MATCH
Communications in Mathematical and in Computer Chemistry

*MATCH Commun. Math. Comput. Chem.* **96** (2026) 495–512
**ISSN:** 0340–6253
**doi:** 10.46793/match.96-2.26425

# Extremal Zagreb Indices of Bicyclic Hypergraphs

**Hong Zhou, Changjiang Bu***

*School of Mathematical Sciences, Harbin Engineering University, Harbin 150001, PR China*

zhouhong777777@163.com, buchangjiang@hrbeu.edu.cn

(Received October 11, 2025)

**Abstract**

In this paper, we determine the hypergraphs with maximum and minimum Zagreb indices among all linear bicyclic uniform hypergraphs with $m$ edges. For a given girth, we also determine the linear bicyclic uniform hypergraphs with $m$ edges that attain the maximum and minimum Zagreb indices.

# 1 Introduction

In 1972, Gutman and Trinajstić [10] proposed the first Zagreb index of graphs. The first Zagreb index of a graph $G$ is defined as the sum of the squares of the degrees of its vertices. The properties of the first Zagreb index were summarized in [9, 14]. Deng [5] characterized the graphs with maximum and minimum first Zagreb indices among all bicyclic graphs with $n$ vertices. Some results on the extremal first Zagreb index have been obtained in the literature: see [1, 2, 9, 15] for trees, [16, 18] for unicyclic

*Corresponding author.

graphs, [11] for $k$-generalized quasi unicyclic graphs, [19] for triangle-free graphs, and [4, 6, 7, 12, 17] for graphs with given parameters.

Let $\mathcal{H}$ be a hypergraph with the vertex set $V(\mathcal{H})$. In [3], the Zagreb index $M(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is given by

$$M(\mathcal{H}) = \sum_{u \in V(\mathcal{H})} (d_{\mathcal{H}}(u))^2,$$

where $d_{\mathcal{H}}(u)$ is the degree of a vertex $u$ in $\mathcal{H}$. The bounds on the Zagreb indices of hypergraphs, weak bipartite hypergraphs, hypertrees, $k$-uniform hypergraphs, $k$-uniform weak bipartite hypergraphs, and $k$-uniform hypertrees were given in [8]. The hypergraphs with maximum and minimum Zagreb indices were determined for both uniform hypertrees and linear unicyclic uniform hypergraphs [20].

In this paper, the hypergraphs with maximum and minimum Zagreb indices among all linear bicyclic uniform hypergraphs with $m$ edges are given. We also determine the hypergraphs with maximum and minimum Zagreb indices among all linear bicyclic uniform hypergraphs with $m$ edges and girth $g$.

## 2 Preliminaries

A hypergraph $\mathcal{H}$ is called *$k$-uniform* if every edge of $\mathcal{H}$ contains exactly $k$ vertices. A vertex of $\mathcal{H}$ is called a *cored vertex* if its degree is 1. An edge $e$ of $\mathcal{H}$ is called a *pendant edge* if it contains exactly $|e| - 1$ cored vertices. A cored vertex in a pendant edge is also called a *pendant vertex*. A hypergraph $\mathcal{H}$ is called *linear* if any two edges intersect into at most one vertex. The *girth* of $\mathcal{H}$ is the minimum length of the hypercycles of $\mathcal{H}$. A connected $k$-uniform hypergraph with $n$ vertices and $m$ edges is called bicyclic if $n = m(k-1) - 1$.

Throughout this paper, all hypergraphs are considered $k$-uniform ($k \geq 3$) unless otherwise stated. The linear bicyclic $k$-uniform hypergraph containing no pendant edges has exactly the following six cases [13].

Let $C_1$ and $C_2$ be linear $k$-uniform hypercycles of length $p$ and $q$, respectively. Suppose that $v_{1,1} \in V(C_1), v_{2,1} \in V(C_2)$ are two vertices with degree 1, and $v_{1,2} \in V(C_1), v_{2,2} \in V(C_2)$ are two vertices with degree 2. Let $P = u_0e_1u_1 \cdots e_lu_l$ be a $k$-uniform hyperpath of length $l$. Without loss of generality, let $q \geq p \geq 3$.

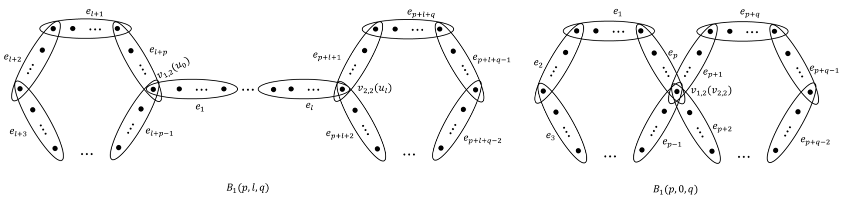


**Figure 1.** The hypergraphs $B_1(p,l,q)(l > 0)$ and $B_1(p,0,q)$.

Let $B_1(p,l,q)$ be the $k$-uniform bicyclic hypergraph obtained by identifying $v_{1,2}$ with $u_0$, and identifying $v_{2,2}$ with $u_l$ (see Fig. 1).

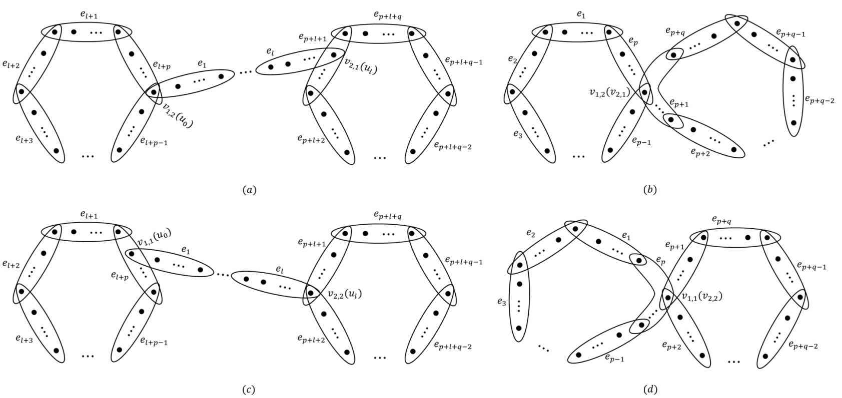


**Figure 2.** Cases (a) and (c) of the hypergraph $B_2(p,l,q)$ for $l > 0$, and cases (b) and (d) for $l = 0$.

Let $B_2(p,l,q)$ be the $k$-uniform bicyclic hypergraph obtained by either identifying $v_{1,2}$ with $u_0$ and identifying $v_{2,1}$ with $u_l$ (see (a) and (b) in Fig. 2), or identifying $v_{1,1}$ with $u_0$ and identifying $v_{2,2}$ with $u_l$ (see (c) and (d) in Fig. 2).

