MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **95** (2026) 919–934
ISSN: 0340–6253
doi: 10.46793/match.95-3.17425

Extremal Elliptic Sombor Index in Tricyclic Graphs

Jie Li[a], **Xuewu Zuo**[b,*]

[a] *Faculty of Mathematics and Statistics, SuZhou University, SuZhou, AnHui, 234000, China*

[b] *General Education Department, Anhui Xinhua University, Hefei, 230088, China*

ljie202405@163.com, xinhuazxw@163.com

(Received July 15, 2025)

Abstract

For a simple graph G with vertex degrees $d_G(\cdot)$, the elliptic Sombor index (ESO) is defined as $\mathrm{ESO}(G) = \sum_{xy\in E(G)}(d_G(x) + d_G(y))\sqrt{d_G^2(x) + d_G^2(y)}$. We completely solve the extremal problem for this index among connected tricyclic graphs of order n, characterizing both the unique graph achieving the minimum ESO value and all graphs attaining the maximum value. These results provide sharp bounds for the elliptic Sombor index in this important graph class and reveal new extremal structural properties not observed in other degree-based topological indices.

1 Introduction

In this article, we consider only simple, connected, and undirected graphs, denoted by G. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The cardinalities of these sets are $n = |V(G)|$

*Corresponding author.

and $m = |E(G)|$. For any vertex $u \in V(G)$, the set of its neighbors is denoted by $N_G(u)$, and its degree is $d_G(u) = |N_G(u)|$. A vertex u is called a *pendant vertex* if $d_G(u) = 1$. The cycle, star, and path graphs of order n are denoted by C_n, S_n, and P_n, respectively. The graph obtained by removing a vertex $x \in V(G)$ along with all its incident edges is denoted by $G - x$, while the graph obtained by removing an edge $xy \in E(G)$ is denoted by $G - xy$. Similarly, adding an edge $xy \notin E(G)$ between two existing vertices $x, y \in V(G)$ is denoted by $G + xy$. A connected graph G is called a *tree*, *unicyclic*, *bicyclic*, or *tricyclic* if $m = n - 1$, $m = n$, $m = n+1$, or $m = n+2$, respectively. Let Γ_n denote the class of connected tricyclic graphs of order n and size $n + 2$. A *brace* of a tricyclic graph is obtained by deleting all pendant vertices from the graph. The set of all such braces is denoted by Γ_n^0 and is illustrated in Figure 6. Additionally, Γ_n^1 and Γ_n^2 denote the classes of tricyclic graphs shown in Figures 5 and 7, respectively.

2 Preliminaries

In graph theory, a *graph invariant* is a property that remains unchanged under graph isomorphisms. When such invariants take numerical values, they are commonly referred to as *topological indices* in chemical graph theory. Many topological indices are defined as simple functions of the degree sequence of a (molecular) graph.

Topological indices are numerical descriptors that capture the structural features of a graph and are invariant under isomorphism. These indices can often be derived analytically from the molecular structure and are widely used in the study of chemical compounds.

In [2], the *elliptic Sombor index* (ESO) of a graph G is defined as

$$ESO(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y)) \sqrt{d_G^2(x) + d_G^2(y)}.$$

The study of extremal values of topological indices is a central topic in mathematical chemistry and has attracted significant attention. In particular, research concerning the ESO index has progressed rapidly. In [1], the extremal values of ESO were investigated for chemical graphs. Rada et al. [5] examined the elliptic Sombor index for benzenoid systems. Tang et al. [6] characterized graphs that attain the maximum ESO among trees and unicyclic graphs. Qi et al. [4] extended these results to bicyclic graphs.

Analyzing the variation of such indices under structural transformations plays a crucial role in understanding the chemical properties of molecular graphs. As demonstrated in [3,7], analogous results for tricyclic graphs are already established for the ordinary Sombor index. In this paper, we focus on tricyclic graphs and apply edge-swapping techniques to investigate how the ESO index changes under different transformations. Based on these observations, we determine the extremal values of the elliptic Sombor index.

3 Some transformations

In this section, we investigate the behavior of the elliptic Sombor index under specific graph transformations. These transformations involve either swapping edges between vertices or modifying the graph by adding or removing edges in a way that alters the degrees of the affected vertices. Our focus is on such operations within the class of tricyclic graphs, observing how they influence the value of the ESO index—increasing or decreasing it accordingly.

