MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **96** (2026) 325–352
ISSN: 0340–6253
doi: 10.46793/match.96-1.07025

Characterization of Trees with a Certain Wiener Complexity

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(Received April 5, 2025)

Abstract

Let u be a vertex of a simple connected graph G. Transmission of u, $Tr_G(u)$ is the sum of all distances between u and other vertices in G. The Wiener index of G, $W(G)$, is half of the sum of the transmission of all vertices. The Wiener complexity of G is the number of different vertex transmissions of G. In this paper, we characterize trees with Wiener complexity at most three, while we discuss the structure of trees with Wiener complexity four and illustrate many cases that arise. The trees of Wiener complexity four have been identified within 16 categories.

1 Introduction

All considered graphs are simple and connected. Let $G(V(G), E(G))$ be a graph. The order and size of G are denoted by $n(G)$ and $m(G)$ respectively. We denote by $\deg(u)$, degree of vertex u. A vertex of degree 1 is called a pendant vertex and an edge is said to be a pendant edge (or a leaf) if one of its end vertices is a pendant vertex. Distance between two vertices u and v in G, $d_G(u, v)$, (shortly $d(u, v)$) is the length of the shortest path

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between u and v. For a vertex u and a positive integer i, $\Gamma_i(u)$ denotes the set of vertices at distance i from u. We denote by $N_G(u)$ (briefly $N(u)$), the set of adjacent vertices to u in G, i.e., $N(u) = \Gamma_1(u)$. The maximum distance from a vertex v to all other vertices is called the eccentricity of v and is denoted by $\varepsilon_G(v)$. Center of G, $C(G)$, is the vertex set of minimum eccentricity. Diameter, diam(G), and radius, rad(G), are the maximum and minimum eccentricity of vertices of G, respectively. Transmission of v, $Tr_G(v)$ is the sum of all distances between v and other vertices of G. Imbalance transmission of an edge uv, $I_G(uv)$, is defined as $I_G(uv) = |Tr(u) - Tr(v)|$. We denote by $Tr(G)$, the set of vertex transmission of G. The Wiener complexity of G, $C_w(G)$, is defined as the cardinality of $Tr(G)$ [2]. Transmission in graphs has introduced several metric concepts in graph theory. For instance, the well-known topological index, Wiener index [16] can be defined as half of the sum of vertex transmission, i.e.,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} Tr(v).$$

Also, the Mostar index has been introduced in [1] as:

$$Mo(G) = \sum_{e \in E(G)} I(e).$$

Furthermore, interesting graphs have been proposed based on transmission and Wiener complexity in several investigations. Graphs with the Wiener complexity 1 are called transmission regular graphs [14]. Transmission irregular graphs (Briefly TI) have the Wiener complexity equal to their order [4]. Some particular families of TI- starlike trees in [7, 11, 13], 2-connected and 3-connected TI graphs in [9, 10] were identified. Interval transmission graphs [6] are a subclass of $TI-$ graphs in which the set of vertex transmissions form a sequence of consecutive positive integers. A graph G is said to be a stepwise irregular graph (SI for short) [12] if $I_G(e) = 1$ for each edge $e \in E(G)$. Generalized SI-graphs, $k-$SI graphs, introduced in [5], are the graphs in which $I_G(e) = k$ holds for each edge e of the graph. Extremal results on STI graphs concerning the diameter, the Wiener index and the eccentricity index were characterized in [3]. In this paper, we characterize all classes of trees with Wiener complexity of at most 3 and identify the structure of all trees with Wiener complexity of 4. Several examples of such trees are also illustrated. It has been proven that the Wiener complexity of a tree is at least equal to its radius.

2 Main results

We first refer to some basic concepts and properties of vertex transmission in simple graphs. Next, we try to characterize all trees with a given small Wiener complexity at most 4. It is well-known that the center of a tree T, is a single vertex or two adjacent vertices. Let us denote by $T \in C_r(x)$ and $T \in C_r(x, y)$ if T is a tree with $\mathrm{rad}(T) = r$ and its center $C(T) = \{x\}$ and $C(T) = \{x, y\}$, respectively.

Lemma 1. *[8] Let u and v be two adjacent vertices of G. Then $Tr(u) - Tr(v) = n_v - n_u$, where n_u denotes the number of all vertices which are closer to u than v in the graph G and n_v is defined similarly.*

Lemma 2. *Let G be a graph of order n. If uv is an edge of G, then $|Tr(u) - Tr(v)| \leq n - 2$, with equality holds if and only if uv is a pendant edge.*

Proof. Without loss of generality, suppose that $Tr(u) \geq Tr(v)$. From Lemma 1,

$$Tr(u) - Tr(v) = n_v - n_u \leq n - 2\deg(u) \leq n - 2.$$

The equality holds if and only if $\deg(u) = 1$. This means that uv is a pendant edge. ∎

The next result shows that the Wiener complexity of a tree is greater than or equal to Its radius. Of course, this is not true in general.

Theorem 1. *If* T *is a tree with* $\mathrm{rad}(T) = r$, *then*

$$C_W(T) \geq \begin{cases} \mathrm{rad}(T) + 1 & \text{if } T \in C_r(x) \\ \\ \mathrm{rad}(T) & \text{if } T \in C_r(x, y) \end{cases}$$

Proof. We must consider two cases. First: $T \in C_r(x)$. Let P be a diametrical path. Since any diametrical path contains central vertices in a tree, consider the path P as $P = v_r - v_{r-1} \cdots v_1 - x - u_1 - u_2 \cdots u_r$. Let T_1 be the connected component of $T - xv_1$ containing v_1 and T_2 be the connected component of $T - xu_1$ containing u_1. Without loss of generality, suppose that $n(T_1) \leq n(T_2)$. Using Lemma 1, we get

$$Tr(v_1) - Tr(x) = n_x - n_{v_1} \geq (n(T_2) + 1) - n(T_1) \geq 1.$$

Moreover, for any pair of adjacent vertices v_i and v_{i+1} of the path P, we have

$$\begin{aligned} Tr(v_{i+1}) - Tr(v_i) = n(v_i) - n(v_{i+1}) \quad &\geq \quad (n(T_2) + i + 1) - (n(T_1) - i) \\ &= n(T_2) - n(T_1) + 2i + 1. \end{aligned}$$

This turn yields the following strictly increasing sequence as

$$Tr(x) < Tr(v_1) < Tr(v_2) \cdots < Tr(v_t).$$

Second: $T \in C_r(x, y)$. Consider a diametrical path as $P = v_r - v_{r-1} \cdots v_1 - x - y - u_1 - u_2 \cdots u_r$. Let $T - xy = T_1 \cup T_2$ with $n(T_1) \leq n(T_2)$ and T_1 be the component containing x . Applying an analogous argument, we get

$$Tr(x) < Tr(v_1) < \cdots < Tr(v_r).$$

The proof is completed. ■

Now, we are going to determine the structure of trees with $1 \leq C_w \leq 4$.