Let $B_3(p,l,q)$ be the $k$-uniform bicyclic hypergraph obtained by identifying $v_{1,1}$ with $u_0$, and identifying $v_{2,1}$ with $u_l$ (see Fig. 3).

Let $P_p = u_1e_1u_2 \cdots e_pu_{p+1}, P_q = v_1f_1v_2 \cdots f_qv_{q+1}$ and $P_l = w_1g_1w_2, \cdots, g_lw_{l+1}$ be $k$-uniform hyperpaths of length $p$, $q$ and $l$, respectively.

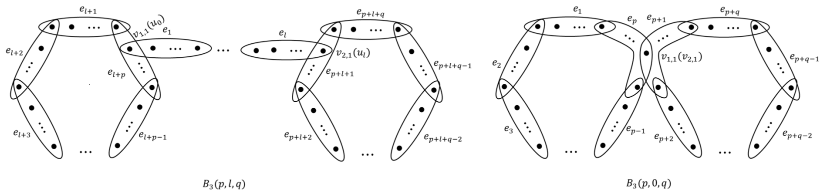


**Figure 3.** The hypergraphs $B_3(p, l, q)(l > 0)$ and $B_3(p, 0, q)$.

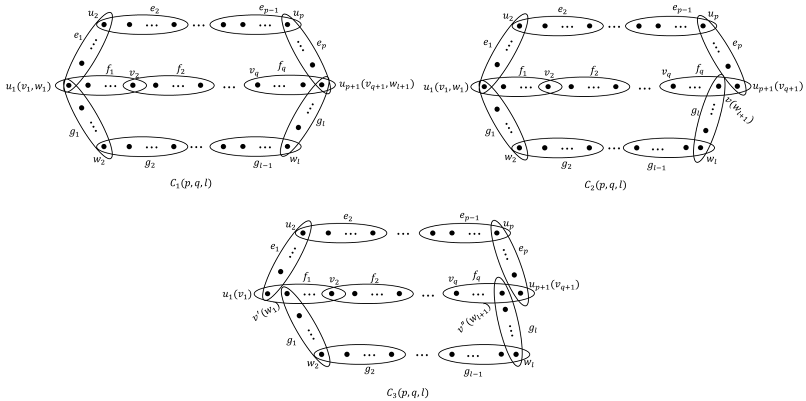


**Figure 4.** The hypergraphs $C_i(p, q, l), i = 1, 2, 3$.

For $p = 1, 1 < q \leq l$ or $1 < p \leq q \leq l$, let $C_1(p, q, l)$ be the $k$-uniform bicyclic hypergraph obtained from $P_p, P_q$ and $P_l$ by identifying three vertices $u_1, v_1, w_1$, and identifying three vertices $u_{p+1}, v_{q+1}, w_{l+1}$ (see Fig. 4).

For $q = 1, 1 < p \leq l$ or $q > 1, 1 \leq p \leq q - 1 \leq l$, let $C_2(p, q, l)$ be the $k$-uniform bicyclic hypergraph obtained from $P_p, P_q$ and $P_l$ by identifying three vertices $u_1, v_1, w_1$, identifying $u_{p+1}$ with $v_{q+1}$, and identifying $w_{l+1}$ with $v$, respectively (see Fig. 4), where $v \in f_q \setminus \{v_q, v_{q+1}\}$.

For $q = 1, k > 3, 1 < p \leq l$ or $q = 2, 1 \leq p \leq l$ or $q > 2, 1 \leq p \leq q - 2 \leq l$, let $C_3(p, q, l)$ be the $k$-uniform bicyclic hypergraph obtained from $P_p, P_q$ and $P_l$ by identifying $u_1$ with $v_1$, identifying $u_{p+1}$ with $v_{q+1}$, identifying $w_1$ with $v'$, and identifying $w_{l+1}$ with $v''$, respectively (see Fig. 4), where $v' \in f_1 \setminus \{v_1, v_2\}$ and $v'' \in f_q \setminus \{v_q, v_{q+1}\}$ (when $q = 1$, we choose $v' \neq v''$).

All linear bicyclic $k$-uniform hypergraphs with $m$ edges are classified into the following two sets $\mathcal{B}$ and $\mathcal{C}$ [13]. For $i \in \{1,2,3\}$, let $\mathcal{B}_i(p,l,q)$ and $\mathcal{C}_i(p,q,l)$ be the sets of $m$-edge $k$-uniform bicyclic hypergraphs each of which contains $B_i(p,l,q)$ and $C_i(p,q,l)$ as a sub-hypergraph, respectively. Let $\mathcal{B} = \bigcup_{i=1}^{3}\{\mathcal{B}_i(p,l,q) \mid q \geq p \geq 3, l \geq 0\}$ and $\mathcal{C} = \{\mathcal{C}_1(p,q,l) \mid p = 1, 1 < q \leq l \text{ or } 1 < p \leq q \leq l\} \bigcup \{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \leq l \text{ or } q > 1, 1 \leq p \leq q-1 \leq l\} \bigcup \{\mathcal{C}_3(p,q,l) \mid q = 1, k > 3, 1 < p \leq l \text{ or } q = 2, 1 \leq p \leq l \text{ or } q > 2, 1 \leq p \leq q-2 \leq l\}$. Note that the girths of hypergraphs in $\mathcal{B}_i(p,l,q)$ and $\mathcal{C}_i(p,q,l)$ are $p$ and $p+q$, respectively.

In the following, we present the transformation used to prove our main result.

**Transformation 1**: Let $\mathcal{H}$ be a linear $k$-uniform hypergraph, $u, v \in V(\mathcal{H})$, $e_1, \ldots, e_t \in E(\mathcal{H})$ and $u \in e_i, v \notin e_i$ for $i = 1, 2, \ldots, t$. Let $d_{\mathcal{H}}(u) \geq 2$ and $d_{\mathcal{H}}(v) > d_{\mathcal{H}}(u) - t$. Write $e_i' = (e_i \setminus \{u\}) \bigcup \{v\}$ for $i = 1, 2, \ldots, t$. Let $\mathcal{H}'$ be the hypergraph with $V(\mathcal{H}') = V(\mathcal{H})$ and $E(\mathcal{H}') = (E(\mathcal{H}) \setminus \{e_1, \ldots, e_t\}) \bigcup \{e_1', \ldots, e_t'\}$. We say that $\mathcal{H}'$ is obtained from $\mathcal{H}$ by moving edges $(e_1, \ldots, e_t)$ from $u$ to $v$.

