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On Hyperbolic Sombor Index and Other Topological Indices

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Abstract

The Hyperbolic Sombor index $HSO(G)$ is a degree–based invariant obtained by assigning to each edge a weight that depends on the degrees of its end vertices and summing these contributions over the edge set. More precisely,

$$\text{HSO}(G) = \sum_{v_1 v_2 \in E(G)} \frac{(d_G(v_1)^2 + d_G(v_2)^2)^{1/2}}{min\{d_G(v_1), d_G(v_2)\}}.$$

where $d_G(v_1)$ denotes the degree of vertex v_1 in the vertex set of G. We give new bounds for Hyperbolic Sombor index and some well-known topological indices namely, geometric arithmetic index, symmetric division degree index, forgotten index, Albertson index, σ-irregularity index, diminished Sombor index.

1 Introduction

Throughout this paper, let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v_0 \in V(G)$, the

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degree of v_0 is denoted by $d_G(v_0)$. The minimum and maximum degrees of G are denoted by

$$\delta = \delta(G) = \min_{v_0 \in V(G)} d_G(v_0), \qquad \Delta = \Delta(G) = \max_{v_0 \in V(G)} d_G(v_0).$$

For an edge $v_1v_2 \in E(G)$, we shall frequently use the notation

$$\alpha_1 = \min\{d_G(v_1), d_G(v_2)\}, \qquad \alpha_2 = \max\{d_G(v_1), d_G(v_2)\},$$

so that $\delta \leq \alpha_1 \leq \alpha_2 \leq \Delta$.

Chemical graph theory provides a powerful framework for modeling molecular structures by means of graph-theoretic concepts, where vertices represent atoms and edges correspond to chemical bonds. Within this framework, topological indices play a central role in relating structural properties of molecular graphs to various physicochemical characteristics of chemical compounds. Among these descriptors, degree-based topological indices have received sustained attention due to their computational simplicity and strong predictive capabilities [3, 12, 28].

Degree–based topological indices provide quantitative links between molecular structure and graph invariants. Among these, Sombor-type indices have recently attracted attention due to their sensitivity to degree variation [1, 5–8, 10, 14, 17, 18, 21, 22, 24, 29].

We next recall several degree–based topological indices used in this paper. The Zagreb index is among the earliest and most fundamental of them and is defined as follows:

$$M_1(G) = \sum_{v_0 \in V(G)} d_G(v_0)^2.$$

Motivated by the Randić connectivity index of a graph, Vukicevic and Furtula [26] gave the definition of geometric–arithmetic (GA) index as following:

$$GA(G) = \sum_{v_1v_2 \in E(G)} \frac{2(d_G(v_1)d_G(v_2))^{1/2}}{d_G(v_1) + d_G(v_2)}.$$

One of the well-known Adriatic index is symmetric division degree index (SDD) defined by [27]:

$$\mathrm{SDD}(G) = \sum_{v_1v_2 \in E(G)} \left(\frac{d_G(v_1)}{d_G(v_2)} + \frac{d_G(v_2)}{d_G(v_1)} \right).$$

A graph invariant, namely forgotten index (F) [11], similar to first Zagreb index is defined as following:

$$F(G) = \sum_{v_0 \in V(G)} d_G(v_0)^3.$$

A graph is called regular if all its vertices have equal degree. A graph that is not regular, that is, one containing at least two vertices of different degrees, is referred to as irregular. Several numerical measures have been introduced to quantify graph irregularity. Among these, the most thoroughly studied is the Albertson index (Alb) also referred to as the irregularity index [2] defined as

$$\mathrm{Alb}(G) = \sum_{v_1v_2 \in E(G)} |d_G(v_1) - d_G(v_2)|.$$

In order to quantify structural irregularity in graphs, [13] introduced the σ-irregularity index, a degree–based invariant that reflects the imbalance between vertex degrees along edges. It is defined as

$$\sigma(G) = \sum_{v_1v_2 \in E(G)} (d_G(v_1) - d_G(v_2))^2.$$

A modified form of the Sombor index was proposed in [25] and subsequently referred to as the diminished Sombor index in [23]. The Diminished Sombor Index (DSO) is defined by

$$\mathrm{DSO}(G) = \sum_{v_1v_2 \in E(G)} \frac{(d_G(v_1)^2 + d_G(v_2)^2)^{1/2}}{d_G(v_1) + d_G(v_2)}.$$