2.1 Trees of the Wiener complexity 1

As an immediate consequence of Theorem 1, the unique tree of Wiener complexity 1 is determined as follows.

Corollary. *Let T be a tree then $C_W(T) = 1$ if and only if $T = P_2$.*

Proof. Let xy be a leaf of a tree T with $\deg(x) = 1$. Then $Tr(x) = Tr(y) + n - 2$. Obviously $C_W(T) = 1$ if and only if $n = 2$. The proof is complete. ■

2.2 Trees of the Wiener complexity 2

Next, we show that the trees of Wiener complexity 2 belong to only two families; stars S_n, $(n \geq 3)$ or double stars $S_{a,a}$ for some positive integer $a \geq 2$. Recall that a double star $S_{a,b}$ is formed by joining the centers of two stars S_a and S_b.

Theorem 2. *Let T be a tree of order $n \geq 3$. Then $C_W(T) = 2$ if and only if $T \in \{S_n, S_{\frac{n}{2},\frac{n}{2}}\}$*

Proof. From Theorem 1, $\mathrm{rad}(T) \leq 2$. This shows that T is a star when $\mathrm{rad}(T) = 1$ and T is a double star when $\mathrm{rad}(T) = 2$. If T is a double star with $C(T) = \{x, y\}$, Lemma 1 follows that all pendant vertices joining to the center of T get the same transmission. For $v \in N(x)$ and $w \in N(y)$ we have

$$\begin{aligned} Tr(v) &= 1 + 2(\deg(x) - 1) + 3(\deg(y) - 1), \\ Tr(w) &= 1 + 2(\deg(y) - 1) + 3(\deg(x) - 1). \end{aligned}$$

Thus, $\deg(x) = \deg(y)$ and then $T = S_{\frac{n}{2},\frac{n}{2}}$. ■

2.3 Trees of the Wiener complexity 3

Let $a_0, a_1, \cdots, a_r$ be positive integers. Suppose that $T^k(a_0, a_1 \cdots, a_r)$, for $k = 1, 2$; denotes trees with k central vertices, whose vertices at distance j from the center of $T^i(a_0, a_1 \cdots, a_r)$ have the same degree a_j for $0 \leq j \leq r$.

Theorem 3. *Let T be a tree. Then $C_W(T) = 3$ if and only if $T \in \{T^1(a_0, a_1, 1), T^2(a_0, a_1, 1)\}$ for some integers $a_0, a_1 \geq 2$.*

Proof. By Theorem 1, $\text{rad}(T) \leq 3$. **First**, consider the case $T \in C_2(x)$ with $P : z_1 - y_1 - x - y_2 - z_2$ as a diametrical path. Suppose that $\deg(y_1) \leq \deg(y_2)$. By Lemma 1 the following relations hold. For $i = 1, 2$

$$\begin{aligned} Tr(z_i) &= Tr(y_i) + n - 2, \\ Tr(y_i) &= Tr(x) + n - 2\deg(y_i), \\ Tr(y_1) &- Tr(y_2) = 2(\deg(y_2) - \deg(y_1)), \\ Tr(z_1) &- Tr(z_2) = 2(\deg(y_2) - \deg(y_1)). \end{aligned} \tag{1}$$

This implies that $Tr(z_i) > Tr(y_i) > Tr(x)$ for $i = 1, 2$. Since $C_W(T) = 3$, $Tr(z_1) = Tr(z_2)$ and $Tr(y_1) = Tr(y_2)$, consequently $\deg(y_1) = \deg(y_2)$. Therefor, $T \cong T(a_0, a_1, 1)$ where $a_0 = \deg(x)$ and $a_1 = \deg(y)$. Note that if $w \in N(x)$ is a pendant vertex, then $Tr(w) - Tr(x) = n - 2$. Since $C_w(T) = 3$ and by Theorem 1, $Tr(w) = Tr(z_1)$. Thus by (1)

$$n - 2 + Tr(x) = Tr(w) = Tr(z_1) = Tr(x) + n - 2 + n - 2\deg(y_1).$$

So we get $\deg(y_1) = \frac{n}{2}$, that is a contradiction. **Second**, assume that $T \in C_2(x, y)$. Such a tree is a double star, let $T = S_{a,b}$. If $a = b$ then $C_W(T) = 2$ and if $a \neq b$ then $C_W(T) = 4$. **Third** case is $T \in C_3(x, y)$ by Theorem1. Let $P : x_2 - x_1 - x - y - y_1 - y_2$ be a diametrical path. let $T - xy = T_1 \cup T_2$ where $x \in V(T_1)$ and $y \in V(T_2)$ with $n(T_1) \leq n(T_2)$. By a similar argument, we infer that $Tr(x_2) > Tr(x_1) > Tr(x)$. By lemma 1, $Tr(x) - Tr(y) = n(T_2) - n(T_1) \geq 0$. Since $C_W(T) = 3$ so $Tr(x) = Tr(y)$ and consequently $n(T_1) = n(T_2)$. This implies that $Tr(y_2) > Tr(y_1) > Tr(y)$. Since $C_W(T) = 3$ we get $Tr(x_2) = Tr(y_2)$ and $Tr(x_1) = Tr(y_1)$. Moreover

$$\begin{aligned} Tr(x_1) - Tr(x) &= n - 2\deg(x_1), \\ Tr(y_1) - Tr(y) &= n - 2\deg(y_1). \end{aligned}$$

Thus $\deg(x_1) = \deg(y_1)$. If central vertices have some pendant adjacent vertices, say $w \in N(x)$, then theorem 1 yields that $Tr(w) = Tr(x_2)$. Thus

$$\begin{aligned}(Tr(x_2) - Tr(x_1)) + (Tr(x_1) - Tr(x)) &= Tr(w) - Tr(x),\\ (n-2) + (n - 2\deg(x_1)) &= n-2.\end{aligned}$$

Thus $\deg(x_1) = \frac{n}{2}$, a contradiction. Moreover, we have

$$0 = Tr(x) - Tr(y) = \deg(x)(\deg(x_1) - 1) - \deg(y)(\deg(y_1) - 1).$$

The equality $Tr(x) = Tr(y)$ yields that $\deg(x) = \deg(y)$. Therefore

$$T = \begin{cases} T^1(a_0, a_1, 1) & \text{if } T \in C_2(x) \\ T^2(a_0, a_1, 1) & \text{if } T \in C_3(x, y) \end{cases}$$

where $a_0 = \deg(x)$ and $a_1 = \deg(x_1)$. ■

2.4 Trees of the Wiener complexity 4

In this section, the structure of the trees of Wiener complexity 4 is verified. In particular, several examples of such trees with different structures are also provided. Let T be a tree with $C_W(T) = 4$. From Theorem 1, $\text{rad}(T) \leq 4$. It is necessary to consider the trees in 5 cases concerning their center and radius.