**Lemma 1.** *Let $\mathcal{H}'$ be obtained from $\mathcal{H}$ by Transformation 1. Then $M(\mathcal{H}') > M(\mathcal{H})$.*

*Proof.* By the definition of the Zagreb index, we have

$$\begin{aligned} M(\mathcal{H}') - M(\mathcal{H}) &= d^2_{\mathcal{H}'}(v) + d^2_{\mathcal{H}'}(u) - d^2_{\mathcal{H}}(v) - d^2_{\mathcal{H}}(u) \\ &= (d_{\mathcal{H}}(v) + t)^2 + (d_{\mathcal{H}}(u) - t)^2 - d^2_{\mathcal{H}}(v) - d^2_{\mathcal{H}}(u) \\ &= 2t(t + d_{\mathcal{H}}(v) - d_{\mathcal{H}}(u)) > 0. \end{aligned}$$

■

# 3 Main results

In this section, we determine the hypergraphs with maximum and minimum Zagreb indices among all linear bicyclic uniform hypergraphs with $m$ edges, and the hypergraphs with maximum and minimum Zagreb indices among all linear bicyclic uniform hypergraphs with $m$ edges and girth $g$.

The following Theorem gives all hypergraphs with the minimum Zagreb index among all linear bicyclic uniform hypergraphs with $m$ edges.

**Theorem 1.** *The hypergraph $\mathcal{H}$ has the minimum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges if and only if the maximum degree of $\mathcal{H}$ is 2.*

*Proof.* Let $\mathcal{H}$ be a linear bicyclic $k$-uniform hypergraph with $n$ vertices and $m$ edges. Let $n_t$ be the number of vertices of $\mathcal{H}$ whose degree is equal to $t$, and $\Delta_{\mathcal{H}}$ be the maximum degree of $\mathcal{H}$. Then

$$\sum_{t=1}^{\Delta_{\mathcal{H}}} n_t = n, \sum_{t=1}^{\Delta_{\mathcal{H}}} t n_t = km, \text{ and } M(\mathcal{H}) = \sum_{t=1}^{\Delta_{\mathcal{H}}} t^2 n_t.$$

By the above Equations, we have

$$M(\mathcal{H}) = \sum_{t=1}^{\Delta_{\mathcal{H}}} ((t-1)(t-2) + 3t - 2) n_t = \sum_{t=1}^{\Delta_{\mathcal{H}}} (t-1)(t-2) n_t + 3km - 2n.$$

Therefore, when $\Delta_{\mathcal{H}} = 2$, $\mathcal{H}$ has the minimum Zagreb index, and $M(\mathcal{H}) = 3km - 2n$.

Let $\mathcal{H}'$ be a linear bicyclic $k$-uniform hypergraph with $n$ vertices and $m$ edges that attains the minimum Zagreb index. Then

$$M(\mathcal{H}') = \sum_{t=1}^{\Delta_{\mathcal{H}'}} (t-1)(t-2) n_t + 3km - 2n = 3km - 2n.$$

Hence, $\Delta_{\mathcal{H}'} = 2$. ■

From the proof of Theorem 1, we obtain the following Corollary.

**Corollary.** *The hypergraph $\mathcal{H}$ has the minimum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$ if and only if the maximum degree of $\mathcal{H}$ is 2.*

In what follows, we determine the hypergraph with the maximum Zagreb index among all linear bicyclic uniform hypergraphs with $m$ edges and girth $g$, and the hypergraph with the maximum Zagreb index among all linear bicyclic uniform hypergraphs with $m$ edges. We proceed in three steps.

Firstly, we give the bicyclic hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{B}$, and give the bicyclic hypergraph with the maximum Zagreb index in $\mathcal{B}$. Let $D(p,q)$ denote the $m$-edge $k$-uniform bicyclic hypergraph obtained from $B_1(p,0,q)$ by attaching $m-p-q$ pendant edges at the unique vertex with degree 4 (see Fig. 5).

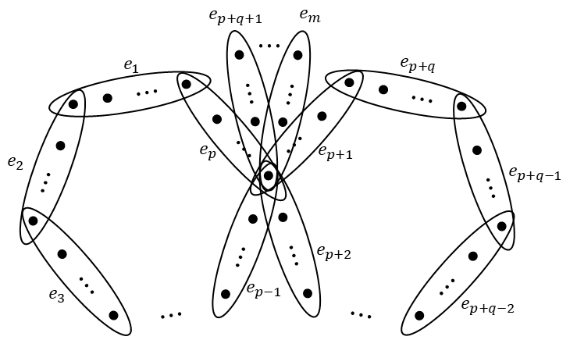


**Figure 5.** The hypergraph $D(p,q)$.

**Theorem 2.** *The hypergraph $D(g,g)$ has the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{B}$.*

*Proof.* We distinguish the following 4 cases to prove our result.

Case 1. We consider the hypergraph in $\mathcal{B}_1(g,l,q)$.

When $l>0$, let $\mathcal{H}\in\mathcal{B}_1(g,l,q)$. If there exist $u\in V(\mathcal{H})$ and $t\neq 0$ such that $u$ is incident with $t$ pendant edges and $d_{\mathcal{H}}(u)=t+1$, then we move $t$ pendant edges from $u$ to a vertex adjacent to $u$ that has degree greater than 1 (see (a) in Fig. 6). Repeating this operation, $\mathcal{H}$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_0$ such that all the edges not in $E(B_1(g,l,q))$ are pendant edges incident with non-cored vertices of $B_1(g,l,q)$. If there exists no vertex $u$ such that $u$ is incident with $t$ pendant edges and $d_{\mathcal{H}}(u)=t+1$, then $\mathcal{H}$ itself is a $k$-uniform bicyclic hypergraph such that all the edges not in $E(B_1(g,l,q))$ are pendant edges incident with non-cored vertices of $B_1(g,l,q)$, and we denote it by $\mathcal{H}_0$. Let $v$ be a vertex with the maximum degree in $\mathcal{H}_0$. If there exist pendant edges not incident with $v$, then we move them from their non-pendant vertices to $v$ (see (b) in Fig. 6). If $v$ is not a vertex with the maximum degree in $B_1(g,l,q)$, then we move all pendant edges from $v$ to a vertex with the maximum degree in $B_1(g,l,q)$. Hence, any hypergraph in $\mathcal{B}_1(g,l,q)$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_1$ obtained from $B_1(g,l,q)$ by attaching $m-g-q-l$ pendant edges at a vertex with degree 3. By Lemma 1, the above 3 operations of moving edges strictly increase the Zagreb index.

