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Revisiting the Diminished Sombor Index

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Abstract

The diminished Sombor index of a graph G with edge set $E(G)$ is defined as

$$DSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v},$$

where d_u denotes the degree of vertex u. In this paper, we revisit and refine some of the results reported in the recent paper [MATCH Commun. Math. Comput. Chem. 95 (2026) 141–162]. One of the obtained refined results guarantees that $DSO(G)$ decreases when

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any of the edges of G is removed. Also, one of the new results gives the graphs minimizing DSO over the class $\mathcal{G}_{m,n}$ of all connected graphs of order n and size m for $3n \geq 2m \geq 2(n+2)$. The paper is concluded with an open problem concerning the graphs minimizing DSO over $\mathcal{G}_{m,n}$ for $m \geq \max\{n+3, \lceil 3n/2 \rceil\}$.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are denoted by $n = |V(G)|$ and $m = |E(G)|$, respectively. For a vertex $v \in V(G)$, the degree of v, denoted by $d_v(G)$, is the number of edges incident to v. A graph is connected if there is a path between every pair of its vertices. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. When there is no confusion about the graph under consideration, we drop the symbol "(G)" from the notations $d_v(G)$, $\delta(G)$ and $\Delta(G)$. An edge $e = uv$ is called a pendent edge if one of its endvertices has degree 1. A nontrivial path $P = u_0u_1\ldots u_t$ in a graph G is referred to as a pendent path if $\max\{d_{u_0}(G), d_{u_t}(G)\} \geq 3$, $\min\{d_{u_0}(G), d_{u_t}(G)\} = 1$ and $d_{u_i}(G) = 2$ when $1 \leq i \leq t-1$. We refer the reader to [7, 11] for undefined notations and terminologies.

Topological indices are numerical graph invariants. In chemical graph theory, these indices are used to predict physicochemical properties of chemical compounds. Among the immense number of vertex-degree-based topological indices, the Sombor index, introduced by Gutman in 2021 [5], has received significant attention. Inspired by the geometric interpretation of the Sombor index, several variants have been proposed in the literature. For instance, the Euler–Sombor index [10], the elliptic Sombor index [6], the augmented Sombor index [3], and the diminished Sombor index [9] are among the notable modifications that have been introduced and studied recently. For a graph G, the diminished Sombor index (DSO) is defined as

$$DSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}.$$

Some mathematical properties and bounds for this index have been investigated in recent studies [2, 4, 8, 9].

In [9], the extremal values of the DSO index were investigated, and several theorems regarding its behavior under graph transformations were presented. We revisit the proofs of Theorem 3, Theorem 4 and Proposition 12 given in [9] and observe that they are incomplete. In this paper, we present refined versions of Theorems 3 and 4 in [9]. We observe that Theorems 3 and 4 of [9] follow from these refinements. We also present alternative proofs for the results that depended on Proposition 12 of [9]. Furthermore, corresponding to Proposition 12 of [9], we provide a result that gives the graphs minimizing DSO over the class $\mathcal{G}_{m,n}$ of all connected graphs of order n and size m for $3n \geq 2m \geq 2(n+2)$. We conclude the paper with an open problem concerning the graphs minimizing DSO over $\mathcal{G}_{m,n}$ for $m \geq \max\{n+3, \lceil 3n/2 \rceil\}$.

2 Results and discussion

We start with the following result appeared in [9].

Theorem 1 (Theorem 3 in [9]). *Let G be a graph of order n. Then*

(i) $DSO(G) > DSO(G-e) + \dfrac{|d_u - d_v|}{\sqrt{2}(2n-2)}$, *for any edge $e = uv \in E(G)$;*

(ii) $DSO(G+e) > DSO(G) + \dfrac{|d_u - d_v|}{\sqrt{2}(2n-2)}$, *where $e = uv$ such that the vertices u and v are not adjacent in G.*

The proof provided for Theorem 1 is imperfect. For instance, the inequality

$$DSO(G) > DSO(G-e) + \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \tag{1}$$

is invalid for the graph depicted in Figure 1 when $e = uv$ is taken to be the unique bridge of the considered graph G.