1. $T \in C_2(x)$**.**
   Analogous Theorem 3, the relations (1) hold.
   First, suppose that all adjacent vertices to x are of degree at least 2. By (1), vertices in $N(x)$ with the same degree get the same transmission. Since $C_w(T) = 4$, there are two vertices, say y_1 and y_2, in $N(x)$ with different degrees. Without loss of generality suppose that $P : z_1 - y_1 - x - y_2 - z_2$ be a diametrical path where $1 < \deg(y_1) < \deg(y_2)$. From Lemma 1, $Tr(z_1) > Tr(y_1) > Tr(x)$. Consider the following cases:

   (a) $\deg(y_2) < \frac{n}{2}$. The relations (1) hold and then

   $$Tr(z_1) > Tr(y_1) > Tr(y_2) > Tr(x).$$

Further, $Tr(z_1) > Tr(z_2)$. Since $C_w(T) = 4$, $Tr(z_2) = Tr(y_1)$. This follows

$$\begin{aligned} Tr(z_2) - Tr(y_2) &= Tr(y_1) - Tr(y_2), \\ n - 2 &= 2(\deg(y_2) - \deg(y_1)). \end{aligned}$$

Thus $\deg(y_1) < 1$, which is a contradiction.

(b) $\deg(y_2) = \frac{n}{2}$**.** Using (1), $Tr(y_2) = Tr(x)$. Further

$$Tr(z_2) - Tr(y_1) = 2(\deg(y_1) - 1) > 0$$

Thus

$$Tr(T) = \{Tr(z_1) > Tr(z_2) > Tr(y_1) > Tr(x) = Tr(y_2)\}$$

Also y_2 is the unique vertex of $N(x)$ with degree $\deg(y_2) = \frac{n}{2}$ and the other vertices in $N(x)$ are of the same degree $\deg(y_1)$, where

$$\deg(y_1)(\deg(x) - 1) = \frac{n}{2} - 1. \tag{2}$$

An example of such a tree is illustrated in Figure 1.

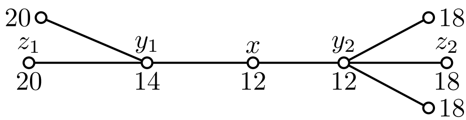

Figure 1. A tree of $C_2(x)$ with $\deg(y_2) = \frac{n}{2}$.

(c) $\deg(y_2) > \frac{n}{2}$**.** Immediately we get $Tr(y_2) < Tr(x)$ and $Tr(z_1) > Tr(z_2)$ by (1). Since $C_w(T) = 4$ then $Tr(z_2) = Tr(y_1)$. Thus

$$Tr(T) = \{Tr(z_1) > Tr(z_2) = Tr(y_1) > Tr(x) > Tr(y_2)\}.$$

Moreover

$$\begin{aligned} Tr(z_2) - Tr(y_2) &= Tr(y_1) - Tr(y_2), \\ n - 2 &= 2(\deg(y_2) - \deg(y_1)). \end{aligned}$$

This follows that

$$\deg(y_1) = \deg(y_2) - \frac{n}{2} + 1. \tag{3}$$

See Figure 2 for an instance of such trees.

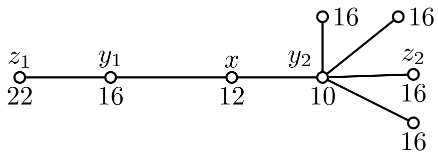

Figure 2. A tree of $C_2(x)$ with $\deg(y_2) > \frac{n}{2}$.

Second, assume that x has a pendant adjacent vertex, say w. We get the following by Lemma 1; for $i = 1, 2$

$$\begin{aligned} Tr(w) - Tr(x) &= n - 2, \\ Tr(z_i) - Tr(w) &= n - 2\deg(y_i), \\ Tr(w) - Tr(y_i) &= 2(\deg(y_i) - 1). \end{aligned}$$

This follows that $Tr(z_1) > Tr(w) > Tr(y_1) > Tr(x)$. Since $C_w(T) = 4$, we have

$$Tr(T) = \{Tr(z_1) > Tr(w) > Tr(y_1) > Tr(x)\}.$$

If $\deg(y_2) \geq \frac{n}{2}$, then Lemma 1 and Lemma 2 imply that $Tr(z_2) \in \{Tr(z_1), Tr(w)\}$. $Tr(z_2) = Tr(z_1)$ follows that $\deg(y_2) = \deg(y_1)$ that is impossible. So $Tr(z_2) = Tr(w)$ and by Lemma 2 we have

$Tr(y_2) = Tr(x)$. Thus $\deg(y_2) = \frac{n}{2}$ and then

$$Tr(T) = \{Tr(z_1) > Tr(w) = Tr(z_2) > Tr(y_1) > Tr(x) = Tr(y_2)\}.$$

Such a tree is shown in Figure 3.

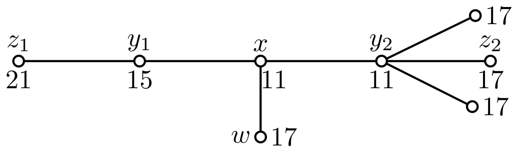

Figure 3. A Tree of $C_2(x)$ with pendant vertices adjacent to the center and $\deg(y_2) = \frac{n}{2}$

Note that if $\deg(y_2) < \frac{n}{2}$ then $Tr(y_2) > Tr(x)$ and by Lemma 2 $Tr(y_2) = Tr(y_1)$ and $Tr(z_2) = Tr(z_1)$. This implies that $\deg(y_2) = \deg(y_1)$. Therefore, if $N(x)$ includes pendant vertices, then other non-pendant vertices in $N(x)$ have the same degree as $\deg(y_1)$. Further

$$Tr(T) = \{Tr(z_i) > Tr(w) > Tr(y_i) > Tr(x)\}.$$

Figure 4 illustrates such a tree with $\deg(y_1) = \deg(y_2)$.

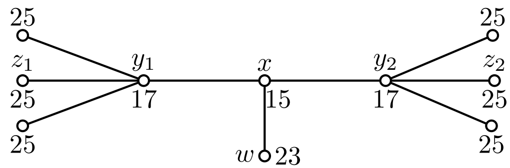

Figure 4. A Tree of $C_2(x)$ with pendant vertices adjacent to the center and $\deg(y_1) = \deg(y_2)$

2. $T \in C_2(x_1, x_2)$.
Note that in this case, T is a double star. Let $C(T) = \{x_1, x_2\}$. x_1 and x_2 have different degrees, because if $\deg(x_1) = \deg(x_2)$ then $Tr(x_1) = Tr(x_2)$ and consequently $C_W(T) = 2$, a contradiction. Without loss of generality suppose that $\deg(x_1) < \deg(x_2)$ and y_i be

adjacent vertex to x_i for $i = 1, 2$. Then

$$\begin{aligned} Tr(x_i) &= 2(n-1) - \deg(x_i), \\ Tr(y_i) &= 3n - 4 - \deg(x_i). \end{aligned}$$

Moreover

$$Tr(y_2) - Tr(x_1) = n - 2 - \deg(x_2) + \deg(x_1) > 0.$$

Therefore, $Tr(T) = \{Tr(y_1) > Tr(y_2) > Tr(x_1) > Tr(x_2)\}$. A double star of Wiener complexity 4 with vertex transmission next to each vertex is illustrated in Figure 5.