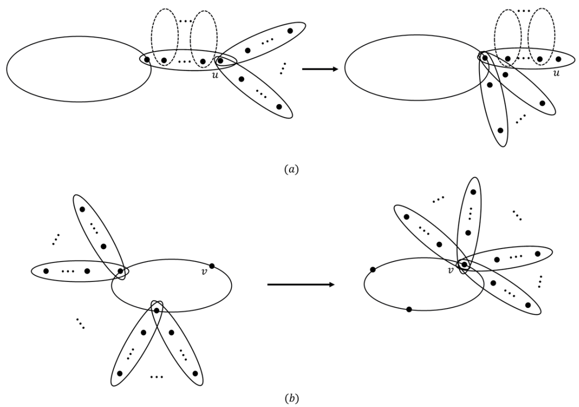


**Figure 6.** Two illustrations of Transformation 1

Without loss of generality, let $d_{\mathcal{H}_1}(v_{2,2}) = 3, d_{\mathcal{H}_1}(v_{1,2}) \geq 3$. Suppose that $\mathcal{H}_2$ is obtained from $\mathcal{H}_1$ by moving 2 edges incident with $v_{2,2}$ in $E(C_2)$ from $v_{2,2}$ to $v_{1,2}$. By Lemma 1, we have $M(\mathcal{H}_2) > M(\mathcal{H}_1)$. If $\mathcal{H}_2 \neq D(g,q)$, then we move the pendant edge not incident with $v_{1,2}$ in $\mathcal{H}_2$ from the non-pendant vertex to $v_{1,2}$. Repeating the above operation of moving edges, $\mathcal{H}_2$ can be changed into $D(g,q)$. By Lemma 1, we have $M(D(g,q)) > M(\mathcal{H}_2)$.

When $l = 0$, similar to the first 3 operations of moving edges in the $l > 0$ subcase of Case 1, any hypergraph in $\mathcal{B}_1(g,0,q)$ can be changed into $D(g,q)$.

The hypergraph $D(g,q-1)$ can be obtained from $D(g,q)$ by moving an edge not incident with $v_{1,2}$ in $E(C_2)$ from a vertex with degree 2 adjacent to $v_{1,2}$ to $v_{1,2}$. By Lemma 1, we have $M(D(g,q-1)) > M(D(g,q))$. When $q = g+s$ and $s > 0$, similar to the above operation of moving edges, we have $M(D(g,q)) < \cdots < M(D(g,q-s+1)) < M(D(g,g))$. Therefore, $D(g,g)$ is the hypergraph with the maximum Zagreb index in $\{\mathcal{B}_1(g,l,q) \mid q \geq g, l \geq 0\}$.

Case 2. We consider the hypergraph in $\mathcal{B}_2(g,l,q)$.

Similar to the first 3 operations of moving edges in the $l > 0$ subcase of Case 1, any hypergraph in $\mathcal{B}_2(g,l,q)$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_3$ obtained from $B_2(g,l,q)$ by attaching $m-g-q-l$ pendant edges at the vertex with degree 3.

Without loss of generality, let $B_2(g,l,q)$ be the sub-hypergraph of $\mathcal{H}_3$ obtained by identifying $v_{1,2}$ with $u_0$, and identifying $v_{2,1}$ with $u_l$. Let $\mathcal{H}_4$ be obtained from $\mathcal{H}_3$ by moving all edges incident with $v_{2,1}$ in $E(\mathcal{H}_3) \setminus E(C_2)$ from $v_{2,1}$ to $v_{2,2}$. By Lemma 1, we have $M(\mathcal{H}_4) > M(\mathcal{H}_3)$. Obviously, $\mathcal{H}_4 \in \mathcal{B}_1(g,l,q)$. Therefore, $D(g,g)$ is the hypergraph with the maximum Zagreb index in $\bigcup_{i=1}^{2} \{\mathcal{B}_i(g,l,q) \mid q \geq g, l \geq 0\}$.

Case 3. We consider the hypergraph in $\mathcal{B}_3(g,l,q)$.

Similar to the first 2 operations of moving edges in the $l > 0$ subcase of Case 1, any hypergraph in $\mathcal{B}_3(g,l,q)$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_5$ obtained from $B_3(g,l,q)$ by attaching $m-g-q-l$ pendant edges at a vertex with degree 2.

Let $\mathcal{H}_6$ be obtained from $\mathcal{H}_5$ by moving all edges incident with $v_{1,1}$ in $E(\mathcal{H}_5) \setminus E(C_1)$ from $v_{1,1}$ to $v_{1,2}$. By Lemma 1, we have $M(\mathcal{H}_6) > M(\mathcal{H}_5)$. Obviously, $\mathcal{H}_6 \in \mathcal{B}_2(g,l,q)$.

Therefore, $D(g,g)$ is the hypergraph with the maximum Zagreb index in $\bigcup_{i=1}^{3} \{\mathcal{B}_i(g,l,q) \mid q \geq g, l \geq 0\}$. ■

**Theorem 3.** *The hypergraph $D(3,3)$ has the maximum Zagreb index in $\mathcal{B}$.*

*Proof.* By Theorem 2, we know that $D(g,g)$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{B}$. For $3 \leq g \leq \frac{m}{2}$, we have

$$\begin{aligned} M(D(g,g)) &= 2g(k-2) + (m-2g)(k-1) + 8(g-1) + (m-2g+4)^2 \\ &= -10g + mk + 7m + 8 + m^2 + 4g^2 - 4mg. \end{aligned}$$

Let $f(x) = -10x + mk + 7m + 8 + m^2 + 4x^2 - 4mx, x \in [3, \frac{m}{2}]$. Since $\frac{df(x)}{dx} = -10+8x-4m < 0$, $f(x)$ is a strictly monotone decreasing function. Then $M(D(g,g)) \leq M(D(3,3))$ for $3 \leq g \leq \frac{m}{2}$, and equality holds if and only if $g = 3$. Hence, $D(3,3)$ is the hypergraph with the maximum Zagreb index in $\mathcal{B}$. ■

Secondly, we give the bicyclic hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{C}$, and give the bicyclic hypergraph with the maximum Zagreb index in $\mathcal{C}$. For $i \in \{1, 2\}$, let $F_i(p, q, l)$ denote the $m$-edge $k$-uniform bicyclic hypergraph obtained from $C_i(p, q, l)$ by attaching $m - p - q - l$ pendant edges at the vertex with degree 3 (see Fig. 7).

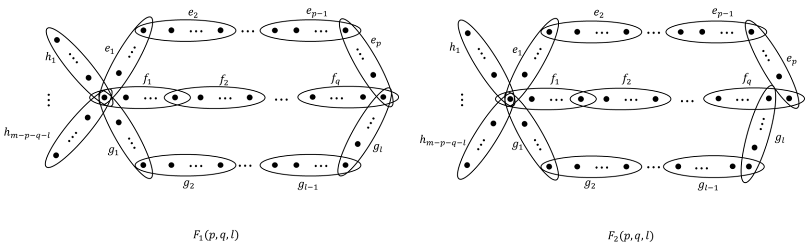


**Figure 7.** The hypergraphs $F_i(p, q, l)$, $i = 1, 2$.

**Theorem 4.** *For $m \geq \frac{3g}{2}$, when $g$ is even, $F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{C}$. When $g$ is odd, $F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{C}$.*

*Proof.* We distinguish the following 3 cases to prove our result.

