MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **96** (2026) 899–906
ISSN: 0340–6253
doi: 10.46793/match.96-3.24925

A Lower Bound on the Energy of Graphs in Terms of Degree Sequence

Saieed Akbari, Akbar Jahanbani*

Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

s_akbari@sharif.edu, akbar.jahanbani92@gmail.com

(Received September 28, 2025)

Abstract

This paper gives an affirmative answer to a conjecture proposed by Goldberg (2014) relating to graph energy which says that for a graph G with m edges,

$$\varepsilon(G) \geq \frac{2m^2}{\sum\limits_{xy \in E(G)} \sqrt{d_x d_y}}.$$

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. The *adjacency matrix* $A(G)$ is an $n \times n$ symmetric matrix with entries $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A(G)$.

The *energy* of G is defined in [10] as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

*Corresponding author.

This work is licensed under a Creative Commons “Attribution 4.0 International” license.

For further details and related results, we refer the reader to [2,3,11].

The energy of a vertex $x_i \in V(G)$ is denoted by $\varepsilon(x_i)$ and defined in [5] by,

$$\varepsilon(x_i) = |A(G)|_{ii}, \quad \text{for all } i \in \{1, \ldots, n\},$$

where $|A| = (AA^*)^{1/2}$ and $A(G)$ is the adjacency matrix of the graph G. Let G be a graph and $V(G) = \{x_1, \ldots, x_n\}$. The energy of a graph is the sum of its vertex energies,

$$\varepsilon(G) = \sum_{i=1}^{n} \varepsilon(x_i).$$

Let $e = xy \in E(G)$. By [6], one can consider the energy of an edge as,

$$\varepsilon(e) = \frac{\varepsilon(x)}{d_x} + \frac{\varepsilon(y)}{d_y}.$$

Thus the energy of G is

$$\varepsilon(G) = \sum_{e \in E(G)} \varepsilon(e) = \sum_{e=xy \in E(G)} \left(\frac{\varepsilon(x)}{d_x} + \frac{\varepsilon(y)}{d_y} \right).$$

The following theorem establishes an upper bound for the energy of any vertex in a graph in terms of its degree.

Theorem 1. *[5] Let G be a graph and let x be an arbitrary vertex of G with degree d_x. Then,*

$$\varepsilon(x) \leq \sqrt{d_x}.$$

Furthermore, equality occurs if and only if the connected component containing x is isomorphic to S_n and x is its center.

Theorem 2. *[5] Let G be a graph containing at least one edge. Then, for every vertex $x \in V(G)$, the following inequality holds:*

$$\varepsilon(x) \geq \sqrt{\frac{d_x}{\Delta(G)}},$$

where $\Delta(G)$ is the maximum degree in the graph G.

Theorem 3. *[4] Let x and y be adjacent vertices in a simple undirected graph G. Then*

$$\varepsilon(x)\varepsilon(y) \geq 1.$$

The following conjecture was proposed in [8].

Conjecture 1. *Let G be a connected graph with m edges. Then the energy $\varepsilon(G)$ satisfies*

$$\varepsilon(G) \geq \frac{2m^2}{\sum_{uv\in E(G)} \sqrt{d_u d_v}}.$$

Remark. If G is an r-regular graph of order n, then $m = \frac{nr}{2}$. Hence, we find that

$$\frac{2m^2}{\sum_{xy\in E(G)} \sqrt{d_x d_y}} = \frac{2m^2}{mr} = \frac{2m}{r} = n.$$

It is a known result that for any r-regular graph, $\mathcal{E}(G) \geq n$, for instance see [9]. Therefore, Conjecture 1 is valid for all regular graphs.

In this paper, we prove Conjecture 1.

2 Proof of Conjecture 1

In this section, we provide a proof for Conjecture 1. Before starting, we need to review several topological indices.

