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A New Deficiency Zero Theorem for Power Law Non-Reactant-Determined Kinetic Systems via PL-TIK Decomposition

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Abstract

In this paper, we provide a novel Deficiency Zero Theorem (DZT) for Power Law Non-Reactant-Determined Kinetic (PL-NDK) systems via the notion of Power Law $\hat{T}$-independent Kinetic (PL-TIK) decomposition, a decomposition class such that each sub-network has zero kinetic reactant deficiency. In order to this, we first note that any PL-NDK system admits a PL-TIK decomposition by using the PL-TIK Decomposition Algorithm. Also, to ensure the existence of Zero Deficiency Decomposition (ZDD), we extend the proposed ZDD Algorithm (for MAK systems) to any Deficiency Zero PL-NDK system. Through ZDD and PL-TIK decomposition classes, we show that any weakly reversible, deficiency zero PL-NDK system admits positive equilibria. Lastly, we applied the main theorem of this paper to Schmitz' Global Carbon Cycle Model, which is a well-known and well-studied Deficiency Zero PL-NDK system.

1 Introduction

The concept of deficiency in a Chemical Kinetic System (CKS) is as vital as the presence of oxygen in the human respiration process. This nonnegative

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integer possesses numerous pieces of information for us to describe the dynamical behavior of existing biochemical systems theory (BST) models. The first Deficiency Zero Theorem (DZT) was established by Feinberg in [5] for Mass Action Kinetic (MAK) systems. The graph-theoretic concept of weak reversibility is essential in describing the set of positive equilibria of weakly reversible, deficiency zero MAK systems.

Not until a few decades later did Müller and Regensburger establish the first DZT result beyond Mass Action, namely, on Generalized Mass Action Kinetic (GMAK) systems [9]. In their paper, they used the concept of kinetic deficiency to conclude the existence of a positive equilibrium in cycle terminal GMAK systems with zero kinetic deficiency.

On the other hand, Talabis et al. introduced the concept of kinetic reactant deficiency and established the Zero Kinetic Reactant Deficiency Theorem (ZKRDT) [11], namely, a DZT result on a class of Power Law reactant-Determined Kinetic (PL-RDK) systems having a column-rank maximal, augmented matrix of kinetic orders (that is, on Power Law $\hat{T}-$independent Kinetic (PL-TIK) systems).

In addition, Fortun et al. [6] provided the first analogous DZT result beyond PL-RDK systems, namely, Power Law Non-Reactant-Determined Kinetic (PL-NDK) systems) by employing weakly reversible independent decomposition. The concept of independent decomposition (and network decomposition theory, in general) was first studied by Feinberg in [3] to show that a positive equilibrium in the whole kinetic system is also a positive equilibrium in each of its sub-networks. Farinas et al. generated an analogous result in [2] via Incidence-independent decomposition, whereby a complex balanced equilibrium in the whole kinetic system is also a complex balanced equilibrium in each of its sub-networks. They also introduced the concept of Zero Deficiency Decomposition (ZDD) for any chemical kinetic system, that is, each sub-network has zero deficiency [2].

On the other hand, Tiongson et al. in [12] presented kinetics-based decomposition classes such as the Zero Kinetic Deficiency decomposition (ZKDD) and the PL-TIK Decomposition, wherein the latter was used to characterize the existence of positive equilibria of weakly reversible PL-RDK systems.

The main goal of this paper is to present a new DZT result on a class of PLK systems by applying PL-TIK decomposition to weakly reversible, Deficiency Zero PL-NDK systems. As an application, we considered the well-studied Schmitz' Global Carbon Cycle Model and showed that it can admit complex balanced equilibria, and hence positive equilibria, via the main result of this study.

2 Foundations of chemical kinetic systems

In this section, we briefly discuss some important fundamental concepts in chemical kinetic systems (CKS) and known results on the existence of positive steady states. In addition, we also discuss some important network and kinetics-based decomposition classes and their properties. For a more comprehensive discussion of other fundamental concepts, the reader is advised to read the article in [12].

