ and $y = d(v)$ for $uv \in E(G)$. Since $\delta \le x, y \le \Delta$, we have

$$\begin{aligned} \delta^2 \le xy \le \Delta^2 \\ \implies \frac{1}{\Delta^2} \le \frac{1}{xy} \le \frac{1}{\delta^2} \\ \implies \frac{1}{\Delta^2} \cdot \sqrt{x^4 + y^4} \le \frac{\sqrt{x^4 + y^4}}{xy} \le \frac{1}{\delta^2} \cdot \sqrt{x^4 + y^4} \end{aligned}$$

By substituting $x = d(u)$ and $y = d(v)$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned} \frac{1}{\Delta^2} \cdot \sum_{uv \in E(G)} \sqrt{d(u)^4 + d(v)^4} &\le \sum_{uv \in E(G)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u)d(v)} \\ &\le \frac{1}{\delta^2} \cdot \sum_{uv \in E(G)} \sqrt{d(u)^4 + d(v)^4}. \end{aligned}$$

$$\implies \frac{1}{\Delta^2} \cdot FSO(G) \le DRSO(G) \le \frac{1}{\delta^2} \cdot FSO(G)$$

When G is regular, i.e $d(u) = d(v) = \delta = \Delta = k$, then

$$\begin{aligned} \frac{1}{k^2} \cdot FSO(G) \le DRSO(G) \le \frac{1}{k^2} \cdot FSO(G) \\ \frac{1}{k^2} \cdot \sum_{uv \in E(G)} \sqrt{2k^4} \le \sum_{uv \in E(G)} \sqrt{2} \le \frac{1}{k^2} \cdot \sum_{uv \in E(G)} \sqrt{2k^4} \\ \implies \sum_{uv \in E(G)} \sqrt{2} \le \sum_{uv \in E(G)} \sqrt{2} \le \sum_{uv \in E(G)} \sqrt{2} \end{aligned}$$

which implies that the equality holds for all regular graphs. ■

Theorem 5. *Let G be a simple connected graph and Δ and δ be the* *maximum and minimum degrees of G, respectively. Then*

$$\frac{1}{\sqrt{2}\Delta} \cdot M_1(G) \leq DRSO(G) < \frac{1}{\delta} \cdot M_1(G)$$

Moreover, equality on the left-hand side holds when G is regular.

Proof. Consider the Cauchy-Schwartz inequality $(a+b)^2 \leq 2(a^2+b^2)$. By letting $a = x^2$ and $b = y^2$, we have

$$\begin{aligned}
\frac{x^2+y^2}{\sqrt{2}} &\leq \sqrt{x^4+y^4} \\
\frac{1}{\sqrt{2}} \cdot \frac{x^2+y^2}{xy} &\leq \frac{\sqrt{x^4+y^4}}{xy} \\
\frac{1}{\sqrt{2}} \cdot \left(\frac{x}{y}+\frac{y}{x}\right) &\leq \frac{\sqrt{x^4+y^4}}{xy}
\end{aligned}$$

Since $\frac{x}{\Delta} \leq \frac{x}{y}$ and $\frac{y}{\Delta} \leq \frac{y}{x}$, we have

$$\begin{aligned}
\frac{x+y}{\Delta} &\leq \frac{x}{y}+\frac{y}{x} \\
\implies \quad \frac{1}{\sqrt{2}\Delta} \cdot (x+y) &\leq \frac{\sqrt{x^4+y^4}}{xy}
\end{aligned}$$

By letting $x = d(u)$ and $y = d(v)$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}\Delta} \cdot \sum_{uv \in E(G)} (d(u)+d(v)) &\leq \sum_{uv \in E(G)} \frac{\sqrt{d(u)^4+d(v)^4}}{d(u)d(v)} \\
\implies \quad \frac{1}{\sqrt{2}\Delta} \cdot M_1(G) &\leq DRSO(G)
\end{aligned}$$

Consider a simple inequality $a^2+b^2 < (a+b)^2$. By letting $a = x^2$ and $b = y^2$, we have

$$\begin{aligned}
\sqrt{x^4+y^4} &< x^2+y^2 \\
\frac{\sqrt{x^4+y^4}}{xy} &< \frac{x^2+y^2}{xy} = \frac{x}{y}+\frac{y}{x}
\end{aligned}$$