The Randić index, the Sombor index, the first Zagreb index and the forgotten index of G are defined as follows, respectively:

$$R(G) = \sum_{xy\in E(G)} \frac{1}{\sqrt{d_x d_y}};$$

$$SO(G) = \sum_{xy\in E(G)} \sqrt{d_x^2 + d_y^2};$$

$$M_1(G) = \sum_{xy\in E(G)} (d_x + d_y);$$

$$F(G) = \sum_{xy\in E(G)} (d_x^2 + d_y^2).$$

In [4], the authors obtained the following bound on the energy as follows:

$$\varepsilon(G) \geq 2R(G). \tag{1}$$

In previous works [12, 13], several bounds for the energy of graphs were established. Subsequently, Akbari et al. [1] improved these bounds and derived the following new inequalities:

$$\varepsilon(G) \geq \frac{M_1(G)}{\Delta^2(G)}; \tag{2}$$

$$\varepsilon(G) \geq \frac{F(G)}{\Delta^3(G)}; \tag{3}$$

$$\varepsilon(G) \geq \frac{(SO(G))^2}{m\Delta^3(G)}. \tag{4}$$

Since for any vertices $x, y \in V(G)$, the inequalities $(d_x + d_y)\sqrt{d_x d_y} \leq 2\Delta^2$ and $(d_x^2 + d_y^2)\sqrt{d_x d_y} \leq 2\Delta^3$ hold, it follows that:

$$\frac{d_x + d_y}{\Delta^2} \leq \frac{2}{\sqrt{d_x d_y}} \quad \text{and}$$
$$\frac{d_x^2 + d_y^2}{\Delta^3} \leq \frac{2}{\sqrt{d_x d_y}}.$$

Summing these inequalities over all edges $xy \in E(G)$ yields:

$$\sum_{xy \in E(G)} \frac{d_x + d_y}{\Delta^2} \leq \sum_{xy \in E(G)} \frac{2}{\sqrt{d_x d_y}} \quad \text{and}$$
$$\sum_{xy \in E(G)} \frac{d_x^2 + d_y^2}{\Delta^3} \leq \sum_{xy \in E(G)} \frac{2}{\sqrt{d_x d_y}}.$$

A direct comparison shows that the bound in inequality (1) is sharper than the bounds given in inequalities (2) and (3).

Also, by [7], for any graph with m edges and spectral radius λ_1, we have

$$R(G) \geq \frac{m}{\lambda_1}.$$

Furthermore, using the inequality $\lambda_1 \leq \Delta(G)$, we conclude

$$R(G) \geq \frac{m}{\Delta(G)}.$$

Since

$$\frac{(\mathrm{SO}(G))^2}{m\Delta^3} \leq \frac{2\Delta^2 m^2}{m\Delta^3} = \frac{2m}{\Delta}.$$

Thus,

$$2R(G) \geq \frac{(\mathrm{SO}(G))^2}{m\Delta^3}.$$

Therefore, we conclude that the bound in inequality (1) is better than the bound in inequality (4).

In the following theorem, we present a new lower bound that is stronger than inequality (1). Specifically, it improves upon all previously mentioned bounds for graph energy.

Theorem 4. *If G is a connected graph, then*

$$\varepsilon(G) \geq 2R(G) + \frac{1}{3} \sum_{xy \in E(G)} \left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}} \right)^2.$$

Proof. For two positive numbers a and b, define $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$, and $N(a,b) = \frac{a+b+\sqrt{ab}}{3}$. By inequality $a + b \geq 2\sqrt{ab}$, we have

$$3(a+b) \geq 2(a+b+\sqrt{ab}),$$

which implies

$$A(a,b) = \frac{a+b}{2} \geq \frac{a+b+\sqrt{ab}}{3} = N(a,b).$$

Cearly, $N(a,b) \geq G(a,b)$.

Moreover, a direct computation shows that

$$A(a,b) - N(a,b) = \frac{(\sqrt{a} - \sqrt{b})^2}{6}.$$

Now, for an edge $e = xy$, we have

$$\varepsilon(e) = \frac{\varepsilon(x)}{d_x} + \frac{\varepsilon(y)}{d_y} = 2A\left(\frac{\varepsilon(x)}{d_x}, \frac{\varepsilon(y)}{d_y}\right).$$

Using the inequality $A(x,y) \geq N(x,y)$, we get

$$\begin{aligned}
\varepsilon(e) &= 2A\left(\frac{\varepsilon(x)}{d_x}, \frac{\varepsilon(y)}{d_y}\right) \\
&= 2N\left(\frac{\varepsilon(x)}{d_x}, \frac{\varepsilon(y)}{d_y}\right) + \frac{1}{3}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2 \\
&\geq 2G\left(\frac{\varepsilon(x)}{d_x}, \frac{\varepsilon(y)}{d_y}\right) + \frac{1}{3}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2.
\end{aligned}$$

