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On the Local Laplacian Energy of Graphs

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Abstract

For a simple graph G with n vertices, m edges, and Laplacian eigenvalues $\mu_1, \dots, \mu_n$, the Laplacian energy $LE(G)$ is defined as $LE(G) = \sum_{k=1}^{n} |\mu_k - \frac{2m}{n}|$. In this paper, we derive an upper bound for the variation in Laplacian energy resulting from a removal of a vertex and characterize the graphs that attain this bound. Furthermore, we define the local Laplacian energy of a graph and establish its relationship with the Laplacian energy of the graph.

1 Introduction

Let G be a simple graph of order n and size m. The *adjacency matrix* of G, denoted by $A(G) = (a_{ij})$, is an $n \times n$ matrix where the entry $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of G, denoted by $L(G)$, is defined as the difference between the degree matrix $D(G)$ and the adjacency matrix $A(G)$, that is,

$$L(G) = D(G) - A(G).$$

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Let d_k be the *degree* of the vertex v_k for $k = 1, \dots, n$. Then the *average degree* of G, denoted by $\bar{d}$, is

$$\bar{d} = \frac{1}{n}\sum_{k=1}^{n} d_k = \frac{2m}{n}.$$

The eigenvalues of G are the eigenvalues of $A(G)$, which are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and the Laplacian eigenvalues of G are the eigenvalues of $L(G)$, which are denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The concept of the energy of a graph was first introduced by Gutman in 1978 [4]. Later, in 2006, Gutman and Zhou extended this concept by introducing the notion of Laplacian energy [5]. We now present the definitions provided by them. The *energy* of the graph G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$:

$$E(G) = \sum_{k=1}^{n} |\, \lambda_k \,|.$$

The *Laplacian energy* of the graph G, denoted by $LE(G)$, is defined as the sum of the absolute deviations of the Laplacian eigenvalues from the average degree $\bar{d}$:

$$LE(G) = \sum_{k=1}^{n} \left|\, \mu_k - \bar{d} \,\right|.$$

A central focus in the study of graph energy is understanding how the energy of a graph changes when specific subgraphs are removed. This problem, commonly referred to as the graph energy change problem, has been extensively studied, particularly in the context of analyzing the effects of edge deletions on graph energy [1,2,8–10]. In 2024, Espinal and Rada [3] introduced a novel perspective on this area of research by providing insights into how graph energy is affected by vertex removal. They proposed the concepts of the local energy of a graph at a vertex and the local energy of a graph, which are defined as follows. Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. Denote by $G - v$ the graph obtained by removing the vertex v and all its incident edges from G. The *local energy of G at a* *vertex* v, denoted by $E_G(v)$, is defined as

$$E_G(v) = E(G) - E(G - v),$$

where $E(G)$ and $E(G - v)$ represent the energy of the graph G and the energy of the graph $G - v$, respectively. The *local energy* of G, denoted by $e(G)$, is then defined as the sum of the local energies at all vertices:

$$e(G) = \sum_{k=1}^{n} E_G(v_k).$$

A relationship between $e(G)$ and $E(G)$ was established in [3], given by the inequality $e(G) \leq 2E(G)$. More recently, Rakshith and Das [7] introduced an infinite class of graphs for which $e(G) < 2E(G)$.

In this paper, we investigate the variation in Laplacian energy resulting from the removal of a vertex. Specifically, we extend the concepts of the local energy graphs introduced by Espinal and Rada to the framework of local Laplacian energy graphs. To this end, we provide formal definitions for the local Laplacian graph energy at a vertex and the local Laplacian energy of a graph. Furthermore, we derive an upper bound for the local Laplacian graph energy at a vertex and identify the class of graphs that achieve this bound. In contrast to the local energy of a graph, the local Laplacian energy of a graph is not bounded by twice the Laplacian energy of the graph. To illustrate this distinction, we provide a counterexample and establish an upper bound for the local Laplacian energy of a graph.