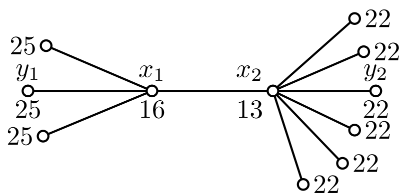

Figure 5. A double star with $C_w = 4$

3. $T \in C_3(x)$.
   Let $P : v_1 - z_1 - y_1 - x - y_2 - z_2 - v_2$ be a diametrical path and $T - x = T_1 \cup T_2$ where T_i is the subtree containing y_i for $i = 1, 2$. Let n_i denotes the order of T_i with $n_1 \leq n_2$. The following relations hold, for $i = 1, 2$,

$$\begin{aligned} Tr(v_i) - Tr(z_i) &= n - 2 > 0, \\ Tr(z_i) - Tr(y_i) &= n - 2\deg(z_i) > 0, \\ Tr(y_1) - Tr(x) &= n - 2n_1 > 0. \end{aligned} \tag{4}$$

   That follows

$$Tr(T) = \{Tr(v_1) > Tr(z_1) > Tr(y_1) > Tr(x)\}.$$

Since $Tr(y_1) - Tr(y_2) = n_2 - n_1 \geq 0$, so $Tr(y_2) \in \{Tr(x), Tr(y_1)\}$. If $Tr(y_2) = Tr(x)$ then

$$n_1 - n_2 = Tr(y_2) - Tr(y_1) = Tr(x) - Tr(y_1) = 2n_1 - n.$$

Thus $n = n_1 + n_2$, a contradiction. Therefore $Tr(y_2) = Tr(y_1)$, that implies $n_1 = n_2$. Lemma 1 follows that

$$Tr(v_2) > Tr(z_2) > Tr(y_2) > Tr(x).$$

Thus $Tr(v_2) = Tr(v_1)$ and $Tr(z_2) = Tr(z_1)$. From (4), $\deg(z_1) = \deg(z_2)$. Thus all adjacent vertices of y_1 and y_2 are of the same degree, this follows $\deg(y_1) = \deg(y_2)$. So

$$\begin{aligned}
&Tr(y_1) = Tr(y_2),\\
&(\deg(y_2) - 1)(k-1) = (\deg(y_1) - 1)(k-1),\\
&\Rightarrow \deg(y_1) = \deg(y_2).
\end{aligned}$$

Hence, vertices on a diametrical path with the same distance from the center have the same degree. If there is a pendant vertex say u, adjacent to x, then $Tr(u) - Tr(x) = n - 2 > 0$. From (4) we get $Tr(y_1) = Tr(x) + n - 2n_1$, consequently $Tr(u) > Tr(y_1)$. Thus $Tr(u) \in \{Tr(v_1), Tr(z_1)\}$.

* If $Tr(u) = Tr(v_1)$, by (4)

$$\begin{aligned}
&Tr(u) - Tr(x) = Tr(v_1) - Tr(x),\\
&n - 2 = Tr(v_1) - Tr(y_1) + n - 2n_1,\\
&n - 2 = Tr(v_1) - Tr(z_1) + n - 2\deg(z_1) + n - 2n_1,\\
&n - 2 = n - 2 + n - 2\deg(z_1) + n - 2n_1.
\end{aligned}$$

  Thus $\deg(z_1) = n - n_1$, a contradiction.

* If $Tr(u) = Tr(z_1)$, we get

$$\begin{aligned} Tr(u) - Tr(x) &= Tr(z_1) - Tr(x), \\ n - 2 &= Tr(z_1) - Tr(y_1) + n - 2n_1, \\ n - 2 &= n - 2\deg(z_1) + n - 2n_1. \end{aligned}$$

So

$$\deg(z_1) = \frac{n}{2} - n_1 + 1. \tag{5}$$

Therefore,

$$Tr(T) = \{Tr(v_i) > Tr(z_i) = Tr(u) > Tr(y_i) > Tr(x)\}.$$

In Figures 6 and 7, two examples of trees $T_1, T_2 \in C_3(x)$ are shown in which $N_{T_1}(x)$ has no pendant vertices while $N_{T_2}(x)$ contains pendant vertices.

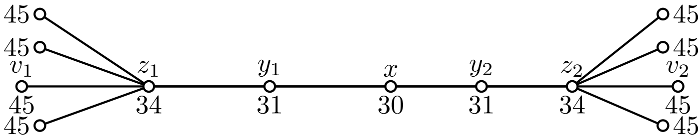

Figure 6. $T_1 \in C_3(x)$ in which $N_{T_1}(x)$ has no pendant vertices

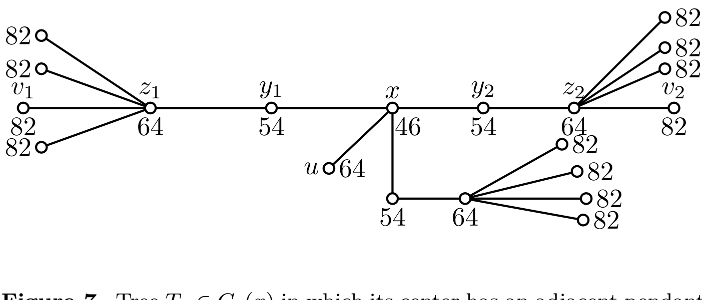

Figure 7. Tree $T_2 \in C_3(x)$ in which its center has an adjacent pendant vertex

If there is a vertex $w \in N(x)$ with $\deg(w) \geq 2$ where $N(w) \setminus \{x\}$ is a set of pendant vertices. Let $p \in N(w) \setminus \{x\}$. The following relations hold.

$$\begin{aligned} Tr(p) - Tr(w) &= n - 2 > 0, \\ Tr(w) - Tr(x) &= n - 2\deg(w), \\ Tr(y_1) - Tr(x) &= n - 2n_1, \\ Tr(y_1) - Tr(w) &= 2(\deg(w) - n_1). \end{aligned}$$

Since $C_W(T) = 4$ and $Tr(v_1) > Tr(z_1) > Tr(y_1) > Tr(x)$, three cases on $Tr(w)$ must be considered:

* First: $Tr(w) = Tr(x)$; that yields $\deg(w) = \frac{n}{2}$. From Lemma 2, $Tr(p) > Tr(y_1)$. Therefore, $Tr(p) = Tr(z_1)$ or $Tr(p) = Tr(v_1)$. If $Tr(p) = Tr(z_1)$ then

$$\begin{aligned} Tr(z_1) - Tr(y_1) &= Tr(p) - Tr(y_1), \\ n - 2\deg(z_1) &= Tr(p) - Tr(x) - n + 2n_1, \\ n - 2\deg(z_1) &= 2(n - \deg(w) - 1) - n + 2n_1. \end{aligned}$$

  Since $\deg(w) = \frac{n}{2}$, so $\deg(z_1) = \frac{n}{2} - n_1 + 1 > n_1 + 1$, that is a contradiction. In the case $Tr(p) = Tr(v_1)$, we have

$$Tr(v_1) - Tr(z_1) = n - 2 = Tr(p) - Tr(w).$$

  That follows $Tr(w) = Tr(z_1)$, a contradiction.

