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Resolving the Conjecture of Tricyclic Graphs Minimizing the Diminished Sombor Index

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Abstract

The diminished Sombor index of a graph G is defined by

$$\mathrm{DSO}(G) = \sum_{v_iv_j \in E(G)} \frac{\sqrt{d_i^2 + d_j^2}}{d_i + d_j},$$

where d_i and d_j denote the degrees of vertices v_i and v_j, respectively. Very recently, Movahedi et al. [**Diminished Sombor Index, *MATCH Commun. Math. Comput. Chem.* 95 (2026) 141–162**] conjectured that among all tricyclic graphs, the minimum value of the diminished Sombor index is attained by the graph formed by connecting two disjoint cycles with two edges to form a quadrangle.

In this paper, we first demonstrate that this conjecture does not hold. We then determine the graph with the minimum diminished

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Sombor index within a specific class of tricyclic graphs, thereby providing a corrected characterization of extremal structures in this context.

1 Introduction

Chemical graph theory is a field that blends graph-theoretical concepts with chemical structure analysis. In this approach, molecules are modeled as graphs, where vertices represent atoms and edges represent chemical bonds. This abstraction provides a convenient mathematical structure for investigating various molecular properties.

A central concept in chemical graph theory is that of a *topological index*–a numerical value computed from the graph that reflects aspects of the molecule's structure, such as its connectivity, branching, or symmetry. These indices are particularly valuable because they remain invariant under graph isomorphism, offering consistent and reliable molecular characterizations.

Topological indices have proven effective in fields such as QSAR and QSPR. They are used to predict biological activity or physicochemical properties like boiling point and solubility. Their predictive power and computational simplicity make them an indispensable part of modern theoretical chemistry. Over time, a wide variety of such indices have been introduced, based on parameters like degree, distance, eccentricity, and spectral properties.

Among these, *degree-based indices* have attracted special attention due to their ease of computation and strong correlations with chemical phenomena. Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Denote by d_i the degree of the vertex v_i and let Δ represent the maximum degree of graph G. A *pendent vertex* is a vertex of degree one. A notable and recently introduced degree-based index is the *Sombor index*, proposed by Gutman [12], defined as

$$\mathrm{SO}(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2},$$

where d_i and d_j denote the degrees of the adjacent vertices v_i and v_j. Various mathematical properties of the Sombor index, including bounds, extremal graphs, and correlation behavior, have been explored in depth (see, [1–12, 16]; for a review, see [13]).

To enhance the structural sensitivity of the classical Sombor index, a normalized variant known as the *diminished Sombor index* (DSO) was introduced by Rajathagiri [15], and was later investigated in detail by Movahedi et al. [14]. It is defined as

$$\mathrm{DSO}(G) = \sum_{v_i v_j \in E(G)} \frac{\sqrt{d_i^2 + d_j^2}}{d_i + d_j},$$

that is,

$$\mathrm{DSO}(G) = \sum_{v_i v_j \in E(G)} f(d_i, d_j),$$

where

$$f(d_i, d_j) = \frac{\sqrt{d_i^2 + d_j^2}}{d_i + d_j}. \tag{1}$$

In a subsequent study, Movahedi et al. [14] investigated extremal values of the diminished Sombor index (DSO) within the classes of trees, unicyclic graphs, and bicyclic graphs. Toward the end of their paper, they proposed the following conjecture concerning tricyclic graphs:

Conjecture 1. [14] *Among all tricyclic graphs of a given order, the minimum value of the DSO is attained by the graph formed by joining two disjoint cycles with two edges forming a quadrangle.*

Figure 1. Two graphs H_1 and H_2.

Let H_1 be one of the tricyclic graphs described in Conjecture 1 (see, Fig. 1). Let K_4' denote the graph obtained from the complete graph K_4 by removing a single edge. Let H_2 denote a tricyclic graph of order n, formed by attaching a path P_{n-4} to K_4' such that the two pendent vertices of P_{n-4} are each connected to one of the two vertices of degree 2 in K_4' (see, Fig. 1). The diminished Sombor indices of these graphs are given by:

$$\mathrm{DSO}(H_1) = \frac{n}{\sqrt{2}} + \frac{4\sqrt{13}}{5} - \sqrt{2}, \quad \mathrm{DSO}(H_2) = \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5}.$$

Upon direct comparison, it is evident that $\mathrm{DSO}(H_1) > \mathrm{DSO}(H_2)$, thereby providing a counterexample to Conjecture 1.

Motivated by this observation, we prove that H_2 attains the minimum diminished Sombor index within a particular class of tricyclic graphs, thereby providing a revised and accurate characterization of the associated extremal structures.