Since $\frac{x}{y} \leq \frac{x}{\delta}$ and $\frac{y}{x} \leq \frac{y}{\delta}$, we have

$$\begin{aligned} \frac{x}{y} + \frac{y}{x} &\leq \frac{x+y}{\delta} \\ \implies \frac{\sqrt{x^4+y^4}}{xy} &< \frac{1}{\delta} \cdot (x+y) \end{aligned}$$

By letting $x = d(u)$ and $y = d(v)$ and summing over all edges $uv \in E(G)$, we have

$$\begin{aligned} \sum_{uv \in E(G)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u)d(v)} &< \frac{1}{\delta} \cdot \sum_{uv \in E(G)} (d(u) + d(v)) \\ \implies DRSO(G) &< \frac{1}{\delta} \cdot M_1(G) \end{aligned}$$

Therefore

$$\frac{1}{\sqrt{2}\Delta} \cdot M_1(G) \leq DRSO(G) < \frac{1}{\delta} \cdot M_1(G)$$

When G is regular, i.e $d(u) = d(v) = \delta = \Delta = k$, we have

$$\begin{aligned} \frac{1}{\sqrt{2}\Delta} \cdot M_1(G) &\leq DRSO(G), \\ \frac{1}{\sqrt{2}k} \cdot \sum_{uv \in E(G)} 2k &\leq \sum_{uv \in E(G)} \sqrt{2}, \\ \implies \sum_{uv \in E(G)} \sqrt{2} &\leq \sum_{uv \in E(G)} \sqrt{2}. \end{aligned}$$

Hence, the left equality holds when G is regular. ■

Lemma 1. *Let P_n, C_n, and S_n be the path, cycle, and star graphs on n vertices, respectively. Then for $n \geq 3$,*

$$\begin{aligned} DRSO(P_n) &= \sqrt{17} + (n-3)\sqrt{2} \\ DRSO(C_n) &= \sqrt{2}n, \\ DRSO(S_n) &= \sqrt{(n-1)^4 + 1}. \end{aligned}$$

Proof. The path graph P_n has $|V(P_n)| = n$ and $|E(P_n)| = n - 1$ and two types of edges $E_{12} = \{uv \in E(P_n)|d(u) = 1, d(v) = 2\}$ and $E_{22} = \{uv \in E(P_n)|d(u) = 2, d(v) = 2\}$ such that $|E_{12}| = 2$ and $|E_{22}| = n - 3$. Therefore,

$$\begin{aligned} DRSO(P_n) &= \sum_{uv \in E(P_n)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u)d(v)} \\ &= 2\left(\frac{\sqrt{1^4 + 2^4}}{1 \cdot 2}\right) + (n-3)\left(\frac{\sqrt{2^4 + 2^4}}{2 \cdot 2}\right) \\ &= 2\left(\frac{\sqrt{17}}{2}\right) + (n-3)\left(\frac{4\sqrt{2}}{4}\right) \\ \implies DRSO(P_n) &= \sqrt{17} + (n-3)\sqrt{2} \end{aligned}$$

Also, we have $|V(C_n)| = |E(C_n)| = n$. Each vertex in C_n has degree 2. Thus,

$$\begin{aligned} DRSO(C_n) &= \sum_{uv \in E(C_n)} \sqrt{\left(\frac{d(u)}{d(v)}\right)^2 + \left(\frac{d(v)}{d(u)}\right)^2} \\ &= n \cdot \sqrt{\left(\frac{2}{2}\right)^2 + \left(\frac{2}{2}\right)^2} \\ \implies DRSO(C_n) &= \sqrt{2}n \end{aligned}$$

Similarly, the star graph S_n has $|V(S_n)| = n$ and $|E(S_n)| = n - 1$. Each edge has degree-point $(d(u), d(v)) = (1, n - 1)$. Thus,

$$\begin{aligned} DRSO(S_n) &= \sum_{uv \in E(S_n)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u)d(v)} \\ &= (n-1) \cdot \frac{\sqrt{(n-1)^4 + 1^4}}{1 \cdot (n-1)} \\ \implies DRSO(S_n) &= \sqrt{(n-1)^4 + 1} \end{aligned}$$

∎

Theorem 6. *Let G be a simple connected graph. Let C_n and S_n be the cycle and star graphs on n vertices, respectively. Then for $n \geq 3$,*

$$DRSO(C_n) \leq DRSO(G) \leq DRSO(S_n)$$

The left and right equalities hold if and only if $G \cong C_n$ and $G \cong S_n$, respectively.