Transformation 1. Let ab be an edge in a simple graph G such that $d_G(b) \geq 2$. Suppose that the set $\{b, x_1, x_2, \ldots, x_t\}$ forms the neighborhood of vertex a, where each x_i $(1 \leq i \leq t)$ is a pendant vertex (i.e., $d_G(x_i) = 1$). Construct a new graph H by removing the edges $\{ax_1, ax_2, \ldots, ax_t\}$ from G and adding the edges $\{bx_1, bx_2, \ldots, bx_t\}$. We refer to H as the graph obtained from G by Transformation 1, illustrated in Figure 1.

We now demonstrate that Transformation 1 results in an increase in the elliptic Sombor index, as shown in the following lemma.

Lemma 1. *Let G and H be the graphs defined in Transformation 1. Then,*

$$ESO(G) < ESO(H).$$

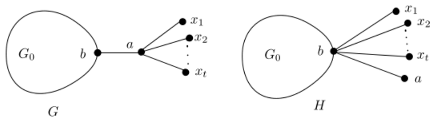

Figure 1. Graphs G and H corresponding to Transformation 1.

Proof. Note that under Transformation 1, the degree of vertex b increases, i.e., $d_G(b) < d_H(b)$, while the sum $d(a) + d(b)$ remains unchanged. Applying the definition of the elliptic Sombor index, we compute:

$$\begin{aligned} ESO(H) - ESO(G) > {} & (d_H(a) + d_H(b))\sqrt{d_H^2(a) + d_H^2(b)} \\ & + \sum_{i=1}^{t} \left((d_H(b) + d_H(x_i))\sqrt{d_H^2(b) + d_H^2(x_i)} \right) \\ & - (d_G(a) + d_G(b))\sqrt{d_G^2(a) + d_G^2(b)} \\ & - \sum_{i=1}^{t} \left((d_G(a) + d_G(x_i))\sqrt{d_G^2(a) + d_G^2(x_i)} \right). \end{aligned}$$

Substituting the degrees: since $d_G(x_i) = d_H(x_i) = 1$ and $d_H(a) = 1$, $d_G(a) = t + 1$, we obtain:

$$\begin{aligned} & ESO(H) - ESO(G) = (1 + d_H(b))\sqrt{1 + d_H^2(b)} \\ & \quad + t(1 + d_H(b))\sqrt{1 + d_H^2(b)} \\ & \quad - ((t+1) + d_G(b))\sqrt{(t+1)^2 + d_G^2(b)} \\ & \quad - t(t+2)\sqrt{(t+1)^2 + 1} \\ & = (t+1)(t+1+d_G(b))\sqrt{1 + (t + d_G(b))^2} \\ & \quad - ((t+1) + d_G(b))\sqrt{(t+1)^2 + d_G^2(b)} \\ & \quad - t(t+2)\sqrt{(t+1)^2 + 1} \\ & = (t+1+d_G(b)) \left(\sqrt{1 + (t + d_G(b))^2} - \sqrt{(t+1)^2 + d_G^2(b)} \right) \\ & \quad + t\left((t+1+d_G(b))\sqrt{1 + (t + d_G(b))^2} - (t+2)\sqrt{(t+1)^2 + 1} \right). \end{aligned}$$

Since $d_G(b) \geq 2$, both terms in the expression above are positive, and hence:

$$ESO(H) - ESO(G) > 0.$$

This completes the proof. ■

Transformation 2. Let G be a connected graph and $a, b \in V(G)$ such that $d_G(a) \geq 3$ and $d_G(b) \geq 3$. Suppose that $P_c = a = x_1, x_2, \ldots, x_c = b$ is a nontrivial path of length c in G connecting vertices a and b. Define the vertex w by fusing a and b, i.e., $w = a + b$. Construct a new graph

$$H = G - \{x_1x_2, x_2x_3, \ldots, x_{c-1}x_c\} + \{wx_1, wx_2, \ldots, wx_{c-1}\}.$$

We say that H is obtained from G by Transformation 2, as illustrated in Figure 2.

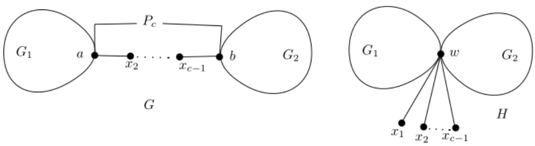

Figure 2. Graphs G and H corresponding to Transformation 2.