Building on earlier studies of the Sombor index and its variants, hyperbolic Sombor index (HSO) was introduced using a geometric approach [4, 9]. HSO is a degree–based invariant obtained by assigning to each edge a weight depending on the degrees of its end vertices, and summing these contributions over the edge set. Its expression is given by

$$\mathrm{HSO}(G) = \sum_{v_1 v_2 \in E(G)} \frac{(d_G(v_1)^2 + d_G(v_2)^2)^{1/2}}{min\{d_G(v_1), d_G(v_2)\}}.$$

We can also write the above definition as

$$\mathrm{HSO}(G) = \sum_{v_1 v_2 \in E(G)} \frac{(d_G(v_1)^2 + d_G(v_2)^2)^{1/2}}{d_G(v_1)}$$

where $0 < d_G(v_1) \leq d_G(v_2)$.

The hyperbolic Sombor index was introduced through a geometric interpretation of degree-based topological indices inspired by the properties of a hyperbola [4].

In [9], the authors revisited the hyperbolic Sombor index by identifying and correcting several inaccuracies in earlier results, thus strengthening the theoretical foundations of this index. They further derived new sharp bounds and extremal characterizations for trees, unicyclic graphs, and bicyclic graphs in terms of basic graph parameters such as size and degree measures.

The hyperbolic Sombor index was investigated with a focus on establishing sharp bounds and identifying the tree structures that attain these extremal values [24].

In [1], the hyperbolic Sombor index and its complementary diminished version were examined to strength several earlier results. The authors analyzed how these indices behave under edge addition and established sharp bounds with precise extremal cases for standard graph classes such as paths, cycles, stars, and regular graphs.

In [3], strong correlations between the hyperbolic Sombor index and several key physicochemical properties, including $\Pi-$electron energy, polarizability, molar refractivity, and heavy atom count were demonstrated.

In [19], the hyperbolic Sombor index was studied with its bounds involving fundamental graph parameters and well-known degree-based topological indices pairs such as $RR(G)$ and $Alb(G)$, $M_1(G)$ and $Alb(G)$.

In [4], a geometric framework for degree-based topological indices was introduced, leading to the definition of the hyperbolic Sombor index. The paper investigates its mathematical properties and compares its predictive power, structural sensitivity, and degeneracy with several classical indices through extensive QSPR analyses on octane, nonane, and decane isomers.

In this paper, we extend the analysis of the HSO index by establishing new bounds expressed in terms of fundamental graph parameters and by examining its relationships with several topological indices. The results presented herein aim to provide a deeper insight into the structural properties of the hyperbolic Sombor index.

2 Main results

Our first theorem establishes upper and lower bounds for the hyperbolic Sombor index in terms of the symmetric division degree index.

Theorem 1. *For any simple connected graph G with m edges, minimum degree $\delta \geq 1$ and maximum degree Δ*

$$\frac{\sqrt{2}\,\Delta}{2\delta}\,\mathrm{SDD}(G) \geq \mathrm{HSO}(G) \geq \frac{m}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\mathrm{SDD}(G)\,.$$

Moreover, equality holds in both inequalities if and only if G is regular.