2 Main results

The following two results were established in the previous article [14].

Lemma 1. [14] *For any edge $v_iv_j \in E(G)$,*

$$\frac{\sqrt{d_i^2 + d_j^2}}{d_i + d_j} \geq \frac{1}{\sqrt{2}}$$

with equality if and only if $d_i = d_j$.

Lemma 2. [14] *Let $f(x, y)$ be a function defined in (1). Then*

$$f(2,1) < f(3,1) < \cdots < f(n-2,1) < f(n-1,1).$$

We now prove the main result of this paper.

Theorem 1. *Let G be a tricyclic graph of order $n > 4$. Then*

$$\mathrm{DSO}(G) \geq \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5} \tag{2}$$

with equality if and only if $G \cong H_2$ (see, Fig. 1).

Proof. Let p be the number of pendent edges in G. Then $p \geq 0$. We consider the following two cases:

Case 1. $p = 0$. Let m_{ij} denote the number of edges in G where one endpoint has degree i and the other endpoint has degree j. Also let Δ be the maximum degree in G. Since G has no pendent edge, we have $d_i \geq 2$ for all $v_i \in V(G)$. Let m be the number of edges in G. Since G is a tricyclic graph of order n, we have $m = n+2$ and hence

$$2\,(n+2) = 2m = \sum_{i=1}^{n} d_i \geq \Delta + 2\,(n-1), \quad \text{that is,} \;\; 3 \leq \Delta \leq 6.$$

We consider the following cases:

Case 1.1. $\Delta = 3$. Since $2m = 2n+4$ and $d_i \geq 2$ for all $v_i \in V(G)$, in this case the degree sequence of G is $(3,\, 3,\, 3,\, 3, \underbrace{2,\, 2, \ldots,\, 2}_{n-4})$. For any edge $v_iv_j \in E(G)$, we have $(d_i,\, d_j) \in \{(3,3),\, (3,2),\, (2,2)\}$, and consequently,

$$m_{22} + m_{23} + m_{33} = m = n+2. \tag{3}$$

Based on the degree sequence of G and the condition $n \geq 5$, there is a vertex of degree 2 and hence $m_{23} \geq 2$ as G is connected. Using this with (3) and Lemma 1, we obtain

$$\begin{aligned}
\mathrm{DSO}(G) &= \sum_{v_iv_j \in E(G)} f(d_i,\, d_j) \\
&= m_{22}\, f(2,2) + m_{23}\, f(2,3) + m_{33}\, f(3,3) \\
&= \frac{m_{23}\,\sqrt{13}}{5} + \frac{n+2-m_{23}}{\sqrt{2}} \\
&= \left(\frac{\sqrt{13}}{5} - \frac{1}{\sqrt{2}}\right) m_{23} + \frac{n+2}{\sqrt{2}}
\end{aligned}$$

$$\geq 2\left(\frac{\sqrt{13}}{5}-\frac{1}{\sqrt{2}}\right)+\frac{n+2}{\sqrt{2}}$$

$$=\frac{n}{\sqrt{2}}+\frac{2\sqrt{13}}{5}.$$

Moreover, the above equality holds if and only if $m_{23} = 2$, that is, the equality holds in (2) if and only if $m_{23} = 2$, $m_{33} = 5$ and $m_{22} = n - 5$ as the degree sequence is $(3, 3, 3, 3, \underbrace{2, 2, \ldots, 2}_{n-4})$, that is, if and only if $G \cong H_2$.

Case 1.2. $\Delta = 4$. Since $m = n + 2$, one can easily see that the degree sequence of G is either $(4, 3, 3, \underbrace{2, 2, \ldots, 2}_{n-3})$ or $(4, 4, \underbrace{2, 2, \ldots, 2}_{n-2})$. First, we assume that the degree sequence of G is $(4, 3, 3, \underbrace{2, 2, \ldots, 2}_{n-3})$. Thus we have $m_{22} + m_{23} + m_{24} + m_{33} + m_{34} = m = n + 2$. In this case we must have $m_{24} \geq 2$. Using the above results with Lemma 1, we obtain

$$\mathrm{DSO}(G) = \sum_{v_iv_j \in E(G)} f(d_i, d_j)$$

$$= m_{22}\, f(2,2) + m_{23}\, f(2,3) + m_{24}\, f(2,4) + m_{33}\, f(3,3)$$

$$+ m_{34}\, f(3,4) \geq m_{24}\, f(2,4) + (m_{22} + m_{23} + m_{33} + m_{34})\,\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{5}}{3}\, m_{24} + \frac{1}{\sqrt{2}}\,(n + 2 - m_{24}) = \left(\frac{\sqrt{5}}{3}-\frac{1}{\sqrt{2}}\right) m_{24} + \frac{n+2}{\sqrt{2}}$$

$$\geq 2\left(\frac{\sqrt{5}}{3}-\frac{1}{\sqrt{2}}\right)+\frac{n+2}{\sqrt{2}} = \frac{2\sqrt{5}}{3}+\frac{n}{\sqrt{2}} > \frac{n}{\sqrt{2}}+\frac{2\sqrt{13}}{5}.$$

The inequality (2) strictly holds.