Case 1. We consider the hypergraph in $\mathcal{C}_1(p, g-p, l)$.

Similar to the first 3 operations of moving edges in the $l > 0$ subcase of Case 1 in Theorem 2, any hypergraph in $\mathcal{C}_1(p, g-p, l)$ can be changed into a $k$-uniform bicyclic hypergraph $F_1(p, g-p, l)$ obtained from $C_1(p, g-p, l)$ by attaching $m - g - l$ pendant edges at a vertex with degree 3. Without loss of generality, let $d_{F_1(p,g-p,l)}(u_1) = m - g - l + 3$.

Note that $g - p \leq l$. When $g - p < l$, $F_1(p, g - p, l - 1)$ can be obtained from $F_1(p, g - p, l)$ by moving $g_2$ from $w_2$ to $u_1$. By Lemma

1, we have $M(F_1(p,g-p,l-1)) > M(F_1(p,g-p,l))$. Similarly, we get $M(F_1(p,g-p,l)) < M(F_1(p,g-p,l-1)) < \cdots < M(F_1(p,g-p,g-p))$.

If $g$ is even, then $p \leq \frac{g}{2}$. When $p < \frac{g}{2}$, we have $M(F_1(p+1,g-p-1,g-p-1)) - M(F_1(p,g-p,g-p)) = (m-2g+p+4)^2+1^2-(m-2g+p+3)^2-2^2 = 2(m-2g+p)+4 > 0$. Similarly, we get $M(F_1(p,g-p,g-p)) < M(F_1(p+1,g-p-1,g-p-1)) < \cdots < M(F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2}))$.

Therefore, when $g$ is even, $F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2})$ has the maximum Zagreb index in $\{\mathcal{C}_1(p,g-p,l) \mid p=1, 1<g-p\leq l \text{ or } 1<p\leq g-p\leq l\}$.

If $g$ is odd, similar to the proof that $g$ is even, we get that $F_1(\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil)$ has the maximum Zagreb index in $\{\mathcal{C}_1(p,g-p,l) \mid p=1, 1<g-p\leq l \text{ or } 1<p\leq g-p\leq l\}$.

Case 2. For $p+q=g$, we consider the hypergraph in $\mathcal{C}_2(p,q,l)$.

Similar to the first 3 operations of moving edges in the $l>0$ subcase of Case 1 in Theorem 2, any hypergraph in $\mathcal{C}_2(p,q,l)$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_1$ obtained from $C_2(p,q,l)$ by attaching $m-p-q-l$ pendant edges at the vertex with degree 3.

If $q=1$ in $\mathcal{H}_1$, then the girth is $p+1$. Let $\mathcal{H}_2$ be obtained from $\mathcal{H}_1$ by moving $g_l$ from $v$ to $v_2$. Obviously, $\mathcal{H}_2 \in \mathcal{C}_1(1,p,l)$ and $g(\mathcal{H}_2)=p+1$. By Lemma 1, we have $M(\mathcal{H}_2) > M(\mathcal{H}_1)$.

If $q>1$ in $\mathcal{H}_1$, then $1\leq p\leq q-1\leq l$.

When $1\leq p<q-1=l$, let $\mathcal{H}_2'$ be obtained from $\mathcal{H}_1$ by moving $g_l$ from $v$ to $v_q$. Obviously, $\mathcal{H}_2' \in \mathcal{C}_1(p+1,q-1,l)$ and $g(\mathcal{H}_2')=p+q$. By Lemma 1, we have $M(\mathcal{H}_2') > M(\mathcal{H}_1)$.

When $1\leq p=q-1=l$, $\mathcal{H}_1 = F_2(\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lfloor\frac{g}{2}\rfloor)$. We have

$$\begin{aligned}
&M(F_2(\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lfloor\frac{g}{2}\rfloor))\\
&= (g+\frac{g-1}{2}-1)(k-2)+(k-3)+(m-g-\frac{g-1}{2})(k-1)+4(g\\
&+\frac{g-1}{2}-1)+(3+m-g-\frac{g-1}{2})^2\\
&= -6g+\frac{23}{4}+mk+6m+m^2+\frac{9g^2}{4}-3mg.
\end{aligned}$$

Since $g(\mathcal{H}_1)=2l+1$, the girth of $\mathcal{H}_1$ is odd. When the girth is odd, $F_1(\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil)$ has the maximum Zagreb index in $\{\mathcal{C}_1(p,g-p,l) \mid p=$

$1, 1 < g-p \le l$ or $1 < p \le g-p \le l\}$. We have

$$\begin{aligned}
&M(F_1(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil)) \\
&= (g + \frac{g+1}{2})(k-2) + (m - g - \frac{g+1}{2})(k-1) + 4(g + \frac{g+1}{2} - 3) + 9 \\
&+ (3 + m - g - \frac{g+1}{2})^2 \\
&= -3g + \frac{19}{4} + mk + 4m + m^2 + \frac{9}{4}g^2 - 3mg.
\end{aligned}$$

Hence, $M(F_1(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil)) - M(F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)) = 3g - 1 - 2m < 0$.

When $1 \le p \le q-1 < l$, let $\mathcal{H}_2''$ be obtained from $\mathcal{H}_1$ by moving $g_l$ from $v$ to $v_{q+1}$. Obviously, $\mathcal{H}_2'' \in \mathcal{C}_1(p,q,l)$ and $g(\mathcal{H}_2'') = p+q$. By Lemma 1, we have $M(\mathcal{H}_2'') > M(\mathcal{H}_1)$.

Therefore, when $g$ is even, $F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\{\mathcal{C}_1(p,q,l) \mid p = 1, 1 < q \le l$ or $1 < p \le q \le l\} \bigcup \{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \le l$ or $q > 1, 1 \le p \le q-1 \le l\}$. When $g$ is odd, $F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\{\mathcal{C}_1(p,q,l) \mid p = 1, 1 < q \le l$ or $1 < p \le q \le l\} \bigcup \{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \le l$ or $q > 1, 1 \le p \le q-1 \le l\}$.

Case 3. For $p+q = g$, we consider the hypergraph in $\mathcal{C}_3(p,q,l)$.

Similar to the first 2 operations of moving edges in the $l > 0$ subcase of Case 1 in Theorem 2, any hypergraph in $\mathcal{C}_3(p,q,l)$ can be changed into a $k$-uniform bicyclic hypergraph $\mathcal{H}_3$ obtained from $C_3(p,q,l)$ by attaching $m-p-q-l$ pendant edges at a vertex with degree 2.