Proof. We first prove the upper bound. Using the definition of the hyperbolic Sombor index and the inequalities $\sqrt{\alpha_1^2+\alpha_2^2} \leq \sqrt{2}\,\alpha_2$ and $\alpha_1 \geq \delta$, we obtain

$$\mathrm{HSO}(G) \leq \frac{\sqrt{2}\,\Delta}{\delta}\,m.$$

On the other hand, we have

$$\frac{d_G(v_1)}{d_G(v_2)} + \frac{d_G(v_2)}{d_G(v_1)} \geq 2 \quad \text{for every } v_1v_2 \in E(G).$$

Summing over all edges yields $\mathrm{SDD}(G) \geq 2m$, that is, $m \leq \frac{1}{2}\,\mathrm{SDD}(G)$.

Combining these two inequalities gives

$$\mathrm{HSO}(G) \;\leq\; \frac{\sqrt{2}\,\Delta}{\delta}\, m \;\leq\; \frac{\sqrt{2}\,\Delta}{2\delta}\,\mathrm{SDD}(G),$$

which proves the upper bound.

Next, we establish the lower bound. As in other articles containing results related to the Sombor index [15, 16, 20, 23] we used following inequalities. For any positive (x, y),

$$(x+y)2^{-1/2} \;\leq\; (x^2+y^2)^{1/2} \;\leq\; x+y.$$

For each edge

$$\frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1} \;\geq\; \frac{\alpha_1+\alpha_2}{\sqrt{2}\alpha_1} = \frac{1}{\sqrt{2}}\Big(1+\frac{\alpha_2}{\alpha_1}\Big).$$

And since $\frac{\alpha_2}{\alpha_1} \geq \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)$, we have

$$\frac{1}{\sqrt{2}}\Big(1+\frac{\alpha_2}{\alpha_1}\Big) \;\geq\; \frac{1}{\sqrt{2}}\Big(1+\frac{1}{2}\Big(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\Big)\Big). \tag{1}$$

Using Equation 1 and the definition of hyperbolic Sombor index, we have

$$\begin{aligned}\mathrm{HSO}(G) &= \sum_{v_1v_2\in E(G)} \frac{\sqrt{d_G^2(v_1)+d_G^2(v_2)}}{d_G(v_1)}\\ &\geq \sum_{v_1v_2\in E(G)} \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \sum_{v_1v_2\in E(G)} \Big(\frac{d_G(v_1)}{d_G(v_2)}+\frac{d_G(v_2)}{d_G(v_1)}\Big)\\ &= \frac{m}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\mathrm{SDD}(G).\end{aligned}$$

For the equality case, both conditions

$$(d_G(v_1)+d_G(v_2))/\sqrt{2} = \sqrt{d_G(v_1)^2+d_G(v_2)^2}$$

and $d_G(v_2)/d_G(v_1) = d_G(v_1)/d_G(v_2)$ must be satisfied. These conditions are fulfilled if and only if $d_G(v_1) = d_G(v_2)$, and hence equality holds when the two vertices have equal degrees. ■

Theorem 1 shows that the hyperbolic Sombor index is tightly controlled by the degree balance of the graph and attains extremal values for regular graphs. Below we give a lower bound for HSO using the geometric-arithmetic index.

Theorem 2. *Let G be a simple connected graph. Then*

$$\mathrm{HSO}(G) \ \geq \ \sqrt{2}GA(G).$$

Furthermore, equality is attained if and only if the graph G is regular.

Proof. Since, $\alpha_1^2 + \alpha_2^2 \geq 2\alpha_1\alpha_2$ we have

$$\frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{\alpha_1} \geq \frac{\sqrt{2\alpha_1\alpha_2}}{\alpha_1} = \sqrt{2}\sqrt{\frac{\alpha_2}{\alpha_1}}.$$

Next, since $\alpha_2 \geq \alpha_1 > 0$, we have $\alpha_2^2 \geq \alpha_1\alpha_2$, and hence

$$\sqrt{\frac{\alpha_2}{\alpha_1}} = \frac{\alpha_2}{\sqrt{\alpha_1\alpha_2}} \ \geq \ \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1 + \alpha_2},$$

Clearly we have

$$\frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{\alpha_1} \ \geq \ \sqrt{2}\frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1 + \alpha_2}.$$