2 Main results

Let M be an $n \times n$ real matrix. The singular values of M are the eigenvalues of $\sqrt{MM^T}$. The *trace norm of* M, denoted by $\|M\|_*$, is the sum of the singular values of M, that is,

$$\|M\|_* = \mathrm{Tr}\left(\sqrt{MM^T}\right) = \sum_{k=1}^{n} \sigma_k(M),$$

where $\sigma_1(M), \ldots, \sigma_n(M)$ are singular values of M. If M is symmetric, the singular values of M are the absolute values of its eigenvalues. Thus, for a graph G of order n, $\sigma_k(L(G) - \bar{d}I_n) = \left| \mu_k - \bar{d} \right|$ for each $k = 1, \ldots, n$, where I_n is the $n \times n$ identity matrix. Hence

$$LE(G) = \sum_{k=1}^{n} \left| \mu_k - \bar{d} \right| = \left\| L(G) - \bar{d}I_n \right\|_* .$$

We extend the notions of local energy graphs introduced by Espinal and Rada to the local Laplacian energy graphs. In the following, we formally define the local Laplacian graph energy at a vertex and the local Laplacian energy of a graph.

Definition 1. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. For a vertex v in G, the *local Laplacian energy of G at v*, denoted by $LE_G(v)$, is

$$LE_G(v) = LE(G) - LE(G - v)$$

and *the local Laplacian energy of G*, denoted by $le(G)$, is

$$le(G) = \sum_{k=1}^{n} LE_G(v_k).$$

We compute the local Laplacian energy at a vertex for the star graph.

Example 1. For a complete bipartite graph $K_{p,q}$, its Laplacian energy is

$$LE(K_{p,q}) = \frac{(p+q)^2 + |\, p - q \,|(2pq - (p+q))}{p+q}.$$

In particular, the Laplacian energy of the star graph $S_n = K_{1,n-1}$ with $n \geq 3$ is

$$LE(S_n) = \frac{2n^2 - 4n + 4}{n}.$$

If v is the center vertex of S_n, then $S_n - v$ is isomorphic to $(n-1)K_1$ and hence the local Laplacian energy of S_n at v is

$$LE_{S_n}(v) = LE(S_n) - (n-1)LE(K_1) = LE(S_n) = \frac{2n^2 - 4n + 4}{n}.$$

For any pendant vertex v of S_n, the local Laplacian energy at v is

$$LE_{S_n}(v) = LE(S_n) - LE(S_{n-1}) = 2 - \frac{4}{n(n-1)}.$$

The following two lemmas are essential for proving our main results.

Lemma 1. *[6, Proposition 2.4] Let A, B be $m \times n$ matrices. Then $\|A+B\|_* = \|A-B\|_* = \|A\|_* + \|B\|_*$ if and only if $A^T B = O$ and $BA^T = O$.*

Lemma 2. *For any $k \in \mathbb{R}$, the spectrum of the $n \times n$ matrix of the following form*

$$\begin{bmatrix} k & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is $\left\{1^{(n-2)}, \frac{k+1 \pm \sqrt{(k-1)^2+4(n-1)}}{2}\right\}$.

Proof. Let M be the matrix defined in the above statement. We consider the matrix $M - I_n$. Since the rank of $M - I_n$ is 2, the matrix $M - I_n$ has eigenvalue 0 with multiplicity $n-2$. Thus the characteristic polynomial of the matrix $M - I_n$ is

$$x^{n-2}(x^2 - a_1 x + a_2),$$

where a_1 and a_2 are real numbers. Let λ_1 and λ_2 be the remaining two eigenvalues of $M - I_n$. Since a_1 is the trace of $M - I_n$ and a_2 is the sum of the 2×2 principal minors of $M - I_n$, we have $a_1 = k-1$ and $a_2 = -(n-1)$. Hence

$$\lambda_1 = \frac{k - 1 - \sqrt{(k-1)^2 + 4(n-1)}}{2}$$

and

$$\lambda_2 = \frac{k - 1 + \sqrt{(k-1)^2 + 4(n-1)}}{2}.$$

Therefore the spectrum of M is

$$\left\{1^{(n-2)},\quad \frac{k+1\pm\sqrt{(k-1)^2+4(n-1)}}{2}\right\}. \qquad \blacksquare$$

In the following theorem, we present an upper bound for the local Laplacian graph energy at a vertex.