Definition 1. [1] A **chemical reaction network** (CRN) is an ordered triple $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ where $\mathcal{S}$ is the finite set of all species or variables in the CRN, $\mathcal{C}$ is the set of all complexes in the CRN, and $\mathcal{R}$ is the set of all reactions involved in the CRN such that

a. $(y, y) \notin \mathcal{R}$ for any $y \in \mathcal{C}$;

b. for each $y \in \mathcal{C}$, there exists $y^{'} \in \mathcal{C}$ such that $(y, y^{'}) \in \mathcal{R}$ or $(y^{'}, y) \in \mathcal{R}$.

The above definition simply states that a CRN has no self-loops while every complex must be an element of any reaction in $\mathcal{R}$. Denote the cardinalities of $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{R}$ by m, n, and r, respectively.

Now, a CRN can induce important matrices that are useful in this study. The **molecularity matrix** Y of a CRN is the $m \times n$ matrix whose entries are the coefficients of the complexes (called the **stoichiometric coefficients**. The **incidence matrix** I_a of a CRN is the $n \times r$ matrix whose entries are $1, -1, 0$, depending if a complex is a reactant complex, a product complex, or neither. The **stoichiometric matrix** N is an $m \times r$ matrix whose entries are the stoichiometric coefficients of the associated

reaction vector of the i^{th} reaction. The **associated reaction vector** of the i^{th} reaction is the difference between the reactant complex and the product complex.

On the other hand, we recalled some graph-theoretic CRNT concepts that are important for us to formulate the results of this study. A **linkage class** $\mathcal{L}$ is a connected subgraph/component of the given CRN. A **connected subgraph/component** implies that there is an undirected path between every pair of complexes. The number of linkage classes is denoted by l. A **strong linkage class** $\mathcal{L}$ is a strong connected subgraph/component of the given CRN. A **strong connected subgraph/component** implies that there must be a directed path between every pair of complexes. The number of strong linkage classes is denoted by sl.

Weak reversibility is an important graph-theoretic structure in CRNT. A CRN is called **weakly reversible** if and only if $sl = l$ [1]. Also, the **stoichiometric subspace**, denoted by S, is the vector space generated by the reaction vectors. Its cardinality is given by s. The deficiency of a CRN is a nonnegative integer that determines the linear independence of a CRN's reactions. The **deficiency** of a CRN is the integer $\delta = n - l - s$ [1]. This concept has the capacity to characterize the existence of positive equilibria in a given chemical kinetic system.

We now proceed to define a Chemical Kinetic System (CKS), which is an ordered pair containing a CRN N coupled with the chosen kinetics K.

Definition 2. [1]. A **kinetics** K for a reaction network $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ associates to each reaction $j \in \mathcal{R}$ a mapping $K_j : \mathbb{R}^m_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $K_j(c) > 0$ if and only if supp y is contained in supp c (positivity condition). The pair $(\mathcal{N}, K)$ is called a chemical kinetic system (CKS).

This paper shall focus on the dynamics of a particular class of chemical kinetic systems.

Definition 3. [1]. The **species formation rate function** (SFRF) $f : \mathbb{R}^m_{\geq 0} \to \mathbb{R}^n$ of a CKS is defined as

$$f(x) = NK(x) = \sum_{y \to y' \in \mathcal{R}} K_{y \to y'}(x)(y' - y), \forall x \in \mathbb{R}^m_{\geq 0}.$$

The ODE system of a CKS is $dx/dt = f(x)$. A zero of f is called an equilibrium or steady state of the kinetic system.

Definition 4. [8]. The **complex formation rate function** (CFRF) $g : \mathbb{R}^m_{\geq 0} \to \mathbb{R}^n$ of a chemical kinetic system is given by

$$g(c) = I_a K(c) = \sum_{y_j \to y_j' \in \mathcal{R}} K_j(c)(\omega_{y_j'} - \omega_{y_j}), \forall c \in \mathbb{R}^m_{\geq 0}$$

where $\omega_{y_j'}$ is the standard basis vector corresponding to the complex y_j'.