By taking $\alpha_1 = d_G(v_1), \alpha_2 = d_G(v_2)$, we get

$$\mathrm{HSO}(G) \geq \sqrt{2} \sum_{v_1v_2 \in E(G)} \frac{2\sqrt{d_G(v_1)d_G(v_2)}}{d_G(v_1) + d_G(v_2)} = \sqrt{2}GA(G).$$

For equality to hold, we must have equality in both steps for every edge. $\frac{\alpha_1^2+\alpha_1^2}{2} \geq \sqrt{\alpha_1\alpha_1}$ is tight only when $\alpha_1 = \alpha_2$. Thus $d_G(v_1) = d_G(v_2)$ for every adjacent pair v_1v_2, which implies G is r-regular. Conversely, if G is regular, then for each edge v_1v_2,

$$\frac{\sqrt{r^2 + r^2}}{r} = \sqrt{2} \quad \text{and} \quad \frac{2\sqrt{rr}}{r + r} = 1,$$

so $\mathrm{HSO}(G) = \sqrt{2}GA(G)$. The proof is thus concluded. ■

From a chemical graph theory perspective, the bound in Theorem 2 reflects the increased sensitivity of the hyperbolic Sombor index to local degree variations along molecular bonds.

In the following theorem, we give an upper and a lower bounds for HSO(G) using the F(G) index.

Theorem 3. *Let G be a simple connected graph with m edges, minimum degree $\delta \geq 1$, and maximum degree Δ. Then*

$$\frac{F(G)}{\sqrt{2}\Delta^2} \;\leq\; \mathrm{HSO}(G) \;\leq\; \frac{1}{\delta}\sqrt{m\,F(G)}.$$

The equality case occurs only when G is regular.

Proof. From [4], we have

$$\frac{1}{\sqrt{2}\Delta} M_1(G) \leq HSO(G)$$

where $M_1(G) = \sum_{v_1 \in V(G)} d_G(v_1)^2$. Since $d_G(v_0) \leq \Delta$ for all $v_0 \in V(G)$, we have

$$d_G(v_1)^3 \;\leq\; \Delta\, d_G(v_1)^2,$$

and therefore

$$F(G) = \sum_{v_1 \in V(G)} d_G(v_1)^3 \;\leq\; \Delta \sum_{v_1 \in V(G)} d_G(v_1)^2 = \Delta\, M_1(G). \tag{2}$$

Consequently,

$$M_1(G) \;\geq\; \frac{F(G)}{\Delta}. \tag{3}$$

For each edge $v_1v_2 \in E(G)$ we have $\min\{d_G(v_1), d_G(v_2)\} \leq \Delta$, hence

$$\begin{aligned}\frac{\sqrt{d_G(v_1)^2 + d_G(v_2)^2}}{\min\{d_G(v_1), d_G(v_2)\}} \;&\geq\; \frac{1}{\Delta}\sqrt{d_G(v_1)^2 + d_G(v_2)^2} \\ &\geq\; \frac{1}{\sqrt{2}\,\Delta}\big(d_G(v_1) + d_G(v_2)\big),\end{aligned}$$

where we used $\sqrt{x^2+y^2} \geq (x+y)/\sqrt{2}$. Summing over all edges gives

$$\begin{aligned}\mathrm{HSO}(G) \;&\geq\; \frac{1}{\sqrt{2}\,\Delta} \sum_{v_1v_2 \in E(G)} \big(d_G(v_1) + d_G(v_2)\big) \\ &= \frac{1}{\sqrt{2}\,\Delta} \sum_{v_1 \in V(G)} d_G(v_1)^2 = \frac{M_1(G)}{\sqrt{2}\,\Delta}.\end{aligned}$$

Combining this with (3) yields

$$\mathrm{HSO}(G) \;\geq\; \frac{1}{\sqrt{2}\,\Delta} \cdot \frac{F(G)}{\Delta} = \frac{F(G)}{\sqrt{2}\,\Delta^2}.$$