Hence,

$$\varepsilon(e) \geq 2\sqrt{\frac{\varepsilon(x)\varepsilon(y)}{d_x d_y}} + \frac{1}{3}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2.$$

Furthermore, an application of Theorem 3 yields the inequality $\varepsilon(x)\varepsilon(y) \geq 1$ for adjacent vertices x and y. Hence, we have

$$\varepsilon(e) \geq \frac{2}{\sqrt{d_x d_y}} + \frac{1}{3}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2.$$

Summing over all edges $e \in E(G)$ gives

$$\begin{aligned}
\varepsilon(G) &\geq 2\sum_{xy\in E(G)} \frac{1}{\sqrt{d_x d_y}} + \frac{1}{3}\sum_{xy\in E(G)}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2 \\
&= 2R(G) + \frac{1}{3}\sum_{xy\in E(G)}\left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}}\right)^2,
\end{aligned}$$

which completes the proof. ■

Theorem 5. *Conjecture 1 holds.*

Proof. If G is a regular graph, then we did it before. Thus, assume that G is a non-regular graph. By the previous result, we have

$$\varepsilon(G) \geq 2R(G) + \frac{1}{3} \sum_{xy \in E(G)} \left(\sqrt{\frac{\varepsilon(x)}{d_x}} - \sqrt{\frac{\varepsilon(y)}{d_y}} \right)^2 \geq 2R(G).$$

By Cauchy-Schwarz inequality, we obtain:

$$\left(\sum_{xy \in E(G)} 1 \right)^2 \leq \left(\sum_{xy \in E(G)} \sqrt{d_x d_y} \right) \left(\sum_{xy \in E(G)} \frac{1}{\sqrt{d_x d_y}} \right).$$

Since $\sum_{xy \in E(G)} 1 = m$, we have

$$R(G) \geq \frac{m^2}{\sum\limits_{xy \in E(G)} \sqrt{d_x d_y}}.$$

Consequently,

$$\varepsilon(G) > 2R(G) \geq \frac{2m^2}{\sum\limits_{xy \in E(G)} \sqrt{d_x d_y}}.$$

This completes the proof. ■

***Acknowledgment*:** This work is based upon research funded by Iran National Science Foundation (INSF) under project No. 4024412.

References

[1] S. Akbari, M. Habibi, S. Rabizadeh, Relations between energy and Sombor index, *MATCH Commun. Math. Comput. Chem.* **92** (2024) 425–435.

[2] S. Akbari, M. Habibi, S. Rouhani, A note on an inequality between energy and Sombor index of a graph, *MATCH Commun. Math. Comput. Chem.* **90** (2023) 765–771.

[3] N. Alawiah, N. Rad, A. Jahanbani, H. Kamarulhaili, New upper bounds on the energy of a graph, *MATCH Commun. Math. Comput. Chem.* **79** (2018) 287–301.

[4] G. Arizmendi, O. Arizmendi, Energy of a graph and Randić index, *Lin. Algebra Appl.* **609** (2021) 332–338.

[5] O. Arizmendi, J. F. Hidalgo, O. Juarez–Romero, Energy of a vertex, *Lin. Algebra Appl.* **557** (2018) 464–495.

[6] T. Došlić, T. Réti, D. Vukičević, On the vertex degree indices of connected graphs, *Chem. Phys. Lett.* **512** (2011) 283–286.

[7] O. Favaron, M. Mahéo, J. F. Saclé, Some eigenvalue properties in graphs, *Discr. Math.* **111** (1993) 197–220.

[8] F. Goldberg, Graph energy estimates via the Chebyshev functional, arXiv:1407.7430 (2014) doi: https://doi.org/10.48550/arXiv.1407.7430.

[9] I. Gutman, S.Z. Firoozabadi, J.A. de la Peña, J. Rada, On the energy of regular graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 435–442.

[10] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[11] B. A. Rather, M. Imran, A note on energy and Sombor energy of graphs, *MATCH Commun. Math. Comput. Chem.* **89** (2023) 467–477.

[12] M. S. Reja, S. Abu. Nayeem, On Sombor index and graph energy, *MATCH Commun. Math. Comput. Chem.* **89** (2023) 451–466.

[13] A. Ülker, A. Gürsoy, N. K . Gürsoy, The energy and Sombor index of graphs, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 51–58.