Theorem 1. *Let G be a graph of order n and size m, and let v be a vertex of G. Then*

$$LE_G(v) \le (n-d_v-1)\left|\bar{d}^{(v)}-\bar{d}\right| + (d_v-1)\left|1+\bar{d}^{(v)}-\bar{d}\right| + \sqrt{(d_v-\bar{d}^{(v)}-1)^2+4d_v},$$

where d_v is the degree of v in G and $\bar{d}^{(v)}$ is the average degree of $G-v$, that is, $\bar{d}^{(v)} = \frac{2(m-d_v)}{n-1}$. Moreover, the equality holds if and only if G is a star graph with v as its center.

Proof. We set $X := L(G)-\bar{d}I_n$ and $X^{(v)} := L(G-v)-\bar{d}^{(v)}I_{n-1}$. Note that $L(G-v)$ is an $(n-1)\times(n-1)$ matrix obtained from $L(G)$ by removing both the row and column corresponding to vertex v, with the diagonal entries reflecting the degrees of the remaining vertices, each reduced by 1 to account for the edges incident to v. Let $V(G)=\{v_1,\dots,v_n\}$ denote the vertex set of the graph G. Without loss of generality, we assume that $v=v_1$, and that the set $\{v_2,\dots,v_{d_v+1}\}$ represents the neighborhood of v. The matrix $Y^{(v)}$ is defined as an $n\times n$ matrix, where the entries in the first row and first column are zeros, and the lower-right $(n-1)\times(n-1)$ submatrix is given by $X^{(v)}$. Specifically, $Y^{(v)}$ can be represented as a block matrix in the following form:

$$Y^{(v)} = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & X^{(v)} \end{bmatrix},$$

where $\mathbf{0}$ denotes the $(n-1)$-dimensional column vector of zeros. Then we have

$$LE_G(v) = LE(G) - LE(G-v) = \left\|X\right\|_* - \left\|X^{(v)}\right\|_*$$

$$= \left\|X\right\|_* - \left\|Y^{(v)}\right\|_* \leq \left\|X - Y^{(v)}\right\|_*.$$

The matrix $X - Y^{(v)}$ can be written in the following form:

$$X - Y^{(v)} = \begin{bmatrix} d_v - \bar{d}^{(v)} & -1 & \cdots & -1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} - \left(\bar{d} - \bar{d}^{(v)}\right)I.$$

By Lemma 2, the spectrum of $X - Y^{(v)}$ is

$$\left\{\left(\bar{d}^{(v)} - \bar{d}\right)^{(n-(d_v+1))}, \left(1 + \bar{d}^{(v)} - \bar{d}\right)^{(d_v-1)}, \lambda_1, \lambda_2\right\},$$

where

$$\lambda_1 = \frac{d_v - \bar{d}^{(v)} + 1 - \sqrt{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}}{2} + \bar{d}^{(v)} - \bar{d}$$

and

$$\lambda_2 = \frac{d_v - \bar{d}^{(v)} + 1 + \sqrt{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}}{2} + \bar{d}^{(v)} - \bar{d}.$$

Thus

$$\left\|X - Y^{(v)}\right\|_* = (n - d_v - 1)\left|\,\bar{d}^{(v)} - \bar{d}\,\right| + (d_v - 1)\left|\,1 + \bar{d}^{(v)} - \bar{d}\,\right| + \left|\lambda_1\right| + \left|\lambda_2\right|.$$

If $\lambda_1\lambda_2 < 0$, then $|\lambda_1| + |\lambda_2| = \sqrt{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}$. Thus, we arrive at the following conclusion.

$$\begin{aligned} LE_G(v) \leq\ & \left\|X - Y^{(v)}\right\|_* \\ &= (n - d_v - 1)\left|\,\bar{d}^{(v)} - \bar{d}\,\right| + (d_v - 1)\left|\,1 + \bar{d}^{(v)} - \bar{d}\,\right| \\ &+ \sqrt{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}. \end{aligned}$$

Now, we show that $\lambda_1\lambda_2 < 0$. Note that

$$\begin{aligned}\lambda_1\lambda_2 &= \left(\frac{d_v + \bar{d}^{(v)} + 1}{2} - \bar{d}\right)^2 - \left(\frac{\sqrt{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}}{2}\right)^2 \\ &= \frac{(d_v + \bar{d}^{(v)} + 1)^2}{4} - \bar{d}(d_v + \bar{d}^{(v)} + 1) + \bar{d}^2 - \frac{(d_v - \bar{d}^{(v)} - 1)^2 + 4d_v}{4} \\ &= \bar{d}^2 - \bar{d}(d_v + \bar{d}^{(v)} + 1) + d_v\bar{d}^{(v)} \\ &= (\bar{d} - d_v)(\bar{d} - \bar{d}^{(v)} - 1) - d_v.\end{aligned}$$