If G is Δ-regular, then each edge contributes $\sqrt{2}$ to $\mathrm{HSO}(G)$, so $\mathrm{HSO}(G) = \sqrt{2}\,m$. Also $F(G) = \sum_{v_0} \Delta^3 = n\Delta^3$ and $m = n\Delta/2$, hence

$$\frac{F(G)}{\sqrt{2}\,\Delta^2} = \frac{n\Delta^3}{\sqrt{2}\,\Delta^2} = \frac{n\Delta}{\sqrt{2}} = \sqrt{2} \cdot \frac{n\Delta}{2} = \sqrt{2}\,m = \mathrm{HSO}(G).$$

Thus equality holds. Conversely, assume that equality holds in $\mathrm{HSO}(G) \geq \frac{F(G)}{\sqrt{2}\,\Delta^2}$. Then equality must hold in both inequalities $\mathrm{HSO}(G) \geq \frac{M_1(G)}{\sqrt{2}\,\Delta}$ and $M_1(G) \geq \frac{F(G)}{\Delta}$. Equality in (2) implies that $d_G(v_0) = \Delta$ for every $v_0 \in V(G)$. Hence, G is Δ-regular.

Since $\min\{d_G(v_1), d_G(v_2)\} \geq \delta$ for every edge $v_1v_2 \in E(G)$, we have

$$\mathrm{HSO}(G) \leq \frac{1}{\delta} \sum_{v_1v_2 \in E(G)} \sqrt{d_G(v_1)^2 + d_G(v_2)^2}.$$

By the Cauchy–Schwarz inequality,

$$\sum_{v_1v_2 \in E(G)} \sqrt{d_G(v_1)^2 + d_G(v_2)^2} \leq \sqrt{m \sum_{v_1v_2 \in E(G)} \big(d_G(v_1)^2 + d_G(v_2)^2\big)}.$$

Finally, we obtain

$$\mathrm{HSO}(G) \leq \frac{1}{\delta}\sqrt{m\,F(G)}.$$

This completes the proof. ■

Theorem 4. *Let G be a simple connected graph with m edges and maximum degree Δ. Then the Hyperbolic Sombor index satisfies*

$$\mathrm{HSO}(G) \;\geq\; \sqrt{2}\,m \;+\; \frac{1}{\sqrt{2}\,\Delta^2}\,\sigma(G).$$

Moreover, equality holds if and only if G is regular.

Proof. As in the proof of the previous theorem, we use

$$\frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1} \;\geq\; \sqrt{2}+\frac{\alpha_2-\alpha_1}{\sqrt{2}\,\alpha_1} \;\geq\; \sqrt{2}+\frac{\alpha_2-\alpha_1}{\sqrt{2}\,\Delta}.$$

Since $0 \leq \alpha_2-\alpha_1 \leq \Delta$, we have

$$\alpha_2-\alpha_1 \;\geq\; \frac{(\alpha_2-\alpha_1)^2}{\Delta}.$$

Therefore,

$$\frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1} \;\geq\; \sqrt{2}+\frac{(\alpha_2-\alpha_1)^2}{\sqrt{2}\,\Delta^2} = \sqrt{2}+\frac{(d_G(v_1)-d_G(v_2))^2}{\sqrt{2}\,\Delta^2}.$$

Summing over all edges yields

$$\mathrm{HSO}(G) \geq \sum_{v_1v_2\in E(G)} \sqrt{2}+\frac{1}{\sqrt{2}\,\Delta^2}\sum_{v_1v_2\in E(G)} (d_G(v_1)-d_G(v_2))^2,$$

which gives

$$\mathrm{HSO}(G) \;\geq\; \sqrt{2}\,m \;+\; \frac{1}{\sqrt{2}\,\Delta^2}\,\sigma(G).$$

If equality holds, then $(d_G(v_1)-d_G(v_2))^2 = 0$ for every edge v_1v_2, so $d_G(v_1)=d_G(v_2)$ on each edge, and hence G is regular. Conversely, if G is regular, then $\sigma(G)=0$ and every edge contributes exactly $\sqrt{2}$ to $\mathrm{HSO}(G)$, so equality holds. ■

Theorem 5. *Let G be a simple graph with minimum degree $\delta > 0$ and maximum degree Δ . Then*

$$\frac{2\Delta}{\delta}\mathrm{DSO}(G) \geq \mathrm{HSO}(G) \;\geq\; 2\,\mathrm{DSO}(G).$$

Moreover, equality holds if and only if G is regular.