Proof. Among all connected graphs with fixed number of vertices $n \geq 3$, only a cycle graph C_n attain the minimum number of edges.

Consider the kernel function of the $DRSO$ index $\psi(x, y) = \frac{\sqrt{x^4+y^4}}{xy}$. Since $\psi(x, y)$ is minimized at $x = y$, then the minimum value of $DRSO(G)$ is obtained when all vertices of G have equal degrees (regular graphs). We know that the cycle graph C_n is the unique regular graph in which every vertex has degree 2. Hence, the lower bound is obtained. That is,

$$DRSO(C_n) \leq DRSO(G)$$

We prove the upper bound by the principle of mathematical induction on trees. Among all connected graphs, the highest value of $DRSO(G)$ is attained by a tree graph. Hence, it is sufficient to show that among all trees, the star graph S_n attain the maximum value of the $DRSO$ index.

Base case: From Lemma 1, S_3 and S_4 verify that $DRSO(S_n)$ is the maximum value for trees with n vertices. So the inequality $DRSO(T) \leq DRSO(S_n)$ holds for $n = 3, 4$.

Induction hypothesis: For $n = k \geq 3$, assume that the inequality $DRSO(T) \leq DRSO(S_k) = \sqrt{(k-1)^4 + 1}$ holds with equality if and only if $T \cong S_k$.

Inductive step: For $n = k + 1$, let T be any tree with $k + 1$ vertices. Thus, T can be obtained from a tree T' on k vertices by adding a new vertex v to some vertex $u \in V(T')$. Let $d(u) = x$. The new edge contribution is given by

$$\psi(x, 1) = f(x) = \frac{\sqrt{x^4 + 1}}{x}$$

The function $f(x)$ is strictly increasing for $x \geq 1$. Therefore, the contribution of the new edge is maximized when x is as large as possible. By the induction hypothesis, the tree T' with the maximum degree is S_k, whose central vertex has degree $k-1$. Hence, $DRSO$ is maximized by adding the new vertex to the centre of S_k, creating S_{k+1}. Thus, the inequality

$$DRSO(T) \leq DRSO(S_{k+1}) = \sqrt{k^4+1}$$

holds with equality if and only if $T \cong S_{k+1}$.

Conclusion: For all $n(\geq 3)$ vertices, among all trees,

$$DRSO(T) \leq DRSO(S_n)$$

Since every connected graph G contains a spanning tree and addition of new edges reduces the value of $DRSO$, then

$$DRSO(G) \leq DRSO(S_n)$$

holds with equality if and only if $G \cong S_n$. Therefore,

$$DRSO(C_n) \leq DRSO(G) \leq DRSO(S_n)$$

with left and right equalities if and only if $G \cong C_n$ and $G \cong S_n$, respectively. ■

4 Applications

In this section, we apply the Euclidean geometric approach, degree-ratio and the related ratio-based coordinates to formulate various novel $DRSO$ variants.

1° The Euclidean distance between a dr-point $\left(\frac{x}{y}, \frac{y}{x}\right)$ and its dual ddr-point $\left(\frac{y}{x}, \frac{x}{y}\right)$ is equals to

$$\sqrt{\left(\frac{x}{y}-\frac{y}{x}\right)^2+\left(\frac{y}{x}-\frac{x}{y}\right)^2} = \sqrt{2}\left|\frac{x}{y}-\frac{y}{x}\right|,$$

which is $\sqrt{2}$ times the *IRDIF* irregularity index defined in [31, 43] as

$$IRDIF(G)=\sum_{uv\in E(G)}\left|\frac{d(u)}{d(v)}-\frac{d(v)}{d(u)}\right|$$

2° For a simple connected graph G with n vertices and m edges, the average (equilibrium) vertex degree is given by $\frac{2m}{n}$. Thus, the average (equilibrium) d-point has the coordinates $\left(\frac{2m}{n},\frac{2m}{n}\right)$. Since $x=y=\frac{2m}{n}$, then the corresponding average (equilibrium) dr-ratio point is $(1,1)$. The Euclidean distance between a dr-point and average (equilibrium) dr-ratio point is given by the kernel function

$$\psi(x,y)=\sqrt{\left(\frac{x}{y}-1\right)^2+\left(\frac{y}{x}-1\right)^2}=\frac{|x-y|\cdot\sqrt{x^2+y^2}}{xy}. \tag{9}$$