By the definitions of $\bar{d}$ and $\bar{d}^{(v)}$, we have

$$\begin{aligned}\lambda_1\lambda_2 &= \left(\frac{2m}{n} - d_v\right)\left(\frac{2m}{n} - \frac{2m - 2d_v}{n-1} - 1\right) - d_v \\ &= \left(\frac{2m - nd_v}{n}\right)\left(\frac{2m(n-1) - n(2m - 2d_v)}{n(n-1)} - 1\right) - d_v \\ &= \left(\frac{2m - nd_v}{n}\right)\left(\frac{nd_v - m\binom{n}{2}}{\binom{n}{2}} - 1\right) - d_v.\end{aligned}$$

If $2m = nd_v$ then $\lambda_1\lambda_2 = -d_v < 0$.

If $2m > nd_v$, then

$$(2m - nd_v)\left(nd_v - m - \binom{n}{2}\right) - nd_v\binom{n}{2} \leq (2m - nd_v)(nd_v - 2m) - nd_v\binom{n}{2} < 0.$$

If $2m < nd_v$, then $nd_v - m > 0$. Hence

$$(2m - nd_v)\left(nd_v - m - \binom{n}{2}\right) - nd_v\binom{n}{2} = (2m - nd_v)(nd_v - m) - 2m\binom{n}{2} < 0.$$

Therefore $\lambda_1\lambda_2 < 0$.

Now, we show that the equality holds if and only if G is a star graph with v as its center. Assume that the equality holds. Then, we have

$$\|X\|_* - \|Y^{(v)}\|_* = \|X - Y^{(v)}\|_*.$$

By Lemma 1, the matrices $X - Y^{(v)}$ and $Y^{(v)}$ satisfy the following conditions:

$$\big(X - Y^{(v)}\big)Y^{(v)} = \big(X - Y^{(v)}\big)^T Y^{(v)} = O.$$

We express the Laplacian matrix of G as follows:

$$\begin{bmatrix} d_v & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ -1 & a_{22} & a_{23} & \cdots & a_{2(d_v+1)} & a_{2(d_v+2)} & \cdots & a_{2n} \\ -1 & a_{32} & a_{33} & \cdots & a_{3(d_v+1)} & a_{3(d_v+2)} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & a_{(d_v+1)2} & a_{(d_v+1)3} & \cdots & a_{(d_v+1)(d_v+1)} & a_{(d_v+1)(d_v+2)} & \cdots & a_{(d_v+1)n} \\ 0 & a_{(d_v+2)2} & a_{(d_v+2)3} & \cdots & a_{(d_v+2)(d_v+1)} & a_{(d_v+2)(d_v+2)} & \cdots & a_{(d_v+2)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{n(d_v+1)} & a_{n(d_v+2)} & \cdots & a_{nn} \end{bmatrix}$$

Let $\mathbf{x}$ denote the $(n-1)-$dimensional vector obtained by removing the first entry of the first column of $L(G)$ and let L denote the $(n-1)\times(n-1)$ submatrix of $L(G)$ obtained by deleting the first row and the first column of $L(G)$. Then the matrices X and $Y^{(v)}$ can be expressed as follows:

$$X = \begin{bmatrix} d_v - \bar{d} & \mathbf{x}^T \\ \mathbf{x} & L - \bar{d}I_{n-1} \end{bmatrix}$$

and

$$Y^{(v)} = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & L - (I_{d_v} \oplus O_{n-1-d_v}) - \bar{d}^{(v)} I_{n-1} \end{bmatrix}.$$