The **set of positive steady states**, denoted as $E_+(\mathcal{N}, K)$, is defined as $E_+(\mathcal{N}, K) = \{x \in \mathbb{R}^m_+ | f(x) = 0\}$ [11]. On the other hand, a positive vector c in $\mathbb{R}^m$ is called **complex balanced** (CB) if and only if $K(c)$ is contained in $\ker I_a$. The **set of complex balanced equilibria** is defined as $Z_+(\mathcal{N}, K) = \{c \in \mathbb{R}^m_+ | I_a \cdot K(c) = 0\}$ [11].

Proposition 1. *[7, 11]. If $Z_+(\mathcal{N}, K) \neq \emptyset$, then $\mathcal{N}$ is weakly reversible.*

Proposition 2. *[4, 10]. If a chemical kinetic system has deficiency 0, then its equilibria are all complex balanced.*

Definition 5. [11]. A kinetics K is a **power-law kinetics** if and only if

$$K_i(x) = k_i \prod_{i=1}^{r} x^{F_{ij}} \quad \forall i \in \overline{1, r}, \text{with } k_i, F_{ij} \in \mathbb{R}_+.$$

Power Law Kinetics (PLK) is defined by an $r \times m$ kinetic order matrix $F = [F_{ij}]$ and a kinetic rate vector $k \in \mathbb{R}^r$. The well-studied **Mass Action Kinetic** (MAK) system is an important subclass of PLK systems, where the kinetic orders are simply the stoichiometric coefficients of the reactant complexes. As our main concern, a class of PLK systems will now be introduced.

Definition 6. [11]. A PLK system has **reactant-determined kinetics (of type PL-RDK)** if and only if for any two reactions i, j with identical reactant complexes, the corresponding rows of kinetic orders in F are identical, i.e., $f_{ik} = f_{jk}$ for $k = 1, ..., m$. Otherwise, the PLK system has **non-reactant-determined kinetics (of type PL-NDK)**.

We then define a PL-TIK system through the two important matrices, namely, the $T-$matrix and the $\hat{T}-$matrix.

Definition 7. [11]. The $m \times n_r$ $T-$matrix is the truncated $\tilde{Y}$ where the non-reactant columns are deleted. The $\tilde{Y}-$matrix is defined below.

$$(\tilde{Y})_{ij} = \begin{cases} (F)_{ki}, & \text{if } j \text{ is a reactant complex of reaction } k \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, define a block matrix $\hat{T} \in \mathbb{R}^{(m+l)\times n_r}$ by $\hat{T} = \begin{bmatrix} T \\ L^T \end{bmatrix}$, where $L = [e^1, ..., e^l]$ and $e^1, ..., e^l \in \{0, 1\}^n$.

Definition 8. [11]. A PL-RDK kinetics is $\hat{T}-$**rank maximal (of type PL-TIK)** if and only if its column-rank is maximal.

Remark. [11]. PL-TIK systems are PLK systems whose reactions with the same reactant complexes have the same kinetic order vectors (called interactions) which are linearly independent per linkage class.

Definition 9. [11]. Let $\mathcal{N}$ be a network with n_r reactant complexes and K a PL-RDK kinetics with T-matrix T. If $\hat{q} = rank(\hat{T})$, then the **kinetic reactant deficiency** $\hat{\delta}$ is defined as $\hat{\delta} = n_r - \hat{q}$.

A PL-TIK system is exactly a PL-RDK system with zero kinetic reactant deficiency, as showed by Talabis et al. in [11].

Proposition 3. *$(\mathcal{N}, K)$ is a PL-TIK system if and only if $\hat{\delta} = 0$.*

Talabis et al. developed and proved the Zero Kinetic Reactant Deficiency Theorem, which concludes the existence of complex balanced equilibria for any weakly reversible PL-TIK system [11].