Here, we provide an extended version of Theorem 1, from which Theorem 1 follows. For this, we first need the following lemma.

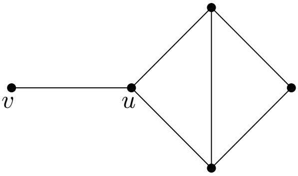

Figure 1. A graph providing counterexamples to the inequalities (1) and (10).

Lemma 1. *Let ℓ_1 and ℓ_2 be integers satisfying $\ell_2 \geq \ell_1 + 1 \geq 3$. Then, it holds that*

$$\frac{\ell_2 - \ell_1 + 1}{(\ell_1 + \ell_2 - 1)^2} \leq \frac{1}{8(\ell_1 - 1)}. \tag{2}$$

Proof. Define the function ϕ on the interval $[\ell_1+1, \infty)$ as $\phi(x) = \frac{x-\ell_1+1}{(x+\ell_1-1)^2}$. The required inequality follows from the fact that the function ϕ is increasing on $[\ell_1 + 1, 3\ell_1 - 3]$ and decreasing on $[3\ell_1 - 3, \infty)$. Hence, $\phi(x) \leq \phi(3\ell_1 - 3)$ for every $x \in [\ell_1 + 1, \infty)$. Particularly, for any integer ℓ_2 with $\ell_2 \geq \ell_1 + 1$, we have $\phi(\ell_2) \leq \phi(3\ell_1 - 3)$, which is equivalent to the required inequality. ■

Theorem 2. *If G is a graph and $uv \in E(G)$, then*

$$DSO(G) > DSO(G - uv) + \frac{4 - \sqrt{2}}{4} f(d_u, d_v), \tag{3}$$

where $f(d_u, d_v) = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$. In particular, $DSO(G) > DSO(G - uv)$.

Proof. We note that

$$\begin{aligned} DSO(G) - DSO(G - uv) = &\sum_{w \in N(u)\setminus\{v\}} \big(\underbrace{f(d_u, d_w) - f(d_u - 1, d_w)}_{\theta_u(w)}\big) \\ &+ \sum_{x \in N(v)\setminus\{u\}} \big(\underbrace{f(d_v, d_x) - f(d_v - 1, d_x)}_{\theta_v(x)}\big) \\ &+ f(d_u, d_v). \end{aligned} \tag{4}$$

Set

$$\Theta_u = \sum_{w \in N(u)\setminus\{v\}} \theta_u(w) \quad \text{and} \quad \Theta_v = \sum_{x \in N(v)\setminus\{u\}} \theta_v(x).$$

If $d_u = d_v = 1$, then $\Theta_u = \Theta_v = 0$, and hence, (4) yields the desired inequality. Next, we consider the case where $\max\{d_u, d_v\} \geq 2$. Without loss of generality, we assume that $d_u \geq d_v$. Using the mean value theorem, we observe that, for any vertex $w \in N(u) \setminus \{v\}$, there exists a real number ξ_w such that $1 \leq d_u - 1 < \xi_w < d_u$ and

$$\theta_u(w) = \frac{d_w(\xi_w - d_w)}{\sqrt{\xi_w^2 + d_w^2}\,(\xi_w + d_w)^2} \begin{cases} < 0 & \text{if } d_w \geq d_u, \\ > 0 & \text{if } d_w \leq d_u - 1. \end{cases} \tag{5}$$