* Second: $Tr(w) = Tr(y_1)$. By Lemma 2, $Tr(p) \neq Tr(z_1)$. Thus $Tr(p) = Tr(v_1)$. It follows that

$$\begin{aligned} n - 2 = Tr(p) - Tr(w) &= Tr(v_1) - Tr(y_1) \\ &> Tr(v_1) - Tr(z_1) = n - 2. \end{aligned}$$

  which is a contradiction.

* Third. $Tr(w) = Tr(z_1)$. We get $Tr(p) = Tr(v_1)$; further

$$\begin{aligned} Tr(z_1) - Tr(y_1) &= Tr(w) - Tr(y_1), \\ n - 2\deg(z_1) &= 2(n_1 - \deg(w)). \end{aligned}$$

Thus

$$\deg(w) = \deg(z_1) + n_1 - \frac{n}{2}. \tag{6}$$

So we have

$$Tr(T) = \{Tr(v_i) = Tr(p) > Tr(z_i) = Tr(w) > Tr(y_i) > Tr(x)\}.$$

Note that by equations (5) and (6), $\Gamma_1(x)$ and $\Gamma_2(x)$ can not both include pendant vertices. For instance, a tree $T \in C_3(x)$ where $\Gamma_2(x)$ includes pendant vertices is shown in Figure 8.

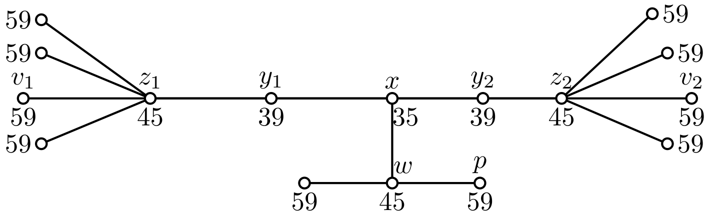

Figure 8. $T \in C_3(x)$ with $\Gamma_2(x)$ including pendant vertices

4. $T \in C_3(x_1, x_2)$.
   Let $T - x_1x_2 = T_1 \cup T_2$ where T_i includes x_i for $i = 1, 2$. Let $P : z_1 - y_1 - x_1 - x_2 - y_2 - z_2$ be a diametrical path. Suppose that $n_1 \leq n_2$. Consider two cases on n_1.

   (a) First. $n_1 < n_2$. Analogously we get

$$Tr(T) = \{Tr(z_1) > Tr(y_1) > Tr(x_1) > Tr(x_2).\}$$

Let $t_1 \in N(x_1) \cap V(T_1)$. Then by Lemma 1,

$$\begin{aligned} Tr(t_1) - Tr(x_1) &= n - 2\deg(t_1) > 0, \\ Tr(t_1) - Tr(y_1) &= 2(\deg(y_1) - \deg(t_1)). \end{aligned}$$

Since $C_w(T) = 4$, $Tr(t_1) = Tr(z_1)$ or $Tr(t_1) = Tr(y_1)$. Note that $Tr(t_1) = Tr(z_1)$ follows that

$$\begin{aligned} Tr(t_1) - Tr(y_1) &= Tr(z_1) - Tr(y_1), \\ 2(\deg(y_1) - \deg(t_1)) &= n - 2. \end{aligned}$$

Thus $\deg(y_1) \geq \frac{n}{2}$, a contradiction. Therefore $Tr(t_1) = Tr(y_1)$ and consequently $\deg(t_1) = \deg(y_1)$. Hence all adjacent vertices to x_1 in T_1 get the same degree. Further, the following relations hold,

$$\begin{aligned} Tr(z_2) &= Tr(y_2) + n - 2, \\ Tr(y_2) &- Tr(x_2) = n - 2\deg(y_2). \end{aligned} \tag{7}$$

We proceed by considering the following conditions on $\deg(y_2)$.

- ❋ $\deg(y_2) > \frac{n}{2}$; This follows $Tr(y_2) < Tr(x_2)$, a contradiction since $C_W(T) = 4$.
- ❋ $\deg(y_2) = \frac{n}{2}$; Thus $Tr(y_2) = Tr(x_2)$. By Lemma 2, we have

  $$Tr(z_2) \in \{Tr(y_1), Tr(z_1)\}$$

  If $Tr(z_2) = Tr(z_1)$ then

  $$\begin{aligned} Tr(z_2) - Tr(y_2) &= Tr(z_1) - Tr(x_2) \\ n - 2 &= Tr(z_1) - Tr(y_1) \\ &+ Tr(y_1) - Tr(x_1) + Tr(x_1) - Tr(x_2) \\ n - 2 &= n - 2\deg(y_1) + n_2 - n_1, \end{aligned}$$

  so $\deg(y_1) = n_2$, a contradiction. Thus $Tr(z_2) = Tr(y_1)$;

It follows $\deg(y_2) = \frac{n}{2}$ and then

$$\deg(y_1) = \frac{n_2 - n_1}{2} + 1. \tag{8}$$

Therefore

$$Tr(T) = \{Tr(z_1) > Tr(z_2) > Tr(x_1) > Tr(x_2)\}.$$

with $Tr(y_1) = Tr(z_2)$ and $Tr(y_2) = Tr(x_2)$. Figure 9 shows a tree of $C_3(x_1, x_2)$ with $\deg(y_2) = \frac{n}{2}$.

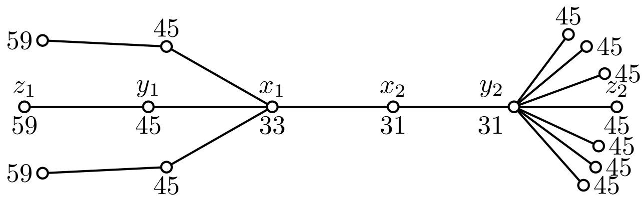

Figure 9. $T \in C_3(x_1, x_2)$ with $\deg(y_2) = \frac{n}{2}$

❊ $\deg(y_2) < \frac{n}{2}$. By (7), $Tr(y_2) > Tr(x_2)$. Hence $Tr(y_2) \in \{Tr(x_1), Tr(y_1)\}$.