Theorem 4. *(Zero Kinetic Reactant Deficiency Theorem (ZKRDT)). Let K be a PL-RDK kinetics with $T-$matrix T on $\mathcal{N}$ and $\hat{\delta} = 0$. Then $\mathcal{N}$ is weakly reversible if and only if $Z_+(\mathcal{N}, K) \neq \emptyset$.*

Lastly, we discuss some important network decomposition classes of a CRN, as well as the equilibria properties induced by these graph-theoretic concepts.

Definition 10. [2]. Let $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ be a CRN. A **covering** of $\mathcal{N}$ is a set of subsets of $\mathcal{R}_i$ whose union is $\mathcal{R}$. A covering is called a **decomposition** of $\mathcal{N}$ if the sets $\mathcal{R}_i$ form a partition of $\mathcal{R}$.

A **linkage class decomposition** is a decomposition class such that each linkage class is considered a sub-network. The next decomposition class is due to Feinberg's initiative. It is obtained by relating the stoichiometric subspace of the parent network and the stoichiometric subspaces of its sub-networks.

Definition 11. [2]. A decomposition is **Independent** if and only if S is the direct sum of the sub-networks' stoichiometric subspaces S_i or equivalently, $s = s_1 + \cdots + s_k$.

Theorem 5. *(Feinberg Decomposition Theorem) [2].*
Let $P(\mathcal{R}) = \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_k$ be a partition of $\mathcal{N}$ and let K be the kinetics. If $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_k$ is a network decomposition of $P(\mathcal{R})$ and $E_+(\mathcal{N}_i, K_i) = \{x \in \mathbb{R}^m_+ | \mathcal{N}_i K_i(x) = 0\}$, then

$$E(\mathcal{N}_1, K_1) \cap \cdots \cap E_+(\mathcal{N}_k, K_k) \subseteq E_+(\mathcal{N}, K).$$

If the network decomposition is independent, then equality holds.

A decomposition class that relates the incidence map of the parent network and the incidence maps of its sub-networks is introduced next.

Definition 12. [2]. A decomposition $\mathcal{N} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k$ is **Incidence-independent** if and only if the image of the incidence map of $\mathcal{N}$ is the direct sum of the images of the incidence maps of $\mathcal{N}_i$'s, or equivalently, $n - l = \sum_{i=1}^{k} (n_i - l_i)$.

Theorem 6. *[2]. Let $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ be a CRN and $\mathcal{N}_i = (\mathcal{S}_i, \mathcal{C}_i, \mathcal{R}_i)$, $i = 1, \cdots, k$ be the sub-networks of a decomposition. Let K be any kinetics and $Z_+(\mathcal{N}, K)$, $Z_+(\mathcal{N}_i, K_i)$ be as defined in Theorem 5. Then,*

i. $\cap Z_+(\mathcal{N}_i, K_i) \subset Z_+(\mathcal{N}, K)$.
If the decomposition is incidence-independent, then

ii. $Z_+(\mathcal{N}, K) = \cap Z_+(\mathcal{N}_i, K_i)$; *and*

iii. if $Z_+(\mathcal{N}, K) \neq \emptyset$, *then* $Z_+(\mathcal{N}_i, K_i) \neq \emptyset$ *for each* i.

In this paper, two essential network decomposition classes based on the sub-networks' deficiencies and kinetics are introduced below.

Definition 13. [2]. A decomposition is a **Zero Deficiency Decomposition** (ZDD) if and only if its decomposition deficiency is zero, or equivalently, each sub-network in the decomposition has zero deficiency.

Tiongson et al. showed in [13] that weakly reversible, non-zero deficiency MAK systems with ZDD can admit complex balanced equilibria.

Definition 14. [12]. A decomposition of a PLK system is called a **PL-TIK Decomposition** if and only if each sub-network has an associated column-rank maximal $\hat{T}_i-$matrix, or equivalently, $\hat{\delta}_i = 0$.

We also recall the established PL-TIK Decomposition algorithm in [12], which states that any PLK system ($\mathcal{N}$, K) has a PL-TIK Decomposition.