Proof. Since $\min\{d_G(v_1), d_G(v_2)\} \leq \frac{d_G(v_1)+d_G(v_2)}{2}$, we have

$$\frac{1}{\min\{d_G(v_1), d_G(v_2)\}} \geq \frac{2}{d_G(v_1) + d_G(v_2)}.$$

Multiplying by the nonnegative quantity $\sqrt{d_G(v_1)^2 + d_G(v_2)^2}$ yields the edgewise inequality

$$\frac{\sqrt{d_G(v_1)^2 + d_G(v_2)^2}}{\min\{d_G(v_1), d_G(v_2)\}} \geq 2\frac{\sqrt{d_G(v_1)^2 + d_G(v_2)^2}}{d_G(v_1) + d_G(v_2)}.$$

Summing over all edges gives $\mathrm{HSO}(G) \geq 2\,\mathrm{DSO}(G)$.

If equality holds, then $\min\{d_G(v_1), d_G(v_2)\} = (d_G(v_1) + d_G(v_2))/2$ for every edge v_1v_2, which forces $d_G(v_1) = d_G(v_2)$ for all $v_1v_2 \in E(G)$. Hence G is regular. Conversely, if G is regular, then every edge contributes $\sqrt{2}$ to $\mathrm{HSO}(G)$ and $\sqrt{2}/2$ to $\mathrm{DSO}(G)$ and therefore $\mathrm{HSO}(G) = 2\,\mathrm{DSO}(G)$.

From Theorem 2 in [4], we have the following:

$$\mathrm{HSO}(G) \leq \frac{1}{\delta}\mathrm{SO}(G)$$

where $\mathrm{SO}(G) = \sum_{v_1v_2 \in E(G)} \sqrt{d_G(v_1)^2 + d_G(v_2)^2}$ denotes the Sombor index. Next, since $d_G(v_1) + d_G(v_2) \leq 2\Delta$ for every edge $v_1v_2 \in E(G)$, we have

$$\frac{1}{d_G(v_1) + d_G(v_2)} \geq \frac{1}{2\Delta}.$$

Multiplying both sides by $\sqrt{d_G(v_1)^2 + d_G(v_2)^2}$ and summing over all edges yields

$$\mathrm{DSO}(G) \geq \frac{1}{2\Delta} \sum_{v_1v_2 \in E(G)} \sqrt{d_G(v_1)^2 + d_G(v_2)^2} = \frac{1}{2\Delta}\mathrm{SO}(G).$$

Consequently,

$$\mathrm{SO}(G) \leq 2\Delta\mathrm{DSO}(G).$$

Combining the above inequalities, we obtain

$$\mathrm{HSO}(G) \leq \frac{2\Delta}{\delta}\mathrm{DSO}(G)$$

and equality condition can be seen easily. ■

3 Conclusion

The present paper contributes to the literature by establishing new bounds for the hyperbolic Sombor index in terms of well-known degree-based topological indices. In particular, we derive two-sided inequalities connecting the hyperbolic Sombor index with the symmetric division degree index, forgotten index, Albertson index and the diminished Sombor index. These bounds provide insight into the structural behavior of the hyperbolic Sombor index and clarify the role of degree balance in determining its extremal values.

The results presented here not only extend existing studies on Sombor-type indices but also strengthen the comparative understanding of degree-based descriptors in chemical graph theory. It is expected that the obtained bounds will be useful in further theoretical investigations and potential applications involving molecular structure analysis.

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