Lemma 2. *Let G and H be the graphs described in Transformation 2. Then,*

$$ESO(H) > ESO(G).$$

Proof. Let $d_G(a) = u \geq 3$ and $d_G(b) = v \geq 3$. By fusing a and b to form w, we have

$$d_H(w) = u + v + c - 1,$$

where $c \geq 2$ is the length of the path P_c.

Case 1: If $c = 2$, then by the definition of the elliptic Sombor index,

$$\begin{aligned} ESO(H) - ESO(G) > &(d_H(w) + d_H(x_1))\sqrt{d_H^2(w) + d_H^2(x_1)} \\ &- (d_G(a) + d_G(b))\sqrt{d_G^2(a) + d_G^2(b)} \end{aligned}$$

$$= (u+v+2)\sqrt{(u+v+1)^2+1}$$
$$-(u+v)\sqrt{(u+1)^2+(v+1)^2} > 0.$$

Case 2: If $c \geq 3$, then

$$\begin{aligned}
ESO(H) - ESO(G) > &\sum_{i=1}^{c-1}(d_H(w)+d_H(x_i))\sqrt{d_H^2(w)+d_H^2(x_i)} \\
&- (d_G(a)+d_G(x_2))\sqrt{d_G^2(a)+d_G^2(x_2)} \\
&- (d_G(b)+d_G(x_{c-1}))\sqrt{d_G^2(b)+d_G^2(x_{c-1})} \\
&- 8(c-3)\sqrt{2} \\
&= (c-1)(u+v+c)\sqrt{(u+v+c-1)^2+1} \\
&- (u+3)\sqrt{(u+1)^2+4} \\
&- (v+3)\sqrt{(v+1)^2+4} - 8(c-3)\sqrt{2}.
\end{aligned}$$

Rewrite this expression as the sum of three terms:

$$ESO(H) - ESO(G) > X + Y + Z,$$

where

$$X = u(c-1)\sqrt{(u+v+c-1)^2+1} - (u+3)\sqrt{(u+1)^2+4},$$

$$Y = v(c-1)\sqrt{(u+v+c-1)^2+1} - (v+3)\sqrt{(v+1)^2+4},$$

$$Z = c(c-1)\sqrt{(u+v+c-1)^2+1} - 8(c-3)\sqrt{2}.$$

Since $c, u, v \geq 3$, note that

$$c - 1 \geq 1 + \frac{3}{u} \implies u(c-1) \geq u+3.$$

Moreover,

$$\sqrt{(u+v+c-1)^2+1} > \sqrt{(u+1)^2+4}.$$

Therefore, $X > 0$. By a similar argument, $Y > 0$.

To analyze Z, observe that

$$\begin{aligned} Z &= c(c-1)\sqrt{(u+v+c-1)^2+1} - 8(c-3)\sqrt{2} \\ &\geq c(c-1)\sqrt{(5+c)^2+1} - 8(c-3)\sqrt{2} \quad (\text{since } u, v \geq 3). \end{aligned}$$

Using the inequality

$$c\sqrt{(5+c)^2+1} \geq 3\sqrt{65} > 8\sqrt{2},$$

we get

$$c(c-1)\sqrt{(5+c)^2+1} > 8\sqrt{2}(c-1),$$

which implies $Z > 0$.

Hence,

$$ESO(H) - ESO(G) > X + Y + Z > 0,$$

which completes the proof. ■

Before presenting Transformation 3, we provide a brief explanation of its purpose. Consider a graph G containing an acyclic subgraph S (a tree) attached to a vertex x_1. This transformation removes the complex structure of S and replaces it with a simpler path connecting x_1 to another neighbor b, while preserving key connections. The aim is to analyze how this simplification affects the elliptic Sombor index, which will be shown to decrease under this transformation.

Transformation 3. Let S be an acyclic subgraph of G with $|V(S)| = t$ attached to vertex x_1 in G, where $d_G(x_1) \geq 4$. Let a and b be two neighbors of x_1 other than those in S. Define

$$H = G - \{S - x_1\} + \{x_1x_2, x_2x_3, \ldots, x_tb\}.$$

We say that H is obtained from G by Transformation 3, as illustrated in Figure 3.