If $q = 1$ in $\mathcal{H}_3$, then $g(\mathcal{H}_3) = p+1$. Let $\mathcal{H}_4$ be obtained from $\mathcal{H}_3$ by moving all edges incident with $v'$ in $E(\mathcal{H}_3) \setminus (E(P_p) \bigcup E(P_q))$ from $v'$ to $v_1$ and moving all edges incident with $v''$ in $E(\mathcal{H}_3) \setminus (E(P_p) \bigcup E(P_q))$ from $v''$ to $v_2$. Obviously, $\mathcal{H}_4 \in \mathcal{C}_1(1,p,l)$ and $g(\mathcal{H}_4) = p+1$. By Lemma 1, we have $M(\mathcal{H}_4) > M(\mathcal{H}_3)$.

If $q = 2, p = l = 1$ of $\mathcal{H}_3$, then $g(\mathcal{H}_3) = 3$. Let $\mathcal{H}_5$ be obtained from $\mathcal{H}_3$ by moving all edges incident with $v'$ in $E(\mathcal{H}_3) \setminus (E(P_p) \bigcup E(P_q))$ from $v'$ to $v_1$. Obviously, $\mathcal{H}_5 \in \mathcal{C}_2(1,2,1)$ and $g(\mathcal{H}_5) = 3$. By Lemma 1, we have $M(\mathcal{H}_5) > M(\mathcal{H}_3)$.

If $q=2, 1=p<l$ (or $q=2, 1<p\leq l$) of $\mathcal{H}_3$, then $g(\mathcal{H}_3)=3$ (or $p+2$). Similar to the proof of $q=1$ of $\mathcal{H}_3$, $\mathcal{H}_3$ can be changed into $\mathcal{H}_4$, $\mathcal{H}_4\in\mathcal{C}_1(1,2,l)$ (or $\mathcal{C}_1(2,p,l)$), $g(\mathcal{H}_4)=3$ (or $p+2$) and $M(\mathcal{H}_4)>M(\mathcal{H}_3)$.

If $q>2$ of $\mathcal{H}_3$, then $1\leq p\leq q-2\leq l$. When $q\geq 2$, $M(\mathcal{H}_3)=-p-q+2-l+mk+3m+m^2+p^2+q^2+l^2-2mp-2mq-2ml+2pq+2pl+2ql$.

When $1\leq p\leq q-2=l$, if $1\leq p=q-2=l$, then $M(\mathcal{H}_3)=9l+mk-m+9l^2-6ml+m^2+4$. Since $l=\frac{1}{2}g-1$, $M(\mathcal{H}_3)=-\frac{9}{2}g+mk+5m+4+m^2+\frac{9}{4}g^2-3mg$.

Since $g(\mathcal{H}_3)=2l+2$, the girth of $\mathcal{H}_3$ is even. When $g$ is even, $F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\{\mathcal{C}_1(p,q,l)\mid p=1, 1<q\leq l \text{ or } 1<p\leq q\leq l\}\bigcup\{\mathcal{C}_2(p,q,l)\mid q=1, 1<p\leq l \text{ or } q>1, 1\leq p\leq q-1\leq l\}$. We have

$$\begin{aligned}
&M(F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2}))\\
&=\frac{3}{2}g(k-2)+(m-\frac{3}{2}g)(k-1)+12(\frac{g}{2}-1)+9+(3+m-\frac{3}{2}g)^2\\
&=-\frac{9}{2}g+mk+5m+6+m^2+\frac{9}{4}g^2-3mg.
\end{aligned}$$

Therefore, $M(F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2}))-M(\mathcal{H}_3)=2>0$.

If $1\leq p<q-2=l$, let $\mathcal{H}_6$ be obtained from $\mathcal{H}_3$ by moving all edges incident with $v'$ in $E(\mathcal{H}_3)\setminus(E(P_p)\bigcup E(P_q))$ from $v'$ to $v_2$. Obviously, $\mathcal{H}_6\in\mathcal{C}_2(p+1,q-1,l)$ and $g(\mathcal{H}_6)=p+q$. By Lemma 1, we have $M(\mathcal{H}_6)>M(\mathcal{H}_3)$.

When $1\leq p\leq q-2<l$, similar to the proof of $q=2, p=l=1$ of $\mathcal{H}_3$, $\mathcal{H}_3$ can be changed into $\mathcal{H}_5$, $\mathcal{H}_5\in\mathcal{C}_2(p,q,l)$, $g(\mathcal{H}_5)=p+q$ and $M(\mathcal{H}_5)>M(\mathcal{H}_3)$.

Therefore, when $g$ is even, $F_1(\frac{g}{2},\frac{g}{2},\frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{C}$. When $g$ is odd, $F_2(\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lfloor\frac{g}{2}\rfloor)$ is the hypergraph with the maximum Zagreb index among all hypergraphs with girth $g$ in $\mathcal{C}$. ∎

**Theorem 5.** *For $m\geq 6$, $F_2(1,2,1)$ is the hypergraph with the maximum Zagreb index in $\mathcal{C}$.*

*Proof.* The following determines the relationship between $m$ and $g$ that guarantees the sets $\{\mathcal{C}_1(p,q,l) \mid p = 1, 1 < q \le l \text{ or } 1 < p \le q \le l\}, \{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \le l \text{ or } q > 1, 1 \le p \le q-1 \le l\}$ and $\{\mathcal{C}_3(p,q,l) \mid q = 1, k > 3, 1 < p \le l \text{ or } q = 2, 1 \le p \le l \text{ or } q > 2, 1 \le p \le q-2 \le l\}$ are non-empty.

For the set $\{\mathcal{C}_1(p,q,l) \mid p = 1, 1 < q \le l \text{ or } 1 < p \le q \le l\}$, since $g = p+q$ and $p \le q$, $g - q \le q$, that is $q \ge \frac{g}{2}$. Since $l \ge q$, we have $l \ge \frac{g}{2}$. Then $p+q+l = g+l \ge \frac{3}{2}g$. Thus, when $m \ge \frac{3}{2}g$, the set is non-empty.