If $Tr(y_2) = Tr(x_1)$ then

$$\begin{aligned} Tr(y_2) - Tr(x_2) &= Tr(x_1) - Tr(x_2), \\ n - 2\deg(y_2) &= n_2 - n_1. \end{aligned}$$

Thus $\deg(y_2) = n_1$. On the other hand $Tr(z_2) = Tr(z_1)$ by Lemma 2. Also, we have

$$\begin{aligned} Tr(z_2) - Tr(y_2) &= Tr(z_1) - Tr(x_1), \\ n - 2 &= Tr(z_1) - Tr(y_1) + Tr(y_1) - Tr(x_1), \\ n - 2 &= n - 2 + n - 2\deg(y_1), \end{aligned}$$

that follows $\deg(y_1) = \frac{n}{2}$, a contradiction.
Also if $Tr(y_2) = Tr(y_1)$, we get $Tr(z_2) = Tr(z_1)$. Moreover

$$\begin{aligned} &Tr(y_2) - Tr(x_1) = Tr(y_1) - Tr(x_1), \\ &Tr(y_2) - Tr(x_2) + Tr(x_2) - Tr(x_1) = n - 2\deg(y_1), \\ &n - 2\deg(y_2) + n_1 - n_2 = n - 2\deg(y_1). \end{aligned} \tag{9}$$

This follows

$$\deg(y_2) = \deg(y_1) - \frac{n_2 - n_1}{2}.$$

Thus

$$Tr(T) = \{Tr(z_i) > Tr(y_i) > Tr(x_1) > Tr(x_2)\}.$$

Figure 10 shows an example of such trees.

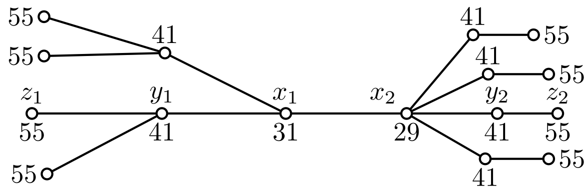

Figure 10. $T \in C_3(x_1, x_2)$ with $\deg(y_2) < \frac{n}{2}$ and $n_1 < n_2$

Recall that all vertices in $N_{T_1}(x_1)$ have the same degree. In the following, we determine the degree of vertices in $N_{T_2}(x_2)$. Let $t_2 \in N_{T_2}(x_2) \setminus \{y_2\}$ where $\deg(t_2) \neq \deg(y_2)$. Consider two cases:

❖ First: $\deg(t_2) > \deg(y_2)$. Then $\deg(y_2) < \frac{n}{2}$ and $Tr(t_2) < Tr(y_2)$. From the above we get $Tr(y_1) = Tr(y_2)$and $Tr(z_1) = Tr(z_2)$. Thus $Tr(t_2) \in \{Tr(x_1), Tr(x_2)\}$. Then we verify the following cases.

➤ If $Tr(t_2) = Tr(x_1)$.

$$\begin{aligned} &Tr(t_2) - Tr(x_2) = Tr(x_1) - Tr(x_2) \\ &n - 2\deg(t_2) = n_2 - n_1 \\ &\Rightarrow \deg(t_2) = n_1 \end{aligned}$$

Let s_2 be an adjacent vertex to t_2 of degree 1. By Lemma 1 and Lemma 2, $Tr(s_2) = Tr(z_1)$. Therefore

$$\begin{aligned} &Tr(s_2) - Tr(t_2) = Tr(z_1) - Tr(t_2), \\ &n - 2 = Tr(z_1) - Tr(x_1) = n - 2 + n - 2\deg(y_1). \end{aligned}$$

Thus $\deg(y_1) = \frac{n}{2}$, a contradiction.

➤ If $Tr(t_2) = Tr(x_2)$, then

$$0 = Tr(t_2) - Tr(x_2) = n - 2\deg(t_2),$$

Thus $\deg(t_2) = \frac{n}{2}$. From (9) and (8), we get $\deg(y_2) = 1$, a contradiction.

❖ Second: $\deg(t_2) < \deg(y_2)$. Then $Tr(t_2) > Tr(y_2)$. Two cases again must be verified.

✦ $\deg(y_2) < \frac{n}{2}$. From (9), $Tr(z_2) = Tr(z_1) > Tr(y_2)$ and then $Tr(t_2) = Tr(z_2)$ that follows

$$\begin{aligned} Tr(t_2) - Tr(y_2) &= Tr(z_2) - Tr(y_2), \\ Tr(t_2) - Tr(x_2) &+ Tr(x_2) - Tr(y_2) = n - 2, \\ n - 2\deg(t_2) &- n + 2\deg(y_2) = n - 2. \end{aligned}$$

Thus $\deg(y_2) = \deg(t_2) + \frac{n}{2} - 1 > \frac{n}{2}$, a contradiction.

✦ $\deg(y_2) = \frac{n}{2}$. Then $\deg(t_2) < \frac{n}{2}$. From (9) and (8) we get

$$\begin{aligned}\deg(y_1) - \deg(t_2) &= \frac{n_2 - n_1}{2},\\ \deg(y_1) &= \frac{n_2 - n_1}{2} + 1.\end{aligned}$$

This implies that $\deg(t_2) = 1$. Therefore, all adjacent vertices to x_2 in T_2 have the same degree as $\deg(y_2)$ when $\deg(y_2) < \frac{n}{2}$, or they are pendant vertices when $\deg(y_2) = \frac{n}{2}$.

Therefore

$$Tr(T) = \{Tr(z_1) > Tr(y_1) > Tr(x_1) > Tr(x_2)\}.$$

Note that $Tr(y_1) = Tr(z_2) = Tr(t_2)$ and $Tr(x_2) = Tr(y_2)$. Figure 11 illustrates an example of such trees.

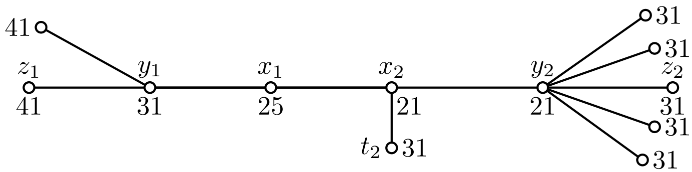

Figure 11. $T \in C_3(x_1, x_2)$ with $\deg(y_2) = \frac{n}{2}$ and $t_2 \in N(x_2)$

(b) Second: $n_1 = n_2$. By Lemma1, $Tr(x_1) = Tr(x_2)$. Assume that $\deg(y_1) \leq \deg(y_2)$. The following holds; for $i = 1, 2$

$$\begin{aligned}&Tr(z_i) - Tr(y_i) = n - 2,\\ &Tr(y_1) - Tr(y_2) = 2(\deg(y_2) - \deg(y_1)),\\ &Tr(z_2) - Tr(z_1) = Tr(y_2) - Tr(y_1).\end{aligned}$$