Lemma 3. *Suppose G and H are graphs described in Transformation 3. Then,*

$$ESO(G) > ESO(H).$$

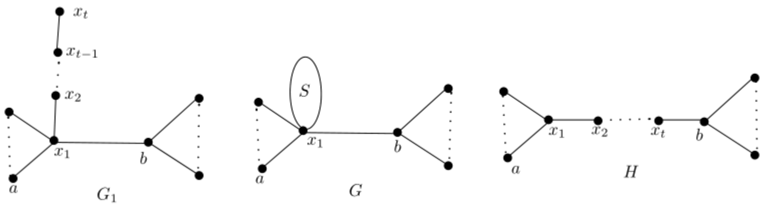

Figure 3. Graphs G_1, G, and H related to Transformation 3.

Proof. By applying Lemma 1, we have

$$ESO(G) > ESO(G_1).$$

Therefore, to complete the proof, it suffices to show that

$$ESO(G_1) > ESO(H).$$

Using the definition of the elliptic Sombor index, we write

$$\begin{aligned} ESO(G_1) - ESO(H) > &(d_{G_1}(x_1) + d_{G_1}(x_2))\sqrt{d_{G_1}^2(x_1) + d_{G_1}^2(x_2)} \\ &+ (d_{G_1}(x_1) + d_{G_1}(b))\sqrt{d_{G_1}^2(x_1) + d_{G_1}^2(b)} \\ &+ (d_{G_1}(x_{t-1}) + d_{G_1}(x_t))\sqrt{d_{G_1}^2(x_{t-1}) + d_{G_1}^2(x_t)} \\ &- (d_H(x_1) + d_H(x_2))\sqrt{d_H^2(x_1) + d_H^2(x_2)} \\ &- (d_H(x_{t-1}) + d_H(x_t))\sqrt{d_H^2(x_{t-1}) + d_H^2(x_t)} \\ &- (d_H(x_t) + d_H(b))\sqrt{d_H^2(x_t) + d_H^2(b)}. \end{aligned}$$

Substituting the degrees, we get

$$\begin{aligned} &(d_{G_1}(x_1) + 2)\sqrt{d_{G_1}^2(x_1) + 4} + (d_{G_1}(x_1) + d_{G_1}(b))\sqrt{d_{G_1}^2(x_1) + d_{G_1}^2(b)} \\ &+ 3\sqrt{5} \quad - (d_H(x_1) + 2)\sqrt{d_H^2(x_1) + 4} - 4\sqrt{8} - (2 + d_H(b))\sqrt{4 + d_H^2(b)}. \end{aligned}$$

Note that

$$d_H(x_1) = d_{G_1}(x_1) - 1, \quad d_H(b) = d_{G_1}(b),$$

so the expression becomes

$$\begin{aligned}
&(d_{G_1}(x_1) + 2)\sqrt{d_{G_1}^2(x_1) + 4} + (d_{G_1}(x_1) \\
&+ d_{G_1}(b))\sqrt{d_{G_1}^2(x_1) + d_{G_1}^2(b)} + 3\sqrt{5} \\
&- (d_{G_1}(x_1) + 1)\sqrt{(d_{G_1}(x_1) - 1)^2 + 4} \\
&- 4\sqrt{8} - (2 + d_{G_1}(b))\sqrt{4 + d_{G_1}^2(b)}.
\end{aligned}$$

Rearranging terms, we have

$$\begin{aligned}
&ESO(G_1) - ESO(H) \\
&= d_{G_1}(x_1)\left(\sqrt{d_{G_1}^2(x_1) + 4} - \sqrt{(d_{G_1}(x_1) - 1)^2 + 4}\right) \\
&\quad + \left((d_{G_1}(x_1) + d_{G_1}(b))\sqrt{d_{G_1}^2(x_1) + d_{G_1}^2(b)} - (2 + d_{G_1}(b))\sqrt{4 + d_{G_1}^2(b)}\right) \\
&\quad + 2\sqrt{d_{G_1}^2(x_1) + 4} - \sqrt{(d_{G_1}(x_1) - 1)^2 + 4} + 3\sqrt{5} - 4\sqrt{8}.
\end{aligned}$$

Since $d_{G_1}(x_1) \geq 4$, it can be shown that this expression is positive. In particular,

$$2\sqrt{d_{G_1}^2(x_1) + 4} - \sqrt{(d_{G_1}(x_1) - 1)^2 + 4} + 3\sqrt{5} - 4\sqrt{8} > 0,$$

which concludes the proof. ■

The goal of this transformation is to reduce the elliptic Sombor index by relocating pendant vertices from a vertex with a smaller neighborhood to another vertex with a larger neighborhood, thus optimizing the graph's structure.