Similarly, if $d_v \geq 2$, then for any $x \in N(v) \setminus \{u\}$, there exists a real number ξ_x such that $1 \leq d_v - 1 < \xi_x < d_v$ and

$$\theta_v(x) = \frac{d_x(\xi_x - d_x)}{\sqrt{\xi_x^2 + d_x^2}\,(\xi_x + d_x)^2} \begin{cases} < 0 & \text{if } d_x \geq d_v, \\ > 0 & \text{if } d_x \leq d_v - 1. \end{cases} \tag{6}$$

Therefore, if $d_w \leq d_u - 1$ and $d_x \leq d_v - 1$ for all $w \in N(u) \setminus \{v\}$ and $x \in N(v) \setminus \{u\}$ (when $d_v \geq 2$), then $\min\{\Theta_u, \Theta_v\} \geq 0$, which together with (4) implies the desired inequality. Now, we consider the case where either $d_w \geq d_u$ or $d_x \geq d_v \geq 2$ for some $w \in N(u) \setminus \{v\}$ or $x \in N(v) \setminus \{u\}$. Then, we have either $\theta_u(w) < 0$ or $\theta_v(x) < 0$ for some $w \in N(u) \setminus \{v\}$ or $x \in N(v) \setminus \{u\}$. Let $A_1(u) = \{w' \in N(u) \setminus \{v\} : d_{w'} \geq d_u\}$, $A_2(u) = (N(u) \setminus \{v\}) \setminus A_1(u)$, $B_1(v) = \{x' \in N(v) \setminus \{u\} : d_{x'} \geq d_v \geq 2\}$ and $B_2(v) = (N(v) \setminus \{u\}) \setminus B_1(v)$. Then, for every $w \in A_1(u)$, using (5) and (2), we have

$$-\theta_u(w) < \frac{d_w - \xi_w}{(\xi_w + d_w)^2} < \frac{d_w - (d_u - 1)}{(d_w + d_u - 1)^2} \leq \frac{1}{8(d_u - 1)}, \tag{7}$$

because $d_u - 1 < \xi_w < d_u \leq d_w$. Similarly, if $d_v \geq 2$ then, for every $x \in B_1(v) \cup B_2(v) = N(v) \setminus \{u\}$, we have

$$-\theta_v(x) < \frac{1}{8(d_v - 1)}. \tag{8}$$

Since $f(d_u, d_v) \geq \frac{1}{\sqrt{2}}$, from (7) and (8) it follows that

$$\sum_{w \in N(u) \setminus \{v\}} \theta_u(w) + \sum_{x \in N(v) \setminus \{u\}} \theta_v(x) > -\frac{1}{4} \geq -\frac{f(d_u, d_v)}{2\sqrt{2}} \tag{9}$$

Now, from (4) and (9), the required inequality follows. ■

We also remark that the proof of Theorem 4 in [9] is imperfect. For instance, the inequality

$$DSO(G) > DSO(G - v) + \sum_{uv \in E(G,v)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \tag{10}$$

is invalid for the graph depicted in Figure 1 when v is taken to be the unique pendent vertex of the considered graph G, where $E(G, v)$ denotes the set of those edges of G that are incident to v. However, we note that Theorem 4 of [9] (when vertex v is not isolated) follows from the following corollary of Theorem 2.

Corollary 1. *Let v be a non-isolated vertex of a graph G. Then*

$$DSO(G) > DSO(G - v) + \frac{4 - \sqrt{2}}{4} \sum_{uv \in E(G,v)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}. \tag{11}$$

Proof. If $v \in V(G)$ is a pendent vertex adjacent to a vertex u, then (3) and (11) are equivalent. If $N(v) = \{v_1, v_2, \ldots, v_s\}$ with $s \geq 2$, then by (3), we have

$$\begin{aligned} DSO(G) &> DSO(G - v_1 v) + \left(\frac{4 - \sqrt{2}}{4}\right) \frac{\sqrt{d_{v_1}^2 + d_v^2}}{d_{v_1} + d_v} \\ &> DSO(G - \{v_1 v, v_2 v, \ldots, v_s v\}) + \frac{4 - \sqrt{2}}{4} \sum_{i=1}^{s} \frac{\sqrt{d_{v_i}^2 + d_v^2}}{d_{v_i} + d_v}, \end{aligned}$$

which yields (11). ■

In [9], the proof of Proposition 12 relied on a specific graph transformation involving an (a, b)-edge, which is an edge connecting vertices of degree a and b. The argument was stated as follows:

> "Let a branch (a tree) Θ be attached to a vertex v. Suppose that Θ has t edges, at least one of which is pendent. Replace one of the pendent edges of Θ with an edge in another part of the graph G, so that it becomes a $(2,2)$-edge."

However, this transformation is not universally applicable to all graphs. The assumption that a pendent edge can always be repositioned to form a $(2,2)$-edge does not hold in general. For instance, consider a graph constructed by attaching a pendent vertex u to a vertex v of the complete graph K_n, where $n \geq 4$. In this case, the edge uv is an $(1,n)$-edge. If we attempt to apply the described transformation by moving this pendent edge, it is not possible to obtain a $(2,2)$-edge.

Although we believe that the statement of Proposition 12 of [9] is likely correct, the given proof is incomplete. Since the proofs of the lower bounds in Theorems 13 and 14 in [9] relied on this proposition, we now provide alternative proofs for these results.

Theorem 3 (see Theorem 13 in [9])**.** *Among unicyclic graphs of order n, $DSO(C_n) \leq DSO(G)$, with equality if and only if $G \cong C_n$.*

Proof. Let p be the number of pendent edges in a unicyclic graph G of order n. If $p = 0$, then G is the cycle graph C_n.
If $p \geq 1$, then

$$DSO(G) = \sum_{uv \in E(G)} f(d_u, d_v) = \sum_{\substack{uv \in E(G) \\ d_u = 1,\, d_v \geq 2}} f(d_u, d_v) + \sum_{\substack{uv \in E(G) \\ d_u,\, d_v \geq 2}} f(d_u, d_v)$$

$$\geq p\, f(1,2) + (n-p)\, f(2,2) = \left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{2}} \right) p + \frac{n}{\sqrt{2}}$$

$$> \frac{n}{\sqrt{2}} = DSO(C_n).$$

■

Lemma 2. *Let G be a connected graph of size m with k pendent paths such that the maximum degree of G is at least 3. Then*

$$DSO(G) \geq \left(\frac{\sqrt{5}}{3} + \frac{\sqrt{13}}{5} - \sqrt{2} \right) k + \frac{1}{\sqrt{2}} m.$$

Proof. Let k_1 and k_2 be the number of pendent paths (in G) of lengths 1 and at least 2, respectively. Then, $k_1 + k_2 = k$. Hence,

$$\begin{aligned} DSO(G) &\geq k_1 f(1,3) + k_2[f(1,2) + f(2,3)] + (m - k_1 - 2k_2) f(2,2) \\ &= (k - k_2) f(1,3) + k_2[f(1,2) + f(2,3)] + (m - k - k_2) f(2,2). \end{aligned} \tag{12}$$

We note that the right-hand side of (12) attains its minimum value when k_2 is maximum; that is, when $k_2 = k$. Hence, from (12), we obtain the required inequality. ■

Theorem 4 (see Theorem 14 in [9])**.** *Among bicyclic graphs of order n, the graphs with minimal DSO are (a) those obtained by inserting an edge into C_n, and (b) those obtained by connecting two disjoint cycles by an edge (when $n \geq 6$), both having DSO equal to*

$$4\,f(3,2) + f(3,3) + (n-4)\,f(2,2) = \frac{1}{\sqrt{2}}\,n + \left(\frac{4}{5}\sqrt{13} - \frac{3}{2}\sqrt{2}\right).$$

Proof. Let G be a bicyclic graph of order n, size m and p pendent paths. Let Δ be the maximum degree of G. We consider the following cases.