Therefore if $\deg(y_1) < \deg(y_2)$,

$$Tr(z_1) > Tr(z_2) > Tr(y_2) > Tr(x_2) = Tr(x_1).$$

Since $Tr(y_1) > Tr(y_2)$ and $C_w(T) = 4$, we have $Tr(y_1) = Tr(z_2)$. This yields that

$$\begin{aligned} Tr(z_2) - Tr(y_2) &= Tr(y_1) - Tr(y_2), \\ n - 2 &= 2(\deg(y_2) - \deg(y_1)). \end{aligned}$$

Consequently $\deg(y_2) = \deg(y_1) + \frac{n}{2} - 1$, that is a contradiction. Hence $\deg(y_1) = \deg(y_2)$. This means all adjacent vertices to the center of T, placed on the diametrical path, get the same degree. So

$$Tr(z_1) = Tr(z_2) > Tr(y_1) = Tr(y_2) > Tr(x_1) = Tr(x_2).$$

Since $C_W(T) = 4$, there is at least a pendant vertex adjacent to a central vertex. Suppose that w_i be such a vertex adjacent to x_i. We have

$$\begin{aligned} Tr(w_i) - Tr(y_i) &= Tr(w_i) - Tr(x_i) + Tr(x_i) - Tr(y_i), \\ &= n - 2 + 2\deg(y_i) - n, \\ &\Rightarrow Tr(w_i) > Tr(y_i). \end{aligned}$$

Further

$$\begin{aligned} Tr(z_i) - Tr(w_i) &= Tr(z_i) - Tr(y_i) + Tr(y_i) - Tr(w_i), \\ &= n - 2 + 2(1 - \deg(y_i)), \\ &\Rightarrow Tr(w_i) < Tr(z_i). \end{aligned}$$

Then

$$Tr(T) = \{Tr(z_i) > Tr(w_i) > Tr(y_i) > Tr(x_i)\}.$$

Trees in Figure 12 are two examples of such trees.
Note that if $\mathrm{rad}(T) = 4$, by Theorem 1, $|C(T)| = 2$. So it is sufficient to verify just the following case.

5. $T \in C_4(x_1, x_2)$.
   Let $P : v_1 - z_1 - y_1 - x_1 - x_2 - y_2 - z_2 - v_2$ be a diametrical path and

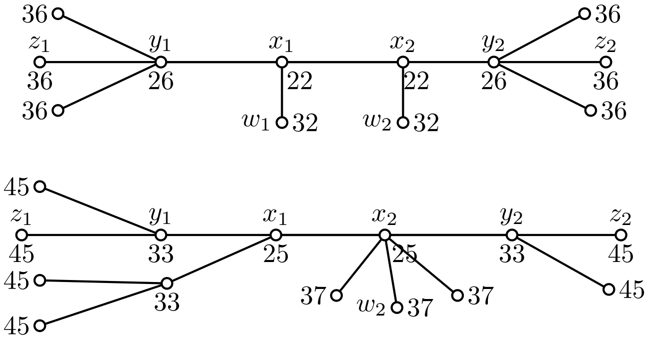

Figure 12. Two trees of $C_3(x_1, x_2)$ with $n_1 = n_2$ and (at least) a central vertex have adjacent pendant vertex

$T - x_1x_2 = T_1 \cup T_2$ with $n_1 = n(T_1) \leq n(T_2) = n_2$. From Lemma 1 and $C_w(T) = 4$, we have

$$Tr(T) = \{Tr(v_1) > Tr(z_1) > Tr(y_1) > Tr(x_1)\}\,.$$

Further, $Tr(x_1) - Tr(x_2) = n_2 - n_1 \geq 0$. Since $C_w(T) = 4$ then $n_1 = n_2$ and consequently

$$Tr(x_1) = Tr(x_2), \quad Tr(y_1) = Tr(y_2),$$
$$Tr(z_1) = Tr(z_2), \quad Tr(v_1) = Tr(v_2).$$

Figure 13 shows a tree of $C_4(x_1, x_2)$ in which central vertices does not have adjacent pendant.

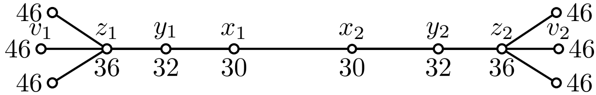

Figure 13. A tree of $C_4(x_1, x_2)$ in which $N(x_i)$ has no pendant vertex for $i = 1, 2$.

Also, we show that vertices on a diametrical path with the same distance from the center have the same degree.

$$\begin{aligned} & Tr(z_1) - Tr(y_1) = Tr(z_2) - Tr(y_2), \\ \Rightarrow \quad & n - 2\deg(z_1) = n - 2\deg(z_2), \\ \Rightarrow \quad & \deg(z_1) = \deg(z_2). \end{aligned}$$

Note that if there is a pendant vertex adjacent to y_i, say u_i, then

$$Tr(u_i) - Tr(y_i) = n - 2 \Rightarrow Tr(u_i) = Tr(z_i) \text{ or } Tr(u_i) = Tr(v_i).$$

By Lemma 2, $Tr(u_i) \neq Tr(z_i)$, and then $Tr(u_i) = Tr(v_i)$. Thus

$$\begin{aligned} Tr(u_i) - Tr(y_i) &= (Tr(v_i) - Tr(z_i)) + (Tr(z_i) - Tr(y_i)), \\ n - 2 &= n - 2 + n - 2\deg(z_i), \\ \Rightarrow \deg(z_i) &= \frac{n}{2}. \end{aligned}$$

That is a contradiction. So all vertices in $\Gamma_1(y_i) \cap \Gamma_2(x_i)$ have the same degree as $\deg(z_i)$, for $i = 1, 2$. Moreover

$$\begin{aligned} & Tr(y_1) - Tr(x_1) = Tr(y_2) - Tr(x_2), \\ & n - 2((\deg(y_1) - 1)\deg(z_1) + 1) = \\ & n - 2((\deg(y_2) - 1)\deg(z_2) + 1), \\ & \Rightarrow \deg(y_1) = \deg(y_2). \end{aligned}$$

Now we investigate the degree of vertices in $N(x_i)$ placed out of the diametrical path. Two cases need to be verified.

