MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **96** (2026) 1097–1099
ISSN: 0340–6253
doi: 10.46793/match.96-3.28425

A Note on the TxGraffiti Conjecture about Harmonic Index and Minimum Maximal Matching Number

Türker Bıyıkoğlu*

Mathematics Research Center, Azerbaijan State Oil and Industry University, AZ1000 Baku, Azerbaijan

tbiyikoglu@gmail.com

(Received October 29, 2025)

Abstract

We show that there exist graphs with harmonic index less than their minimum maximal matching number. This disproves the TxGraffiti conjecture stating that the minimum maximal matching number of a connected graph is at most its harmonic index.

Very recently Randy Davila, Boris Brimkov and Ryan Pepper present the following conjecture generated by the automated conjecturing system TxGraffiti [2]. TxGraffiti is created by Randy Davila and is an automated conjecturing system designed to propose mathematical conjectures, specially in graph theory [3, 8].

Let $G = (V, E)$ be a finite, simple, undirected graph. Let d_x denote the degree of a vertex x of G. The harmonic index $H(G)$ of a graph G is defined by $\sum_{xy\in E(G)} \frac{2}{d_x+d_y}$. Harmonic index is introduced by Siemion Fajtlowicz through his well known automated conjecturing system Graffitti [6]. Harmonic index is also closely related to mathematical chemistry and

*Corresponding author.

chemical graph theory indices such as Randić type indices and general sum-connectivity index (see e.g. [1, 5, 7]).

Matching is a set of edges without common vertices. Maximal matching is a matching that is not extendable to a larger one. $\mu^*(G)$ denotes the minimum cardinality of maximal matching of G. Finding a minimum maximal matching is a well known NP-hard problem. One of the few bounds on $\mu^*(G)$ is based on a conjecture of Graffiti.pc system written by Ermelinda DeLaViña [4].

The following TxGraffiti conjecture deals with harmonic index and minimum maximal matching number.

Conjecture 1 (TxGraffiti 2023 [2])**.** *If G is not a trivial connected graph, then $\mu^*(G) \leq H(G)$.*

We show that there exist graphs with $\mu^*(G) > H(G)$.

Theorem 1. *There exist connected graphs G with minimum degree $d+1$ and $H(G) < \frac{\mu^*(G)}{d}$.*

Proof. Let $M = mK_2$ be the disjoint union of m K_2's and I be an independent set, where I and $V(M)$ are disjoint. Let $G_1 * G_2$ denote join operation that connect the vertices of G_1 with the vertices of G_2, where $V(G_1)$ and $V(G_2)$ are disjoint. We define the graph $G = M * I$, where G is a join of mK_2 and I. The edges of M form a maximal matching in G, thus $\mu^*(G) \leq m$. Assume that there exists a minimum maximal matching S in G that has less edges than m. The edges of S must share at least one vertex from each edge of M. Otherwise, there exists an edge $xy \in E(M)$ such that x and y are not in S, then $S \cup xy$ is a matching and it is a contradiction to the maximality of S. It follows that $\mu^*(G) = m$.

By the definitions of the graph G and harmonic index, $H(G) = \frac{m}{|I|+1} + \frac{4m|I|}{|I|+1+2m}$. If we choose $|I| = d < |V(M)| = 2m$, then we have to show that $\frac{1}{d+1} + \frac{4d}{d+1+2m} < \frac{1}{d}$. After simplifying, we obtain $d(2m + 4d^2 + 5d + 1) = 2md + 4d^3 + 5d^2 + d < d^2 + 2d + 1 + 2md + 2m = (d+1)(d+1+2m)$. Finally, $4d^3 + 4d^2 - d - 1 < 2m = |V(M)|$ must hold. If we choose $4d^3 + 4d^2 - d \leq |V(M)|$ and $|I| = d$, then the graph G is connected with minimum degree $d+1$ and $H(G) < \frac{\mu^*(G)}{d} = \frac{m}{d}$. ∎

We remark that instead of working with the join operation between M and I, we can use any connected (d,k)-semiregular bipartite graph $B = (V(M), I)$ with $3(d+1) < k$. Then the graph $T = (V(M) \cup I, E(B) \cup E(M))$ satisfies $H(T) < \mu^*(T) = m$. The arguments are same as in the proof of Theorem 1.

References

[1] A. Ali, L. Zhong, I. Gutman, Harmonic index and its generalizations: Extremal results and bounds, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 249–311.

[2] R. Davila, B. Brimkov, R. Pepper, In reverie together: ten years of mathematical discovery with a machine collaborator, *arXiv*, 2025, **doi:** 10.48550/arXiv.2507.17780.

[3] R. Davila, Automated conjecturing in mathematics with TxGraffiti, *arXiv*, 2024, **doi:** 10.48550/arXiv.2409.19379.

[4] E. DeLaViña, Q. Liu, R. Pepper, B. Waller, D. B. West, Some conjectures of Graffiti.pc on total domination, *Congr. Num.* **185** (2007) 81–95.

[5] H. Deng, S. Balachandran, S. Elumalai, Some tight bounds for the harmonic index and the variation of the Randić index of graphs, *Discr. Math.* **342** (2019) 2060–2065.

[6] S. Fajtlowicz, On conjectures of Graffiti-II, *Cong. Num.* **60** (1987) 187–197.

[7] J. M. Rodrígueza, J. M. Sigarreta, New results on the harmonic index and its generalizations, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 387–404.

[8] `https://txgraffiti.streamlit.app/` (accessed 20 October 2025).