The simplified expression in Eq. (9) is the product of edge contributions of the Albertson index *Alb* [2] and first Banhatti-Sombor index *BSO* [22]. In this regard, we name a resulting novel index as *Albertson-Banhatti-Sombor index ABSO*, defined as

$$ABSO(G)=\sum_{uv\in E(G)}\frac{|d(u)-d(v)|\cdot\sqrt{d(u)^2+d(v)^2}}{d(u)d(v)} \tag{10}$$

This index is a novel measure of graph irregularity [1, 32, 43].

3° The Euclidean distance between a d-point and dr-point of a graph leads to a novel index, which we designate as the *Degree-Ratio Gap Sombor index DRGSO* defined as

$$DRGSO(G)=\sum_{uv\in E(G)}\sqrt{\left(d(u)-\frac{d(u)}{d(v)}\right)^2+\left(d(v)-\frac{d(v)}{d(u)}\right)^2}. \tag{11}$$

4° The Euclidean distance between a reciprocal d-point $\left(\frac{1}{d(u)},\frac{1}{d(v)}\right)$ and dr-point $\left(\frac{d(u)}{d(v)},\frac{d(v)}{d(u)}\right)$ conceives a novel graph invariant, termed as the

Reciprocal–Ratio Gap Sombor Index RRGSO, defined as

$$RRGSO(G) = \sum_{uv\in E(G)} \sqrt{\left(\frac{1}{d(u)} - \frac{d(u)}{d(v)}\right)^2 + \left(\frac{1}{d(v)} - \frac{d(v)}{d(u)}\right)^2}. \qquad (12)$$

5° Consider a square-root degree-ratio point $\left(\sqrt{\frac{d(u)}{d(v)}}, \sqrt{\frac{d(v)}{d(u)}}\right)$. The novel index proposed in [5] can be interpreted as the additive combination of these coordinates. The absolute difference of these coordinates provides a measure of irregularity of graphs, defined in [41]. The Euclidean distance between the square-root degree-ratio point and origin of the coordinate system gives a novel index, which we hereafter refer to as the *Randić–Sombor index RASO* and it is defined as

$$RASO(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)}{d(v)} + \frac{d(v)}{d(u)}} = \sum_{uv\in E(G)} \sqrt{\frac{d(u)^2 + d(v)^2}{d(u)d(v)}} \qquad (13)$$

The square-root degree-ratio point is equivalent to $\left(\frac{d(u)}{\sqrt{d(u)d(v)}}, \frac{d(v)}{\sqrt{d(u)d(v)}}\right)$. At the edge level, the *RASO* index admits the multiplicative form of the edge contributions of Randić index [30] and Sombor index [12], hence the name. The *Modified Randić–Sombor index* $^{m}RASO$ is defined as

$$^{m}RASO(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\frac{d(u)}{d(v)} + \frac{d(v)}{d(u)}}} = \sum_{uv\in E(G)} \sqrt{\frac{d(u)d(v)}{d(u)^2 + d(v)^2}} \qquad (14)$$

6° Consider a squared degree-ratio point $\left(\left(\frac{d(u)}{d(v)}\right)^2, \left(\frac{d(v)}{d(u)}\right)^2\right)$ of the edge $uv \in E(G)$. Motivated by the Euclidean norm of this point and F-Sombor *FSO* and modified F-Sombor ^{m}FSO indices [23], we define forgotten *DRSO* index *FDRSO* and modified *FDRSO* index $^{m}FDRSO$ as

$$FDRSO(G) = \sum_{uv\in E(G)} \sqrt{\left(\frac{d(u)}{d(v)}\right)^4 + \left(\frac{d(v)}{d(u)}\right)^4} \qquad (15)$$

$$= \sum_{uv\in E(G)} \frac{\sqrt{d(u)^8 + d(v)^8}}{\left(d(u)d(v)\right)^2}$$

and

$$^{m}FDRSO(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\left(\frac{d(u)}{d(v)}\right)^4 + \left(\frac{d(v)}{d(u)}\right)^4}} \tag{16}$$