❉ First: $N(x_i)$ contains a pendant vertex. Let $s_i \in N(x_i)$ be a pendant vertex.
Then by Lemma 2, we have $Tr(s_i) \in \{Tr(z_i), Tr(v_i)\}$.
If $Tr(s_i) = Tr(z_i)$ then

$$\begin{aligned} Tr(s_i) - Tr(x_i) &= (Tr(z_i) - Tr(y_i)) + (Tr(y_i) - Tr(x_i), \\ n - 2 &= (n - 2\deg(z_i)) + (n - 2n_{y_i}), \end{aligned}$$

$$\deg(z_i) = \frac{n}{2} - n_{y_i} + 1. \tag{10}$$

where $n_{y_i} = (\deg(y_i) - 1)\deg(z_i) + 1$. This follows that

$$\deg(z_i)\deg(y_i) = \frac{n}{2}. \tag{11}$$

If $Tr(s_i) = Tr(v_i)$ then we get

$$\begin{aligned}
&(Tr(s_i) - Tr(x_i)) + (Tr(x_i) - Tr(y_i)) = \qquad (12)\\
&(Tr(v_i) - Tr(z_i)) + (Tr(z_i) - Tr(y_i)),\\
&(n-2) + (2n_{y_i} - n) = (n-2) + (n - 2\deg(z_i)),\\
&2((\deg(y_i) - 1)\deg(z_i) + 1) - n = n - 2\deg(z_i),\\
&\Rightarrow \deg(z_i)\deg(y_i) = n - 1.
\end{aligned}$$

Which is impossible, since $\deg(z_i)\deg(y_i) \leq \frac{n}{2}$.Thus

$$Tr(T) = \{Tr(v_i) > Tr(z_i) = Tr(s_i) > Tr(y_i) > Tr(x_i)\}.$$

See Figure 14 of such a tree.

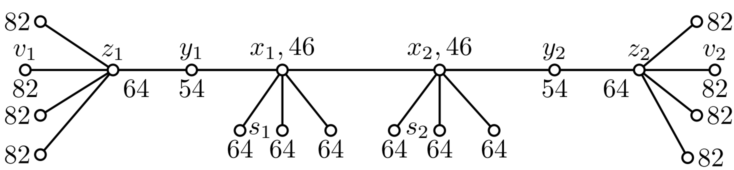

Figure 14. A tree of $C_4(x_1, x_2)$ in which $N(x_i)$ has (at least) a pendant vertex for $i = 1, 2$.

❄ Second: $N(x_i)$ contains a vertex, say p_i with $\deg(p_i) \geq 2$. Let q_i be a pendant vertex in $N(p_i) \setminus \{x_i\}$. We have

$$Tr(q_i) > Tr(p_i) > Tr(x_i).$$

Considering

$$Tr(T) = \{Tr(v_i) > Tr(z_i) > Tr(y_i) > Tr(x_i)\},$$

We infer that

$$Tr(p_i) \in \{Tr(y_i), Tr(z_i)\}$$

and

$$Tr(q_i) \in \{Tr(z_i), Tr(v_i)\}.$$

Thus, we must consider the following cases:

⋆ If $Tr(p_i) = Tr(y_i)$ and $Tr(q_i) = Tr(z_i)$, then

$$\begin{aligned} Tr(q_i) - Tr(p_i) &= Tr(z_i) - Tr(y_i), \\ n - 2 &= n - 2\deg(z_i) \Rightarrow \deg(z_i) = 1. \end{aligned}$$

That is a contradiction.

⋆ If $Tr(p_i) = Tr(y_i)$ and $Tr(q_i) = Tr(v_i)$, then

$$\begin{aligned} Tr(q_i) - Tr(p_i) &= (Tr(v_i) - Tr(z_i)) \\ &\quad +(Tr(z_i) - Tr(y_i)), \\ \Rightarrow n - 2 &= n - 2 + n - 2\deg(z_i), \\ &\Rightarrow \deg(z_i) = \frac{n}{2} \end{aligned}$$

a contradiction again.

⋆ If $Tr(p_i) = Tr(z_i)$ and $Tr(q_i) = Tr(v_i)$, then

$$\begin{aligned} Tr(p_i) - Tr(x_i) &= (Tr(z_i) - Tr(y_i)) \\ &\quad +(Tr(y_i) - Tr(x_i)), \\ n - 2\deg(p_i) &= (n - 2\deg(z_i)) + (n - 2n_{y_i}). \end{aligned}$$

This also follows that

$$\deg(z_i) = \deg(p_i) + \frac{n}{2} - n_{y_i}. \tag{13}$$

Consequently

$$\deg(p_i) = \deg(y_i)\deg(z_i) - \frac{n}{2} - 1.$$

$$Tr(T) = \{Tr(v_i) = Tr(q_i) > Tr(z_i) = Tr(p_i) > Tr(y_i) > Tr(x_i)\}.$$

In Figure 15, such a tree is shown.

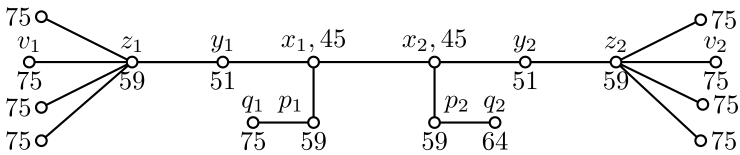

Figure 15. A tree of $C_4(x_1, x_2)$ in which $\deg(p_i) \geq 2$ where $p_i \in N(x_i)$, for $i = 1, 2$.

Let for $i = 1, 2$, s_i and t_i denote the number vertices in $N(x_i)$ with degree 1 and degree at least 2 which are placed out of any diametrical path respectively. We show that the central vertices x_1 and x_2 have the same number of such adjacent vertices, i.e. $s_1 = s_2$ and $t_1 = t_2$. Notice that by Eq(10) and Eq(13), at least one of s_i and t_i is zero. Let k_i denotes the number of subtrees isomorphic to T_i connecting to x_i, in fact $k_i = \deg(x_i) - (s_i + t_i + 1)$. We claim that if x_1 or x_2 have a pendant adjacent vertex, then $k_i = 1$ for $i = 1, 2$. Without loss of generality, assume that $s_1 \geq 1$. By Eq(10), $\deg(z_i) + n_{y_i} - 1 = \frac{n}{2}$ and with the fact $k_i n_{y_i} + (\deg(x_i) - k_i) = \frac{n}{2}$ for $i = 1, 2$, we get

$$k_i n_{y_i} + (\deg(x_i) - k_i) = \deg(z_i) + n_{y_i} - 1,$$
$$(k_i - 1)n_{y_i} + (\deg(x_i) - k_i) + 1 = \deg(z_i).$$

Note that $\deg(z_1) = \deg(z_2)$. Further $\deg(z_i) \leq n_{y_i}$ and $\deg(x_i) > k_i$ that follow $k_i = 1$ for $i = 1, 2$ and then $s_1 = s_2$. Now suppose that $t_i \geq 1$. Then we have $s_i = 0$ for $i = 1, 2$ and $k_i n_{y_i} + t_i \deg(p_i) + 1 = \frac{n}{2}$.

By Eq(13), $\frac{n}{2} = \deg(z_i) - \deg(p_i) + n_{y_i}$. This follows

$$(k_i - 1)n_{y_i} + (t_i + 1)\deg(p_i) + 1 = \deg(z_i).$$

Further $\deg(z_i) \le n_{y_i}$ implies that $k_i = 1$ and then

$$\deg(z_i) = (t_i + 1)\deg(p_i) + 1.$$

Finally, $\deg(z_1) = \deg(z_2)$ follows that $t_1 = t_2$. We are done. ■

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