Transformation 4. Let G_0 be a graph and a, b be two vertices in G_0 with degrees $d_{G_0}(a) = u$ and $d_{G_0}(b) = v$ such that $N_{G_0}(b) \subseteq N_{G_0}(a)$. Let G be the graph obtained by affixing star graphs S_{s+1} and S_{t+1} to vertices a and b of G_0, respectively.

If H is the graph obtained by removing t pendant vertices at b in G and reattaching them to vertex a of G, then H is said to be derived from G by transformation 4, as shown in Figure 4.

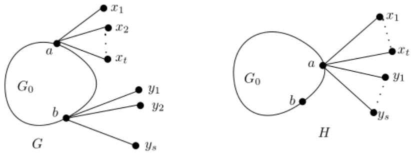

Figure 4. Graphs G and H involved in transformation 4

Lemma 4. *Suppose G and H are the graphs described in transformation 4. Then,*

$$ESO(G) > ESO(H).$$

Proof. For convenience, assume that $d_{G_0}(a) = u$, $d_{G_0}(b) = v$, and $N_{G_0}(b) \subseteq N_{G_0}(a)$ with $u > v$. Using the definition of the elliptic Sombor index, we have:

$$\begin{aligned}
ESO(H) - ESO(G) = &\sum_{i=1}^{t} \Big((d_H(a) + d_H(x_i))\sqrt{d_H^2(a) + d_H^2(x_i)} \\
&- (d_G(a) + d_G(x_i))\sqrt{d_G^2(a) + d_G^2(x_i)} \Big) \\
&+ \sum_{i=1}^{s} \Big((d_H(b) + d_H(y_i))\sqrt{d_H^2(b) + d_H^2(y_i)} \\
&- (d_G(b) + d_G(y_i))\sqrt{d_G^2(b) + d_G^2(y_i)} \Big) \\
&+ \sum_{w \in N_{G_0}(b)} \Big((d_H(a) + d_H(w))\sqrt{d_H^2(a) + d_H^2(w)} \\
&+ (d_H(b) + d_H(w))\sqrt{d_H^2(b) + d_H^2(w)} \Big) \\
&- \sum_{w \in N_{G_0}(b)} \Big((d_G(a) + d_G(w))\sqrt{d_G^2(a) + d_G^2(w)} \\
&+ (d_G(b) + d_G(w))\sqrt{d_G^2(b) + d_G^2(w)} \Big).
\end{aligned}$$

By substituting the degrees after the transformation, this expression becomes

$$\begin{aligned}
ESO(H) - ESO(G) = t\Big(&(u+s+t+1)\sqrt{(u+s+t)^2+1} \\
&- (u+t+1)\sqrt{(u+t)^2+1}\Big) \\
&+ s\Big((u+s+t+1)\sqrt{(u+s+t)^2+1} \\
&- (s+v+1)\sqrt{(s+v)^2+1}\Big) \\
&+ \sum_{w\in N_{G_0}(b)} \Big[(u+s+t+d_H(w))\sqrt{(u+s+t)^2+d_H^2(w)} \\
&+ (v+d_H(w))\sqrt{v^2+d_H^2(w)}\Big] \\
&- \sum_{w\in N_{G_0}(b)} \Big[(u+t+d_G(w))\sqrt{(u+t)^2+d_G^2(w)} \\
&+ (v+s+d_G(w))\sqrt{(v+s)^2+d_G^2(w)}\Big].
\end{aligned}$$

Using the monotonicity properties of the function $f(x) = (x+c)\sqrt{x^2+d}$ for appropriate constants c, d, we find that

$$ESO(H) - ESO(G) > 0,$$

which completes the proof. ∎

4 Main results

In this section, we characterize the extremal tricyclic graphs with respect to the elliptic Sombor index. To achieve this goal, we utilize the graph transformations introduced earlier, which enable us to simplify the structure while preserving or improving the elliptic Sombor index properties.

4.1 Minimum elliptic Sombor index of tricyclic graphs

In this subsection, we investigate the tricyclic graphs that minimize the elliptic Sombor index. By employing a series of graph transformations and comparing their corresponding elliptic Sombor values, we identify the graph structure that yields the smallest possible value for this index. The main result characterizes a unique extremal graph achieving this minimum among all tricyclic graphs of given order.