Case 1. $p = 0$. In this case, following the proof of Theorem 14 in [9], we have the required conclusion.

Case 2. $p = 1$. Since $2m = \sum_{u \in V(G)} d_u$, we have $2(n+1) \geq \Delta + 1 + 2(n-2)$, implying $3 \leq \Delta \leq 5$. Let P be the pendent path in G. First, we consider the possibility where a vertex of degree Δ has at least three neighbors of degree 2. Then,

$$\begin{aligned} DSO(G) &\geq 3\,f(\Delta,2) + f(1,2) + (n-3)\,f(2,2) \\ &\geq 3\,f(3,2) + f(1,2) + (n-3)\,f(2,2) \\ &> \frac{1}{\sqrt{2}}\,n + \left(\frac{4}{5}\sqrt{13} - \frac{3}{2}\sqrt{2}\right). \end{aligned}$$

Now, we consider the possibility where every vertex of degree Δ has at most two neighbors of degree 2. Then, G can be obtained from either $\mathcal{A}_1$ or $\mathcal{A}_2$ (shown in Figure 1 of [9]) by attaching a pendent path, and hence, we have $\Delta \in \{3,4\}$. We observe that G contains at least two edges, not lying on P, having one endvertex of degree 2 and the other endvertex of degree 3. Since $f(1,\Delta) \geq f(1,3) > f(1,2)$, we have

$$\begin{aligned} DSO(G) &\geq 2\,f(3,2) + f(1,2) + (n-2)\,f(2,2) \\ &> \frac{1}{\sqrt{2}}\,n + \left(\frac{4}{5}\sqrt{13} - \frac{3}{2}\sqrt{2}\right). \end{aligned}$$

Case 2. $p \geq 2$. Using Lemma 2, we obtain

$$\begin{aligned} DSO(G) &\geq \left(\frac{\sqrt{5}}{3} + \frac{\sqrt{13}}{5} - \sqrt{2}\right) p + \frac{\sqrt{2}}{2}(n+1) \\ &\geq 2\left(\frac{\sqrt{5}}{3} + \frac{\sqrt{13}}{5} - \sqrt{2}\right) + \frac{\sqrt{2}}{2}(n+1) \\ &> \frac{1}{\sqrt{2}}\,n + \left(\frac{4}{5}\sqrt{13} - \frac{3}{2}\sqrt{2}\right), \end{aligned}$$

which completes the proof. ■

The proof of the next result is similar to the proof of Theorem 3.

Proposition 5. *Let G be a graph with m edges, from which p are pendent, provided that G contains no component isomorphic to the path graph of order 2. Then,*

$$DSO(G) \geq \frac{m-p}{\sqrt{2}} + \frac{\sqrt{5}\,p}{3}$$

with equality if and only if every pendent edge (if exists) is incident to a vertex of degree 2, whereas every non-pendent edge (if exists) has both endvertices of the same degree.

Proof. We note that $f(d_v,1) \geq f(2,1)$ for every vertex $v \in V(G)$ adjacent to a pendent vertex, where the equality holds if and only if $d_v = 2$. Also, $f(d_u,d_w) \geq f(\ell,\ell) = \frac{1}{\sqrt{2}}$ for every non-pendent edge $uw \in E(G)$, where the equality holds if and only if $d_u = d_w = \ell$ for some integer larger ℓ than 1. Hence, $DSO(G) \geq p\,f(2,1) + (m-p)\,f(\ell,\ell)$. ■

We remark here that, for $n \geq 3$, the lower bound of Theorem 10 reported in [9] follows from Theorem 2 and Proposition 5, whereas the upper bound of Theorem 10 in [9] follows from Theorem 2.

For a non-negative integer k, a connected graph G with size $|V(G)| + k - 1$ is known as a k-cyclic graph.