For the set $\{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \le l \text{ or } q > 1, 1 \le p \le q-1 \le l\}$, we have $q \le l+1$ and $p \le l$, which implies $l \ge \frac{g-1}{2}$. Then we have $p+q+l \ge g + \frac{g-1}{2} = \frac{3}{2}g - \frac{1}{2}$. Thus, when $m \ge \frac{3}{2}g - \frac{1}{2}$, the set is non-empty. Since we consider $m \ge \frac{3g}{2}$ in Theorem 4, now we need consider $\frac{3g}{2} - \frac{1}{2} \le m < \frac{3g}{2}$. Let $\mathcal{H} \in \{\mathcal{C}_2(p,q,l) \mid q = 1, 1 < p \le l \text{ or } q > 1, 1 \le p \le q-1 \le l\}$. For $\frac{3g}{2} - \frac{1}{2} \le m < \frac{3g}{2}$, we have $m = \frac{3g}{2} - \frac{1}{2}$ and $g$ is odd. If $q = 1$, $\mathcal{H}$ does not exist. If $q > 1$, $1 \le g - q \le q - 1 \le l$, then $\frac{g}{2} + \frac{1}{2} \le q \le l+1$. Since $m = \frac{3g}{2} - \frac{1}{2}$, $l \ge \frac{g-1}{2}$ and $\frac{g}{2} + \frac{1}{2} \le q \le l+1$, $l = \frac{g-1}{2}$ and $q = \frac{g}{2} + \frac{1}{2}$. Therefore, when $\frac{3g}{2} - \frac{1}{2} \le m < \frac{3g}{2}$, $\mathcal{H} = C_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)$ and $g = \frac{2m}{3} + \frac{1}{3}$ is odd.

For the set $\{\mathcal{C}_3(p,q,l) \mid q = 1, k > 3, 1 < p \le l \text{ or } q = 2, 1 \le p \le l \text{ or } q > 2, 1 \le p \le q-2 \le l\}$, we have $q \le l+2$ and $p \le l$, which implies $l \ge \frac{1}{2}g - 1$. Then we have $p+q+l \ge g + \frac{1}{2}g - 1 = \frac{3}{2}g - 1$. Thus, when $m \ge \frac{3}{2}g - 1$, the set is non-empty. Since we consider $m \ge \frac{3g}{2}$ in Theorem 4, now we need consider $\frac{3g}{2} - 1 \le m < \frac{3g}{2}$. Let $\mathcal{H} \in \{\mathcal{C}_3(p,q,l) \mid q = 1, k > 3, 1 < p \le l \text{ or } q = 2, 1 \le p \le l \text{ or } q > 2, 1 \le p \le q-2 \le l\}$. If $q > 2$, $1 \le g - q \le q - 2 \le l$, then $\frac{g}{2} + 1 \le q \le l+2$. For $\frac{3g}{2} - 1 \le m < \frac{3g}{2}$, when $g$ is even, $m = \frac{3g}{2} - 1$. Since $m = \frac{3g}{2} - 1$, $l \ge \frac{1}{2}g - 1$ and $\frac{g}{2} + 1 \le q \le l+2$, $l = \frac{1}{2}g - 1$ and $q = \frac{g}{2} + 1$. Therefore, $\mathcal{H} = C_3(\frac{g}{2} - 1, \frac{g}{2} + 1, \frac{g}{2} - 1)$ and $g = \frac{2m}{3} + \frac{2}{3}$ is even. When $g$ is odd, $m = \frac{3g}{2} - \frac{1}{2}$. Since $m = \frac{3g}{2} - \frac{1}{2}$, $l \ge \frac{1}{2}g - 1$ and $\frac{g}{2} + 1 \le q \le l+2$, $l = \frac{1}{2}g - \frac{1}{2}$ and $q = \frac{1}{2}g + \frac{3}{2}$. Therefore, $\mathcal{H} = C_3(\frac{1}{2}g - \frac{3}{2}, \frac{1}{2}g + \frac{3}{2}, \frac{1}{2}g - \frac{1}{2})$ and $g = \frac{2m}{3} + \frac{1}{3}$ is odd. If $q = 1$ or 2, for $\frac{3g}{2} - 1 \le m < \frac{3g}{2}$ and $m \ge 6$, $\mathcal{H}$ does not exist. Hence, when $\frac{3g}{2} - 1 \le m < \frac{3g}{2}$ and $\mathcal{H} \in \{\mathcal{C}_3(p,q,l) \mid q = 1, k > 3, 1 < p \le l \text{ or } q = 2, 1 \le p \le l \text{ or } q > 2, 1 \le p \le q-2 \le l\}$, if $g$ is even, $\mathcal{H} = C_3(\frac{g}{2} - 1, \frac{g}{2} + 1, \frac{g}{2} - 1)$ and $g = \frac{2m}{3} + \frac{2}{3}$. If $g$ is odd, $\mathcal{H} = C_3(\frac{1}{2}g - \frac{3}{2}, \frac{1}{2}g + \frac{3}{2}, \frac{1}{2}g - \frac{1}{2})$ and $g = \frac{2m}{3} + \frac{1}{3}$.

When $g$ is even, we have $M(F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})) = -\frac{9}{2}g + mk + 5m + 6 + m^2 +$

$\frac{9}{4}g^2 - 3mg$. Let $f(x) = -\frac{9}{2}x + mk + 5m + 6 + m^2 + \frac{9}{4}x^2 - 3mx, 4 \leq x \leq \frac{2m}{3}$. Since $\frac{df(x)}{dx} = -\frac{9}{2} + \frac{9x}{2} - 3m < 0$, $f(x)$ is a strictly monotone decreasing function. Then when $g$ is even, $M(F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})) \leq M(F_1(2, 2, 2))$ for $4 \leq g \leq \frac{2m}{3}$, and equality holds if and only if $g = 4$. Since the maximum degree of $C_3(\frac{g}{2} - 1, \frac{g}{2} + 1, \frac{g}{2} - 1)$ is 2, $M(F_1(2, 2, 2)) > M(C_3(\frac{g}{2} - 1, \frac{g}{2} + 1, \frac{g}{2} - 1))$ for $g = \frac{2m}{3} + \frac{2}{3}$.

When $g$ is odd, we have $M(F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)) = -6g + \frac{23}{4} + mk + 6m + m^2 + \frac{9g^2}{4} - 3mg$. Let $f(x) = -6x + \frac{23}{4} + mk + 6m + m^2 + \frac{9x^2}{4} - 3mx, 3 \leq x \leq \frac{2m}{3} + \frac{1}{3}$. Since $\frac{df(x)}{dx} = -6 + \frac{9x}{2} - 3m < 0$, $f(x)$ is a strictly monotone decreasing function. Then when $g$ is odd, $M(F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)) \leq M(F_2(1, 2, 1))$ for $3 \leq g \leq \frac{2m}{3}$, and equality holds if and only if $g = 3$. And $M(F_2(1, 2, 1)) > M(C_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor))$ for $g = \frac{2m}{3} + \frac{1}{3}$. Since the maximum degree of $C_3(\frac{1}{2}g - \frac{3}{2}, \frac{1}{2}g + \frac{3}{2}, \frac{1}{2}g - \frac{1}{2})$ is 2, $M(F_2(1, 2, 1)) > M(C_3(\frac{1}{2}g - \frac{3}{2}, \frac{1}{2}g + \frac{3}{2}, \frac{1}{2}g - \frac{1}{2}))$ for $g = \frac{2m}{3} + \frac{1}{3}$.