Our observations reveal that in Transformation 1, converting an internal path vertex to a pendant vertex results in a uniform increase across all considered indices. Similarly, Transformations 2 and 4 demonstrate index increases, leading to the establishment of an upper bound.

Remark. According to Lemmas 1, 2, and 4, any graph in Γ_n attaining the minimum elliptic Sombor index must necessarily be a tricyclic graph without pendant vertices. Specifically, such extremal graphs correspond to the structures depicted in Figures 5 or 6.

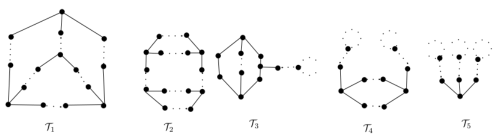

Figure 5. Graphs $\mathcal{T}_i$ in Γ_n^1

Now, we present the minimum value of the elliptic Sombor index among all tricyclic graphs.

Theorem 1. *Let $\mathcal{T}$ be any tricyclic graph of order n. Then*

$$ESO(\mathcal{T}) \geq 8\sqrt{2}(n-10) + 60\sqrt{13},$$

with equality if and only if $\mathcal{T} \cong \Gamma_n^1$.

Proof. Let $\mathcal{T}$ be an arbitrary tricyclic graph. By Lemma 2, $\mathcal{T}$ can be transformed into one of the fifteen base graphs illustrated in Figure 5.

Applying Lemma 3, for each such graph $\mathcal{T}$, there exists a graph $t_i \in \Gamma_n^0$ for $i = 1, 2, \ldots, 15$ such that

$$ESO(t_i) \leq ESO(\mathcal{T}).$$

Direct calculations for $i = 1, 2, \ldots, 5$ yield

$$ESO(t_i) = 8\sqrt{2}(n-10) + 60\sqrt{13}.$$

This completes the proof. ■

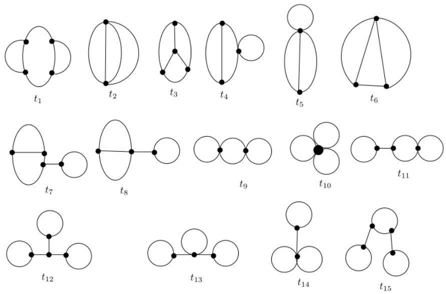

Figure 6. All possible braces t_i of tricyclic graphs in Γ_n^0

4.2 Maximum elliptic Sombor index of tricyclic graphs

Here we establish sharp upper bounds for the elliptic Sombor index in Γ_n, identifying extremal graphs via comparative analysis of three fundamental edge-swapping transformations.

Through Transformation 3, the conversion of a pendant path to an internal path results in a systematic decrease of all topological invariants, consequently providing a lower bound for the index values.

Remark. Lemma 3 establishes that the extremal graphs in Γ_n maximizing the elliptic Sombor index are precisely those tricyclic graphs containing pendant vertices. The family Γ_n contains six distinguished configurations $\{\rho_i\}_{i=1}^6$ with the following pendant vertex distributions:

- Graphs ρ_1 and ρ_5 possess $n-5$ pendant vertices each
- Graph ρ_2 contains $n-4$ pendant vertices
- Graph ρ_3 has $n-6$ pendant vertices
- Graphs ρ_4 and ρ_6 feature $n-7$ pendant vertices each

The complete structural details are illustrated in Figure 7.

Now, we determine the maximum value of the elliptic Sombor index among tricyclic graphs.

Theorem 2. *Let $\mathcal{T}$ be any tricyclic graph of order $n \geq 4$. Then*

$$\begin{aligned} ESO(\mathcal{T}) &\leq n(n-5)\sqrt{(n-1)^2+1} + 3(n+1)\sqrt{(n-1)^2+4} \\ &\quad + (n+3)\sqrt{(n-1)^2+16} + 36\sqrt{5}, \end{aligned}$$

and equality holds if and only if $\mathcal{T} \cong \varrho_1$.

Proof. Let $\mathcal{T}$ be a tricyclic graph as illustrated in Figure 5. By repeatedly applying transformations 2 and 4, we know that $\mathcal{T}$ can be converted into one of the graphs $\varrho_i \in \Gamma_n^2$, for $i = 1, 2, \ldots, 6$.