Since

$$O = \left(X - Y^{(v)}\right) Y^{(v)} = \begin{bmatrix} 0 & \mathbf{x}^T\big(L - (I_{d_v} \oplus O_{n-1-d_v}) - \bar{d}^{(v)} I_{n-1}\big) \\ \mathbf{0} & * \end{bmatrix},$$

it follows that

$$0 = -\sum_{i=2}^{d_v+1} a_{ij} + 1 + \bar{d}^{(v)} \quad \text{for } 2 \le j \le d_v + 1 \tag{1}$$

and

$$0 = -\sum_{i=2}^{d_v+1} a_{ij} \quad \text{for } d_v + 2 \le j \le n. \tag{2}$$

From equation (2), it follows that $a_{ij} = 0$ for $2 \le i \le d_v + 1$ and $d_v + 2 \le j \le n$. By the symmetry property of the Laplacian matrix, we also deduce that $a_{ij} = 0$ for $d_v + 2 \le i \le n$ and $2 \le j \le d_v + 1$. Consequently, the Laplacian matrix $L(G)$ can be expressed in the following block-structured form:

$$\begin{bmatrix} d_v & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ -1 & a_{22} & a_{23} & \cdots & a_{2(d_v+1)} & 0 & \cdots & 0 \\ -1 & a_{32} & a_{33} & \cdots & a_{3(d_v+1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & a_{(d_v+1)2} & a_{(d_v+1)3} & \cdots & a_{(d_v+1)(d_v+1)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{(d_v+2)(d_v+2)} & \cdots & a_{(d_v+2)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n(d_v+2)} & \cdots & a_{nn} \end{bmatrix}$$

Since

$$-1 + \sum_{i=2}^{d_v+1} a_{i2} = 0,$$

it directly follows from equation (1) that $\bar{d}^{(v)} = 0$ and hence $m = d_v$. Thus, graph G is the star graph S_n with v as its center vertex. The converse can be shown using Example 1. ■

It has been shown in [3] that the local energy of a graph G is always at most twice the energy of G. However, this property does not hold for local Laplacian energy. We provide an example to illustrate this fact.

Example 2. For $n \ge 5$, the local Laplacian energy of S_n is greater than twice its Laplacian energy. Let v_1 denote the center and $v_2, \ldots, v_n$ the pendant vertices of S_n. Using the formulas from Example 1, the local Laplacian energy of S_n is given by

$$le(S_n) = LE_{S_n}(v_1) + \sum_{i=2}^{n} LE_{S_n}(v_i) = LE(S_n) + (n-1)\left(2 - \frac{4}{n(n-1)}\right)$$

$$= LE(S_n) + \frac{2n^2 - 4n + 4}{n} + \frac{2n-8}{n} = 2LE(S_n) + \frac{2n-8}{n}.$$

It follows that $le(S_n) > 2LE(S_n)$ for $n \geq 5$. On the other hand, for $n = 4$, we have $le(S_4) = 2LE(S_4)$.

The following theorem establishes a relationship between the local Laplacian energy of a graph and its Laplacian energy.

Theorem 2. *Let G be a graph of order n and size m. Then*

$$le(G) \leq 2LE(G) + \frac{2}{n-1} \sum_{k=1}^{n} \left| d_k - \bar{d} \right|.$$

In particular, if G is regular, then $le(G) \leq 2LE(G)$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ denote vertex set of G. We adopt the notations for X, $X^{(v)}$, and $Y^{(v)}$ from the proof of Theorem 1. For notational simplicity, we use $X^{(k)}$ and $Y^{(k)}$ to denote $X^{(v_k)}$ and $Y^{(v_k)}$, respectively.

We consider the summation of $Y^{(k)}$ over $k = 1, \ldots, n$, that is, $\sum_{k=1}^{n} Y^{(k)}$. We first analyze its off-diagonal entries. The (i,j)-entry of $Y^{(k)}$, denoted by $(Y^{(k)})_{ij}$, is -1 if v_i is adjacent to v_j and $k \neq i, j$, and 0 otherwise. Consequently, the off-diagonal (i,j)-entry of $\sum_{k=1}^{n} Y^{(k)}$ is given by

$$\left(\sum_{k=1}^{n} Y^{(k)} \right)_{ij} = \begin{cases} -(n-2), & \text{if } v_i \text{ is adjacent to } v_j \text{ in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we examine the diagonal entries of $\sum_{k=1}^{n} Y^{(k)}$. The diagonal entries of $Y^{(k)}$ are determined by the following three cases for each k:

1. If $i = k$, then $(Y^{(k)})_{ii} = 0$ by the definition of $Y^{(k)}$.

2. If $i \neq k$ and v_i is adjacent to v_k, then $(Y^{(k)})_{ii} = d_i - 1 - \bar{d}^{(k)}$.

3. If $i \neq k$ and v_i is not adjacent to v_k, then $(Y^{(k)})_{ii} = d_i - \bar{d}^{(k)}$.

Summing these entries over all k, the diagonal (i,j)-entry of $\sum_{k=1}^{n} Y^{(k)}$ is

$$\left(\sum_{k=1}^{n} Y^{(k)} \right)_{ii} = d_i(d_i - 1) + (n - d_i - 1)d_i - \sum_{k=1}^{n} \bar{d}^{(k)} + \bar{d}^{(i)}$$

$$= (n-2)d_i - \sum_{k=1}^{n} \bar{d}^{(k)} + \bar{d}^{(i)}.$$

Thus, we obtain

$$\sum_{k=1}^{n} Y^{(k)} = (n-2)L(G) - \sum_{k=1}^{n} \bar{d}^{(k)} I_n + \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)}).$$

We examine the expression for $(n-2)LE(G)$, given by

$$(n-2)LE(G) = (n-2) \left\| L(G) - \bar{d}I_n \right\|_* = \left\| (n-2)(L(G) - \bar{d}I_n) \right\|_* .$$

By adding and subtracting the matrix

$$\sum_{k=1}^{n} \bar{d}^{(k)} I_n + \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)})$$

inside the trace norm $\left\| (n-2)(L(G) - \bar{d}I_n) \right\|_*$, and applying the triangle inequality, we obtain

$$\begin{aligned}
(n-2)LE(G) &\leq \left\| (n-2)L(G) - \sum_{k=1}^{n} \bar{d}^{(k)} I_n + \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)}) \right\|_* \\
&\quad + \left\| \left(\sum_{k=1}^{n} \bar{d}^{(k)} - (n-2)\bar{d} \right) I_n - \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)}) \right\|_* \\
&= \left\| \sum_{k=1}^{n} Y^{(k)} \right\|_* + \left\| \left(\sum_{k=1}^{n} \bar{d}^{(k)} - (n-2)\bar{d} \right) I_n - \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)}) \right\|_* \\
&\leq \sum_{k=1}^{n} \left\| Y^{(k)} \right\|_* + \left\| \left(\sum_{k=1}^{n} \bar{d}^{(k)} - (n-2)\bar{d} \right) I_n - \operatorname{diag}(\bar{d}^{(1)}, \ldots, \bar{d}^{(n)}) \right\|_* .
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=1}^{n} \bar{d}^{(k)} &= \sum_{k=1}^{n} \frac{2m - 2d_k}{n-1} = \frac{2mn - 2\sum_{k=1}^{n} d_k}{n-1} \\
&= \frac{2mn - 4m}{n-1} = \frac{2m(n-2)}{(n-1)},
\end{aligned}$$

it follows that

$$\sum_{k=1}^{n} \bar{d}^{(k)} - (n-2)\bar{d} - \bar{d}^{(i)} = 2m(n-2)\left(\frac{1}{n-1} - \frac{1}{n}\right) - \bar{d}^{(i)}$$
$$= \frac{2m(n-2)}{n(n-1)} - \frac{2m-2d_i}{n-1} = \frac{2}{n-1}\big(d_i - \bar{d}\,\big)$$

for each i. Therefore, we have

$$(n-2)LE(G) \leq \sum_{k=1}^{n} \left\| Y^{(k)} \right\|_* + \frac{2}{n-1} \sum_{k=1}^{n} \left| d_k - \bar{d} \right|,$$

which implies

$$le(G) = \sum_{k=1}^{n} LE_G(v_k) \leq 2LE(G) + \frac{2}{n-1} \sum_{k=1}^{n} \left| d_k - \bar{d} \right|.$$

If G is regular, then $d_k = \bar{d}$ for all k and thus

$$le(G) \leq 2LE(G).$$ ■

We note that the upper bound given in Theorem 2 is attained when $G = K_2$. Furthermore, for $G = S_4$, we have $le(G) = 2LE(G)$.

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