Next we assume that the degree sequence of G is $(4, 4, \underbrace{2, 2, \ldots, 2}_{n-2})$. Thus we have $m_{22} + m_{24} + m_{44} = m = n + 2$. In this case we must have

$m_{24} \geq 6$. Applying Lemma 1 to the above, we obtain

$$\begin{aligned}
\mathrm{DSO}(G) &= \sum_{v_i v_j \in E(G)} f(d_i,\, d_j) \\
&= m_{22}\, f(2,2) + m_{24}\, f(2,4) + m_{44}\, f(4,4) \\
&= m_{24}\, f(2,4) + (m_{22} + m_{44})\, \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{5}}{3}\, m_{24} + (n + 2 - m_{24})\, \frac{1}{\sqrt{2}} \\
&= \left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{2}} \right) m_{24} + \frac{n+2}{\sqrt{2}} \\
&\geq 6 \left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{2}} \right) + \frac{n+2}{\sqrt{2}} \\
&> \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5}.
\end{aligned}$$

The inequality (2) strictly holds.

Case 1.3. $\Delta = 5$. In this case the degree sequence of G is $(5,\, 3, \underbrace{2,\, 2, \ldots,\, 2}_{n-2})$.
Thus we have $m_{22} + m_{23} + m_{25} + m_{35} = m = n + 2$. In this case we must have $m_{25} \geq 4$. Using the above results with Lemma 1, we obtain

$$\begin{aligned}
\mathrm{DSO}(G) &= \sum_{v_i v_j \in E(G)} f(d_i,\, d_j) \\
&= m_{22}\, f(2,2) + m_{23}\, f(2,3) + m_{25}\, f(2,5) + m_{35}\, f(3,5) \\
&\geq m_{25}\, f(2,5) + (m_{22} + m_{23} + m_{35})\, \frac{1}{\sqrt{2}} \\
&= m_{25}\, \frac{\sqrt{29}}{7} + (n + 2 - m_{25})\, \frac{1}{\sqrt{2}}
\end{aligned}$$

$$= \left(\frac{\sqrt{29}}{7} - \frac{1}{\sqrt{2}}\right) m_{25} + \frac{n+2}{\sqrt{2}}$$

$$\geq 4\left(\frac{\sqrt{29}}{7} - \frac{1}{\sqrt{2}}\right) + \frac{n+2}{\sqrt{2}}$$

$$> \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5}.$$

The inequality (2) strictly holds.

Case 1.4. $\Delta = 6$. In this case the degree sequence of G is $(6, \underbrace{2,\, 2, \ldots,\, 2}_{n-1})$. Thus we have $m_{22} + m_{26} = m = n+2$. In this case we must have $m_{26} = 6$ and $m_{22} = n-4$. Using the above results with Lemma 1, we obtain

$$\mathrm{DSO}(G) = \sum_{v_i v_j \in E(G)} f(d_i,\, d_j) = m_{22}\, f(2,2) + m_{26}\, f(2,6)$$

$$= \frac{n-4}{\sqrt{2}} + \frac{6\sqrt{40}}{8}$$

$$> \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5}.$$

The inequality (2) strictly holds.

Case 2. $p \geq 1$. By Lemmas 1 and 2, we obtain

$$\mathrm{DSO}(G) = \sum_{v_i v_j \in E(G)} f(d_i,\, d_j)$$

$$= \sum_{\substack{v_i v_j \in E(G), \\ d_i > d_j = 1}} f(d_i,\, 1) + \sum_{\substack{v_i v_j \in E(G), \\ d_i \geq d_j \geq 2}} f(d_i,\, d_j)$$

$$\geq p\, f(2,1) + \frac{n+2-p}{\sqrt{2}}$$

$$= \frac{\sqrt{5}}{3}\, p + \frac{n+2-p}{\sqrt{2}}$$

$$= \left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{2}}\right) p + \frac{n+2}{\sqrt{2}}$$

$$\geq \left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{2}}\right) + \frac{n+2}{\sqrt{2}} > \frac{n}{\sqrt{2}} + \frac{2\sqrt{13}}{5}$$

as $p \geq 1$. The inequality (2) strictly holds. This completes the proof of the theorem. ■

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