Step 1. Consider the kinetic order matrix F of ($\mathcal{N}$, K) and get its transpose F^T. Identify the linkage classes from ($\mathcal{N}$, K) and check if there is an NDK-reactant complex for each linkage class $\mathcal{L}_i, i \in \{1, 2, ..., l\}$. If there is no NDK-reactant complex, then proceed to Step 3. Otherwise, proceed to Step 2.

Step 2. If there is at least one NDK-reactant complex, say y, separate the kinetic order vectors v_i's associated to y by choosing appropriate RDK subsets, that is, each subset containing the kinetic order vectors v_i's associated to y forms a PL-RDK system. For each kinetic order vector v_i of the i^{th} branching reaction associated with y, denote these subsets by RDK_{v_i}.

Step 3. Form the associated $\hat{T}-$matrix for each RDK-subset by augmenting the kinetic order vectors $v_1, v_2, ..., v_n$. Denote the augmented kinetic order vectors by $\hat{v}_1, \hat{v}_2, ..., \hat{v}_n$.

Step 4. For each RDK-subset, if the column vectors $\hat{v}_1, \hat{v}_2, ..., \hat{v}_n$ are linearly independent, then we are done. Otherwise, isolate from this set the first $\hat{v}_j$, which is a linear combination of its predecessors $\hat{v}_1, \hat{v}_2, ..., \hat{v}_{j-1}$, and collect the reactions formed by these predecessors to form $\mathcal{N}_1$.

Step 5. If the remaining vectors $\hat{v}_j, \hat{v}_{j+1}, ..., \hat{v}_n$ are linearly independent, then collect the reactions formed by these vectors to form $\mathcal{N}_2$, and we are done. Otherwise, isolate from this set the second linearly dependent column vector $\hat{v}_m, j < m \leq n$, and collect the reactions formed by the predecessors $\hat{v}_j, \hat{v}_{j+1}, ..., \hat{v}_{m-1}$ to form $\mathcal{N}_2$.

Step 6. Continue the isolation process on the remaining vectors until there is no column vector that is a linear combination of its predecessors. The sub-networks formed will comprise our PL-TIK decomposition for the given PLK system.

Tiongson et al. showed that any weakly reversible PL-RDK system (whether it is PL-TIK or not) with PL-TIK decomposition admits complex balanced equilibria [12]. In this study, we will provide an analogous version for Deficiency Zero PL-NDK systems.

2.1 Previous results on PL-TIK decomposition

This subsection highlights the known results on the application of PL-TIK decomposition to different subclasses of PL-RDK systems. The goal of this paper is to provide a new result applying the said decomposition class beyond PL-RDK systems, namely, Deficiency Zero PL-NDK systems.

Tiongson et al. in [12] showed that for MAK systems, ZDD is equivalent to PL-TIK decomposition since the entries of $\hat{T}$ are the stoichiometric coefficients of the complexes, and hence $\hat{T} = \hat{Y}$. The following results for MAK systems can be found in [13]:

Proposition 7. *Let $(\mathcal{N}, K)$ be a MA-TIK system. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_k$ be any decomposition.*

1. *If $\delta = 0$, then $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if $(\mathcal{N}, K)$ has a weakly reversible decomposition.*

2. *If $\delta > 0$, then $Z_+(\mathcal{N}, K) = \emptyset$.*

Theorem 8. *Let $(\mathcal{N}, K)$ be a non-zero deficiency MA-NTIK system. Let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_k$ be a zero deficiency, $\mathcal{C}-$decomposition. Then, $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if the decomposition is weakly reversible.*

Remark. Proposition 7 showed that only Deficiency Zero MA-TIK systems with weakly reversible decomposition are complex balanced. Also, Theorem 8 showed that Non-Zero Deficiency MA-NTIK systems with weakly reversible, Zero Deficiency, $\mathcal{C}-$decomposition (and hence weakly reversible, PL-TIK, $\mathcal{C}-$decomposition) are complex balanced.

Theorem 9. *Let $(\mathcal{N}, K)$ be a PL-RDK system with PL-TIK, $\mathcal{C}-$decomposition. Then $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if the decomposition is weakly reversible.*

Remark. Theorem 9 asserted that any PL-RDK system (whether it is PL-TIK or non-PL-TIK) with weakly reversible, PL-TIK, $\mathcal{C}-$decomposition is complex balanced.