Using Lemmas 2 and 4, for any $\mathcal{T}$ there exists ϱ_i such that

$$ESO(\mathcal{T}) \leq ESO(\varrho_i).$$

We observe that

$$\begin{aligned} ESO(\varrho_1) &= n(n-5)\sqrt{(n-1)^2+1} + 3(n+1)\sqrt{(n-1)^2+4} \\ &\quad + (n+3)\sqrt{(n-1)^2+16} + 36\sqrt{5}, \end{aligned}$$

and

$$ESO(\varrho_2) = n(n-4)\sqrt{(n-1)^2+1} + 3(n+2)\sqrt{(n-1)^2+9} + 54\sqrt{2}.$$

Consider the difference:

$$\begin{aligned} &ESO(\varrho_1) - ESO(\varrho_2) \\ &= 3n\sqrt{n^2-2n+5} + 2n\sqrt{n^2-2n+17} - 3n\sqrt{n^2-2n+10} \\ &\quad - n\sqrt{n^2-2n+2} + 3\sqrt{n^2-2n+5} + 6\sqrt{n^2-2n+17} \\ &\quad - 6\sqrt{n^2-2n+10} + 36\sqrt{5} - 54\sqrt{2} \\ &> 2n\sqrt{n^2-2n+5} + 2n\sqrt{n^2-2n+17} - 3n\sqrt{n^2-2n+10}. \end{aligned}$$

Therefore, it suffices to show

$$2\left(\sqrt{n^2-2n+5}+\sqrt{n^2-2n+17}\right)>3\sqrt{n^2-2n+10}.$$

For $n \geq 4$, this inequality holds.

Next, by direct calculation,

$$ESO(\varrho_3)=n(n-6)\sqrt{(n-1)^2+1}+4(n+1)\sqrt{(n-1)^2+4}$$
$$+(n+2)\sqrt{(n-1)^2+9}+8\sqrt{2}+10\sqrt{13}.$$

Comparing ϱ_1 and ϱ_3, we have

$$ESO(\varrho_1)-ESO(\varrho_3)>0,$$

and similarly,

$$ESO(\varrho_1)>ESO(\varrho_4),\quad ESO(\varrho_1)>ESO(\varrho_5),\quad ESO(\varrho_1)>ESO(\varrho_6).$$

Hence, the maximum elliptic Sombor index among the tricyclic graphs is attained uniquely by ϱ_1. ■

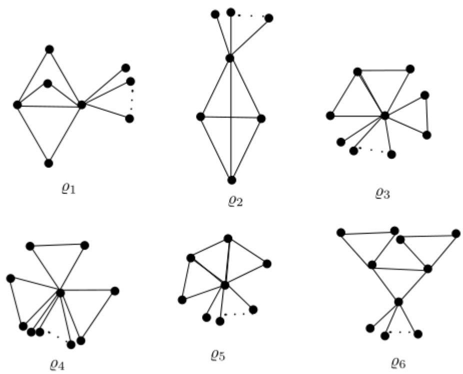

Figure 7. Graphs ϱ_i in Γ_n^2

***Acknowledgment*:** Part of this work was supported by the Major Project of Humanities and Social Sciences Research in Anhui Province (No. 2024AH040374) and Anhui Province Social Science Innovation and Development Research Project (No.2023CX511)

References

[1] C. Espinal, I. Gutman, J. Rada, Elliptic Sombor index of chemical graphs, *Commun. Comb. Optim.* **10** (2025) 989–999.

[2] I. Gutman, B. Furtula, M. S. Oz, Geometric approach to vertex-degree-based topological indices – Elliptic Sombor index, theory and application, *Int. J. Quantum Chem.* **124** (2024) #e27346.

[3] A. E. Hamza, Z. Raza, A. Ali, Z. Alsheekhhussain, On Sombor indices of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* **90** (2023) 223–234.

[4] F. Qi, Z. Lin, Maximal elliptic Sombor index of bicyclic graphs, *Contrib. Math.* **10** (2024) 25–29.

[5] J. Rada, J. M. Rodríguez, J. M. Sigarreta, Sombor index and elliptic Sombor index of benzenoid systems, *Appl. Math. Comput.* **475** (2024) #128756.

[6] Z. Tang, Y. Li, H, Deng, Elliptic Sombor index of trees and unicyclic graphs, *El. J. Math.* **7** (2024) 19–34.

[7] M. Zhang, B. Zhao, Extremal values of the Sombor index in tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* **89** (2023) 741–758.