Proposition 6. *If G is a graph minimizing the DSO index among all k-cyclic graphs of order n, then G does not contain any pendent edge, where $n \geq 2(k-1) \geq 4$.*

Proof. If $n = 2(k-1) \geq 4$, then there exists at least one 3-regular graph. Hence, by Proposition 5, only 3-regular graph(s) minimize(s) the DSO index among all k-cyclic graphs of order n, which proves the required conclusion for the considered case.

For $n > 2(k-1)$, let G^* be the k-cyclic graph of order n obtained from a 3-regular k-cyclic graph of order $2(k-1)$ by inserting $n-2(k-1)$ vertex/vertices of degree 2 on one edge. Then, $m_{3,3}(G^*) = 3k-4$, $m_{2,2}(G^*) = n-2k+1$ and $m_{2,3}(G^*) = 2$. Let p be the number of pendent edges of G. We contrarily assume that $p \geq 1$. Since $|E(G)| = |E(G^*)| = n+k-1$, by Proposition 5, we have

$$DSO(G) \geq \frac{n+k-2}{\sqrt{2}} + \frac{\sqrt{5}}{3} > \frac{n+k-3}{\sqrt{2}} + \frac{2\sqrt{13}}{5} = DSO(G^*),$$

a contradiction to the minimality of $DSO(G)$. Hence, $p = 0$. ■

By Proposition 6 and the observation given in the first paragraph of the concluding remarks section in [2], we have the following result, which supports Proposition 12 of [9].

Proposition 7. *If G is a k-cyclic graph of order n such that $n \geq 2(k-1) \geq 4$, then*

$$DSO(G) \geq \frac{n+k-3}{\sqrt{2}} + \frac{2\sqrt{13}}{5},$$

with equality if and only if G is isomorphic to either a 3-regular graph (when $n = 2k-2$) or the graph G^* (when $n > 2k-2$) defined in the proof of Proposition 6.

An (i,j)-edge in a graph is an edge whose endvertices have degrees i and j. Let U_n be the graph of order n and maximum degree $n-1$. In the proof of Theorem 13 of [9], it was written that "U_n is the unicyclic graph with maximum number of $(n-1,1)$-edges, which implies the upper bound." This implication was based on the fact that the function $f(d_u, d_v) = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$ attains its maximum value on the interval $[1, n-1]$ when $\{d_u, d_v\} = \{n-1, 1\}$. However, the implication under consideration does not hold in general. To disprove it, we define the following new variant of the Sombor index:

$$RRS(G) = \sum_{uv\in E(G)} \frac{\max\{d_u, d_v\}}{\sqrt{d_u^2 + d_v^2}}.$$

Based on the definition of this index, we refer to it as the refined reciprocal Sombor (RRS) index. Clearly, for any $d_u, d_v \in [1, n-1]$, we have

$$\frac{\max\{d_u, d_v\}}{\sqrt{d_u^2 + d_v^2}} \le \frac{n-1}{\sqrt{(n-1)^2 + 1}},$$

where the equality holds if and only if $\{d_u, d_v\} = \{1, n-1\}$. However, for every $n \ge 9$, we have

$$RRS(U_n) < RRS(U_n^\star),$$

where $U_n^\star$ is the graph obtained from U_{n-1} by attaching a pendent vertex to a vertex of degree 2. Based on this discussion, we conclude that the proof of the upper bound of Theorem 13 in [9] is not complete. Similarly, the proof of Theorem 15 in [9] as well as the one for Theorem 1 in [8] are not complete. We remark here that the upper bound of Theorem 13 in [9] follows from Theorem 2.2 in [1] for $n \ge 12$. On the other hand, although we believe that the statement of Theorem 15 in [9] as well as the one for Theorem 1 in [8] are true, we do not have their valid proofs at present.

We end this section by providing a lower bound on DSO. For establishing this lower bound, we require a lemma first.