3 Main result and discussion

In this section, we state an alternative Deficiency Zero Theorem for a class of PLK systems. The result is then applied to Schmitz' Global Carbon Cycle model to validate the existence of the system's positive equilibria.

3.1 A new deficiency zero theorem for deficiency zero PL-NDK systems

Fortun et al. [6] provided the first analogous DZT result on PL-NDK systems by employing a weakly reversible independent decomposition. However, not all kinetic systems have the capacity to generate weakly reversible, independent decompositions. In this subsection, we provide a new DZT via weakly reversible PL-TIK decomposition, allowing the analysis of Deficiency Zero PL-NDK systems using a kinetics-based decomposition class.

Lemma 1. *Let $(\mathcal{N}, K)$ be a Deficiency Zero PL-NDK system. Then there exists a Zero Deficiency Decomposition $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_k$ in $\mathcal{N}$.*

Proof. The claim directly follows from the ZDD algorithm in [13] since the construction of the $\hat{Y}-$matrix does not rely on the power law-based kinetics of $(\mathcal{N}, K)$. ∎

We are now ready to formulate our main result. Based on the above lemma, we can assume ZDD on any Deficiency Zero, PL-NDK system.

Theorem 10. *Let $(\mathcal{N}, K)$ be a Deficiency Zero PL-NDK system with Zero Deficiency, PL-TIK decomposition. Then, $Z_+(\mathcal{N}, K) \neq \emptyset$ if and only if the decomposition is weakly reversible.*

Proof. Suppose $(\mathcal{N}, K)$ is a Deficiency Zero PL-NDK system with Zero Deficiency, PL-TIK decomposition. Suppose $Z_+(\mathcal{N}, K) \neq \emptyset$. Suppose further that the decomposition is not weakly reversible. It follows that at least one sub-network, say N_f, is not weakly reversible. Since $\delta = 0$, it follows from DZT that for any kinetics, $E_+(\mathcal{N}_f, K_f) = \emptyset$. Since $Z_+(\mathcal{N}_f, K_f) \subseteq E_+(\mathcal{N}_f, K_f)$, we have $Z_+(\mathcal{N}_f, K_f) = \emptyset$, or equivalently, $I_{a,f}K_f(c) \neq 0$. Since the decomposition is PL-TIK, we have

$$I_{a,1}K_1(c) + ... + I_{a,f}K_f(c) + ... + I_{a,k}K_k(c) = I_aK(c) \neq 0$$

Hence, $Z_+(\mathcal{N}, K) = \emptyset$, a contradiction, and so the decomposition must be weakly reversible.

For the converse, suppose the decomposition is weakly reversible and $Z_+(\mathcal{N}, K) = \emptyset$. It follows from the proof of Theorem 6.*i* in [2] that

$$0 \neq I_aK(c) = \sum_{i=1}^{k} I_{a,i}K_i(c)$$

Thus, for any vector $c \in \mathbb{R}^m_{\geq 0}$, at least one sub-network, say N_j, yields $0 \neq I_{a,j}K_j(c)$. This means that $Z_+(\mathcal{N}_j, K_j) = \emptyset$. Since the decomposition is PL-TIK, it follows from Theorem 4 that N_j is not weakly reversible, a contradiction. Hence, $Z_+(\mathcal{N}, K) \neq \emptyset$. ■

Remark. The above theorem can be thought of as a "Weak Reversibility Theorem for Deficiency Zero PL-NDK systems", since a weakly reversible parent network follows from a weakly reversible decomposition, which ensures the existence of complex balanced equilibria. Moreover, since $Z_+(\mathcal{N}, K) \subseteq E_+(\mathcal{N}, K)$, it follows that $E_+(\mathcal{N}, K) \neq \emptyset$ if and only if the Zero Deficiency, PL-TIK decomposition of a Deficiency Zero PL-NDK system is also weakly reversible.