When $m \geq 6$, since $M(F_1(2, 2, 2)) - M(F_2(1, 2, 1)) = 16 - 4m < 0$, $F_2(1, 2, 1)$ is the hypergraph with the maximum Zagreb index in $\mathcal{C}$. ■

Finally, we give the hypergraph with the maximum Zagreb index among all linear bicyclic uniform hypergraphs with $m$ edges and girth $g$, and the hypergraph with the maximum Zagreb index among all linear bicyclic uniform hypergraphs with $m$ edges.

**Theorem 6.** *For $m \geq \frac{3g}{2}$, when $g$ is even, $F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$. When $g$ is odd, $F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$.*

*For $m \geq 6$, $F_2(1, 2, 1)$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges.*

*Proof.* For $\frac{3g}{2} \leq m < 2g$, the set $\mathcal{B}$ is empty. We need consider the hypergraphs with girth $g$ in $\mathcal{C}$. Hence, by Theorem 4, when $g$ is even, $F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$. When $g$ is odd, $F_2(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lfloor \frac{g}{2} \rfloor)$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$.

For $m \geq 2g$, when $g$ is even, $M(F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})) - M(D(g,g)) = \frac{11}{2}g - 2m - 2 - \frac{7}{4}g^2 + mg$, $4 \leq g \leq \frac{m}{2}$. Let $f(x) = \frac{11}{2}x - 2m - 2 - \frac{7}{4}x^2 + mx$. The roots of $f(x) = 0$ are easily obtained as $x_1 = \frac{11+2m-2\sqrt{(m-\frac{3}{2})^2+14}}{7}$ and $x_2 = \frac{11+2m+2\sqrt{(m-\frac{3}{2})^2+14}}{7}$. Since $x_1 < \frac{11+2m-2(m-\frac{3}{2})}{7} = 2 < 4$ and $x_2 > \frac{11+2m+2(m-\frac{3}{2})}{7} = \frac{4m+8}{7} > \frac{m}{2}$, when $4 \leq g \leq \frac{m}{2}$, $M(F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})) - M(D(g,g)) > 0$. Therefore, when $g$ is even, $F_1(\frac{g}{2}, \frac{g}{2}, \frac{g}{2})$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$.

When $g$ is odd, $M(F_2(\lfloor\frac{g}{2}\rfloor, \lceil\frac{g}{2}\rceil, \lfloor\frac{g}{2}\rfloor)) - M(D(g,g)) = 4g - m - \frac{9}{4} - \frac{7}{4}g^2 + mg, 3 \leq g \leq \frac{m}{2}$. Let $f(x) = 4x - m - \frac{9}{4} - \frac{7}{4}x^2 + mx$. The roots of $f(x) = 0$ are easily obtained as $x_1 = \frac{8+2m-2\sqrt{(m+\frac{1}{2})^2}}{7} = 1$ and $x_2 = \frac{8+2m+2\sqrt{(m+\frac{1}{2})^2}}{7} = \frac{4m+9}{7}$. Obviously, $x_1 < 3, x_2 > \frac{m}{2}$. So, when $3 \leq g \leq \frac{m}{2}$, $M(F_2(\lfloor\frac{g}{2}\rfloor, \lceil\frac{g}{2}\rceil, \lfloor\frac{g}{2}\rfloor)) - M(D(g,g)) > 0$. Hence, when $g$ is odd, $F_2(\lfloor\frac{g}{2}\rfloor, \lceil\frac{g}{2}\rceil, \lfloor\frac{g}{2}\rfloor)$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges and girth $g$.

From the proof of Theorem 5, we get that $F_2(1,2,1)$ is the hypergraph with the maximum Zagreb index among all linear bicyclic $k$-uniform hypergraphs with $m$ edges. ■

***Acknowledgment*:** The research of the second author is partially supported by the National Natural Science Foundation of China (No. 1207109 7) and the Natural Science Foundation for The Excellent Youth Scholars of the Heilongjiang Province (No. YQ2022A002).

# References

[1] B. Borovićanin, B. Furtula, On extremal Zagreb indices of trees with given domination number, *Appl. Math. Comput.* **279** (2016) 208–218.

[2] B. Borovićanin, T.A. Lampert, On the maximum and minimum Zagreb indices of trees with a given number of vertices of maximum degree, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 81–96.

[3] K. Cardoso, V. Trevisan, Energies of hypergraphs, *El. J. Lin. Algebra* **36** (2020) 293–308.

[4] S. Chen, W. Liu, Extremal Zagreb indices of graphs with a given number of cut edges, *Graphs Comb.* **30** (2014) 109–118.

[5] H. Deng, A unified approach to the extremal Zagreb indices for trees, unicyclic graphs and bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 597–616.

[6] M. Enteshari, B. Taeri, Extremal Zagreb indices of graphs of order $n$ with $p$ pendent vertices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 17–28.

[7] Y. Feng, X. Hu, S. Li, On the extremal Zagreb indices of graphs with cut edges, *Acta Appl. Math.* **110** (2010) 667–684.

[8] W. Gao, The first and second Zagreb indices of hypergraphs, *Trans. Comb.* (2025) **doi:** `https://doi.org/10.22108/toc.2024.141216.2169`.

[9] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.

[10] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total $\pi$-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[11] F. Javaid, M. K. Jamil, I. Tomescu, Extremal $k$-generalized quasi unicyclic graphs with respect to first and second Zagreb indices, *Discr. Appl. Math.* **270** (2019) 153–158.

[12] S. Li, H. Zhou, On the maximum and minimum Zagreb indices of graphs with connectivity at most $k$, *Appl. Math. Lett.* **23** (2010) 128–132.

[13] X. Liu, L. Wang, Distance spectral radii of $k$-uniform bicyclic hypergraphs, *Lin. Multilin. Algebra* **70** (2022) 6190–6210.

[14] S. Nikolić, G. Kovačević, A. Miličević, N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta* **76** (2003) 113–124.

[15] L. Pei, X. Pan, Extremal values on Zagreb indices of trees with given distance $k$-domination number, *J. Inequal. Appl.* **2018** (2018) #16.

[16] F. Xia, S. Chen, Ordering unicyclic graphs with respect to Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **58** (2007) 663–673.

[17] K. Xu, The Zagreb indices of graphs with a given clique number, *Appl. Math. Lett.* **24** (2011) 1026–1030.

[18] S. Zhang, H. Zhang, Unicyclic graphs with the first three smallest and largest first general Zagreb index, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 427–438.

[19] B. Zhou, Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **52** (2004) 113–118.

[20] H. Zhou, C. Bu, Lexicographical ordering of hypergraphs via spectral moments, Available at arXiv: 2309.16925, **doi:** `https://doi.org/10.48550/arXiv.2309.16925`.