Lemma 3. *Let $a, b \ge 1$. Then the following holds:*

$$\frac{\sqrt{a^2 + b^2}}{a+b} \ge \frac{1}{\sqrt{2}} + \frac{(a-b)^2}{4(a+b)^2}.$$

Proof. With no loss of generality, assume that $a \geq b$. Let $t = \frac{a-b}{a+b}$. Then, $t \in [0, 1)$ and

$$\frac{\sqrt{a^2+b^2}}{a+b} = \sqrt{\frac{1+t^2}{2}}.$$

The required inequality is equivalent to

$$\sqrt{1+t^2} \geq 1 + \frac{t^2}{2\sqrt{2}},$$

which, for $t \in (0, 1)$, is equivalent to $t^2 \leq 8 - 4\sqrt{2}$. ■

Now, we have the following corollary.

Corollary 2. *Let G be a graph of size m and maximum degree Δ. If p is the number of those edges of G whose endvertices have different degrees, then*

$$DSO(G) > \frac{m}{\sqrt{2}} + \frac{p}{4(2\Delta - 1)^2}.$$

3 An open problem

Let m and n be positive integers provided that $m \geq n - 1$. Define $DSO_{\min}(m, n)$ as the minimum diminished Sombor index among all connected graphs of order n and size m. By Theorems 3 and 4, Propositions 5 and 7, we know the value of $DSO_{\min}(m, n)$ when (i) $m \in \{n-1, n, n+1\}$, (ii) $3n \geq 2m \geq 2(n+2)$. Also, recall that there is only one graph of order $n \geq 2$ and size m for each $m \in \left\{\binom{n}{2}, \binom{n}{2} - 1\right\}$. This leads to the following problem.

Problem. Let m and n be positive integers satisfying the inequality

$$\max\left\{n+3, \left\lceil \frac{3n}{2} \right\rceil\right\} \leq m \leq \binom{n}{2} - 2.$$

Determine $DSO_{\min}(m, n)$.

This problem seems not to be easy to solve. Based on the computer search for small values of n and m, we expected that the difference between the maximum and minimum degrees of every extremal graph corresponding to the solution of this problem will be at most 1; however, this is not true in general. For instance, let us consider the determination of $DSO_{\min}(30, 14)$. We can construct a candidate extremal graph by minimizing the number of edges with unequal endvertex degrees. Since the function $f(x, y) = \frac{\sqrt{x^2+y^2}}{x+y}$ attains its minimum value of $\frac{1}{\sqrt{2}}$ when $x = y$, the optimal strategy is likely to maximize the number of edges connecting vertices of equal degrees. Ideally, we seek a structure composed of vertex-disjoint regular (or nearly regular) components connected by a minimal number of edges. Specifically, if we can construct a graph with only a single edge connecting vertices of unequal degrees a and b, the DSO index of G would be

$$DSO(G) = (m - 1)\frac{1}{\sqrt{2}} + \frac{\sqrt{a^2 + b^2}}{a + b}.$$

For $n = 14$ and $m = 30$, the average degree is $\frac{30}{7}$. We explore integer degrees a and b close to this average degree satisfying the degree sum equation $a\,n_a + b\,n_b = 2m = 60$, subject to $n_a + n_b = 14$, where n_i denotes the number of vertices of degree i. The pair $(a, b) = (3, 5)$ yields the unique integer solution:

$$3n_a + 5n_b = 60 \quad \text{and} \quad n_a + n_b = 14 \implies n_a = 5,\ n_b = 9.$$

It is graph-theoretically possible to construct such a graph G (shown in Figure 1).

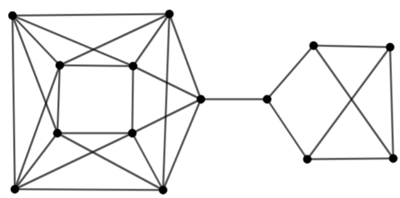

Figure 2. The graph G with minimum DSO for $n = 14$ and $m = 30$.

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