Remark. Unlike the independent decomposition assumption (which may not exist for some PL-NDK systems) of Fortun et al.'s main result in [6], the PL-TIK Decomposition Algorithm in [12] ensures that a PL-TIK decomposition exists for any Deficiency Zero PL-NDK system. Therefore, via Theorem 10, one can easily infer the capacity of this subclass of PL-NDK systems to admit complex balanced equilibria.

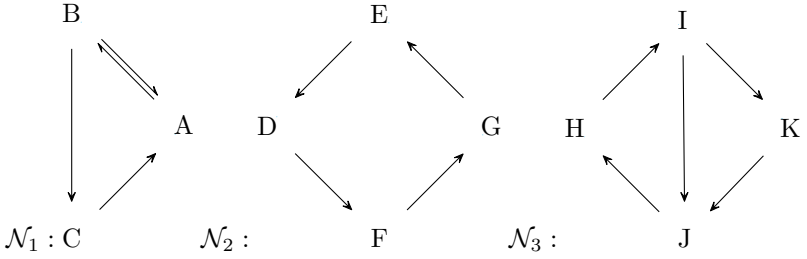
3.2 Application to Schmitz' global carbon cycle model

In this subsection, we consider Schmitz's Global Carbon Cycle model, which is a well-studied Deficiency Zero PL-NDK system. Fortun et al. (2019) showed that a subnetwork of this system admits positive equilibria via weakly reversible, independent decomposition with zero reactant deficiency. However, the authors did not apply the independent decomposition to the whole system. Hence, the main result of this paper validates the existence result in [6] for the whole kinetic system by identifying a weakly reversible, PL-TIK decomposition (Theorem 10). We recall the PL-TIK decomposition of Schmitz's Global Carbon Cycle Model found in [12].

$$\mathcal{N}_1 : \{r_1, r_2, r_3, r_4\},$$

$$\mathcal{N}_2 : \{r_5, r_6, r_7, r_8\},$$

$$\mathcal{N}_3 : \{r_9, r_{10}, r_{11}, r_{12}, r_{13}\}$$



where for
$\mathcal{N}_1 : A = M_1, B = M_5, C = M_6$;
for $\mathcal{N}_2 : D = M_1, E = M_2, F = M_3, G = M_4$; and
for $\mathcal{N}_3 : H = M_1, I = M_2, J = M_3, K = M_4$.

Remark. Observe first that the above decomposition is not independent since $s = 5 \neq 2 + 3 + 3 = s_1 + s_2 + s_3$. Hence, we cannot apply the main theorem in [6]. Now, observe that the above PL-TIK decomposition is also weakly reversible. Then, by applying Theorem 10, it can be inferred that $Z_+(\mathcal{N}, K) \neq \emptyset$ and hence, $E_+(\mathcal{N}, K) \neq \emptyset$. Lastly, since the above model has zero deficiency and is complex balanced, we argue based on Feinberg's ACB Theorem (see Proposition 2) that it is also Absolutely Complex Balanced (ACB), that is, each positive equilibrium is complex balanced.

4 Conclusion and outlook

The results of this paper, as well as some research problems and recommendations, are summarized as follows:

1. We showed the existence of zero deficiency decomposition for any Deficiency Zero PL-NDK system. We plan to modify this algorithm to generate a weakly reversible zero deficiency decomposition.

2. We guaranteed the existence of a PL-TIK decomposition from any PLK system using the PL-TIK Decomposition algorithm from [12]. We also plan to enhance this algorithm to generate a weakly reversible PL-TIK decomposition.

3. We proved our main result involving the characterization of a Deficiency Zero PL-NDK system's positive equilibria via weakly reversible PL-TIK decomposition (Theorem 10) and applied it to Schmitz's Global Carbon Cycle Model. We showed that the model admits positive equilibria and we argued that it is absolutely complex balanced. We then plan to extend the main result of this study by providing analogous results for Non-Zero Deficiency PL-NDK systems. We also plan to apply the proposed extension to complex biochemical systems with the above-mentioned CRN properties.

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