$$= \sum_{uv\in E(G)} \frac{(d(u)d(v))^2}{\sqrt{d(u)^8 + d(v)^8}}$$

7° The equivalent definition in Eq. (4) can be directly interpreted as the Euclidean norm of the ratio-based point of the form $\left(\frac{x^2}{xy}, \frac{y^2}{xy}\right) = \frac{1}{xy}(x^2, y^2)$. Alternatively, we consider the Euclidean norm of another ratio-based point of the form $\left(\frac{x^2}{x+y}, \frac{y^2}{x+y}\right) = \frac{1}{x+y}(x^2, y^2)$. Motivated by the Euclidean norm of this point, diminished Sombor index *DSO* [29], and F-Sombor index *FSO* [23], we formulate a novel index, henceforth referred to as the *Diminished F-Sombor index DFSO*. By letting $x = d(u)$ and $y = d(v)$, the index is defined as

$$DFSO(G) = \sum_{uv\in E(G)} \frac{\sqrt{d(u)^4 + d(v)^4}}{d(u) + d(v)}. \tag{17}$$

The modified *DFSO* index $^{m}DFSO$ is defined as

$$^{m}DFSO(G) = \sum_{uv\in E(G)} \frac{d(u) + d(v)}{\sqrt{d(u)^4 + d(v)^4}}. \tag{18}$$

8° Consider the ratio-based degree points of the form $\left(\frac{d(u)}{\sqrt{d(v)}}, \frac{d(v)}{\sqrt{d(u)}}\right)$ and $\left(\frac{d(u)^{\frac{3}{2}}}{\sqrt{d(v)}}, \frac{d(v)^{\frac{3}{2}}}{\sqrt{d(u)}}\right)$. The Euclidean distance between the origin of the coordinate system and these points conceive the following novel indices:

(i) *Randić-Dharwad index RDh*

$$RDh(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)^3 + d(v)^3}{d(u)d(v)}}. \tag{19}$$

(ii) *Randić-Forgotten Sombor index RFSO*

$$RFSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^4 + d(v)^4}{d(u)d(v)}}. \tag{20}$$

The edge contribution of these indices can be interpreted as the product of the respective edge contributions of the Randić index [30], Dharwad index [18], and F-Sombor index [23], hence the name. Their modified versions are defined as follows:

(i) *Modified Randić-Dharwad index* ${}^{m}RDh$

$${}^{m}RDh(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)d(v)}{d(u)^3 + d(v)^3}}. \tag{21}$$

(ii) *Modified Randić-Forgotten Sombor index* ${}^{m}RFSO$

$${}^{m}RFSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)d(v)}{d(u)^4 + d(v)^4}}. \tag{22}$$

9° Consider a ratio-based degree point of the form $\left(\frac{\sqrt{d(u)}}{d(v)}, \frac{\sqrt{d(v)}}{d(u)}\right)$. The Euclidean distance from this point to the origin of the coordinate system lead to a novel index, referred to as *Normalized Dharwad index NDh*, defined as

$$NDh(G) = \sum_{uv \in E(G)} \frac{\sqrt{d(u)^3 + d(v)^3}}{d(u)d(v)} \tag{23}$$

The modified normalized Dharwad index ${}^{m}NDh$ is defined as

$${}^{m}NDh(G) = \sum_{uv \in E(G)} \frac{d(u)d(v)}{\sqrt{d(u)^3 + d(v)^3}}. \tag{24}$$

10° Consider a ratio-based point of the form $\left(\frac{d(u)^{\frac{3}{2}}}{d(u)+d(v)}, \frac{d(v)^{\frac{3}{2}}}{d(u)+d(v)}\right)$. The Euclidean distance from this point to the origin of the coordinate system leads to a novel index, which we propose the name *Diminished Dharwad index DDh*. The index is defined as

$$DDh(G) = \sum_{uv\in E(G)} \frac{\sqrt{d(u)^3 + d(v)^3}}{d(u) + d(v)} \tag{25}$$

The modified diminished Dharwad index ${}^{m}DDh$ is defined as

$${}^{m}DDh(G) = \sum_{uv\in E(G)} \frac{d(u) + d(v)}{\sqrt{d(u)^3 + d(v)^3}}. \tag{26}$$

11° Consider the following ratio-based degree points:

(i) $\left(\frac{d(u)}{\sqrt{d(u)+d(v)}}, \frac{d(v)}{\sqrt{d(u)+d(v)}}\right)$

(ii) $\left(\frac{d(u)^{\frac{3}{2}}}{\sqrt{d(u)+d(v)}}, \frac{d(v)^{\frac{3}{2}}}{\sqrt{d(u)+d(v)}}\right)$

(iii) $\left(\frac{d(u)^2}{\sqrt{d(u)+d(v)}}, \frac{d(v)^2}{\sqrt{d(u)+d(v)}}\right)$

The Euclidean distance between these points and the origin of the coordinate system leads to the following novel Sombor-type topological indices

(i) *Root-Diminished Sombor index RDSO*

$$RDSO(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)^2 + d(v)^2}{d(u) + d(v)}} \tag{27}$$

(ii) *Modified Root-Diminished Sombor index* ${}^{m}RDSO$

$${}^{m}RDSO(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u) + d(v)}{d(u)^2 + d(v)^2}} \tag{28}$$

(iii) *Root-Diminished Dharwad index RDDh*

$$RDDh(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)^3 + d(v)^3}{d(u) + d(v)}} \tag{29}$$

(iv) *Modified Root-Diminished Dharwad index* ${}^{m}RDDh$

$$^{m}RDDh(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)+d(v)}{d(u)^3+d(v)^3}} \tag{30}$$

(v) *Root-Diminished F-Sombor index* $RDFSO$

$$RDFSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^4+d(v)^4}{d(u)+d(v)}} \tag{31}$$

(vi) *Modified Root-Diminished F-Sombor index* ${}^{m}RDFSO$

$$^{m}RDFSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)+d(v)}{d(u)^4+d(v)^4}} \tag{32}$$

12° Consider the ratio-based coordinates such that $d(u)+d(v) \geq 3$:

(i) $\left(\frac{d(u)}{\sqrt{d(u)+d(v)-2}}, \frac{d(v)}{\sqrt{d(u)+d(v)-2}}\right)$

(ii) $\left(\frac{d(u)^{\frac{3}{2}}}{\sqrt{d(u)+d(v)-2}}, \frac{d(v)^{\frac{3}{2}}}{\sqrt{d(u)+d(v)-2}}\right)$

(iii) $\left(\frac{d(u)^2}{\sqrt{d(u)+d(v)-2}}, \frac{d(v)^2}{\sqrt{d(u)+d(v)-2}}\right)$

Motivated by the Augmented Euler-Sombor indices AEU [27], the classical Atom-Bond Connectivity index ABC [7], Atom-Bond Sum-Connectivity index ABS [3], and the Euclidean distances from $(0,0)$ to these points, we propose augmented Sombor-type indices and their corresponding modified (reciprocal) versions (ABC variants) defined as

(i) *Augmented Sombor index* $AUSO$

$$AUSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^2+d(v)^2}{d(u)+d(v)-2}} \tag{33}$$

(ii) *Atom-Bond Quadratic-Sum Connectivity index ABQS*

$$ABQS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)^2 + d(v)^2}} \tag{34}$$

(iii) *Augmented Dharwad index ADh*

$$ADh(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^3 + d(v)^3}{d(u) + d(v) - 2}} \tag{35}$$

(iv) *Atom-Bond Cubic-Sum Connectivity index ABCS*

$$ABCS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)^3 + d(v)^3}} \tag{36}$$

(v) *Augmented Forgotten Sombor index AFSO*

$$AFSO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^4 + d(v)^4}{d(u) + d(v) - 2}} \tag{37}$$

(vi) *Atom-Bond Forgotten Connectivity index ABFC*

$$ABFC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)^4 + d(v)^4}} \tag{38}$$

The ABS, $ABQS$, $ABCS$, and $ABFC$ indices form a parametric family ABS_k, defined as

$$ABS_k(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)^k + d(v)^k}} \tag{39}$$

for $k = 1, 2, 3, 4$, respectively.

13° To each of the $DRSO$ variants proposed in this paper, it is possible to associate a “reduced” index by replacing $d(u)$ and $d(v)$ by $d_r(u) = d(u) - 1$ and $d_r(v) = d(v) - 1$, respectively.

5 Conclusion

In this paper, the Degree-Ratio Sombor index *DRSO* and its associated variants are introduced based on the Euclidean geometric approach. Basic mathematical properties of the *DRSO* index are established. Mathematical properties of the formulated *DRSO* variants are posed as open problems. The chemical applicability of the *DRSO* indices and their relationship with existing indices await to be demonstrated.

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