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Using Edge Energy to Improve Existing Bounds for Sombor Index of a Graph

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Abstract

Let G be an arbitrary simple graph. In this paper, we correct the proof of some stated inequalities between energy and Sombor index of G in the existing literature and improve them. Also, we state a new inequality between energy and the Randić index of G.

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected chemical graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. If $|E(G)| = m$, then G is called a graph of *size* m. The maximum and the minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent and 0 otherwise. The *energy* of a graph G (i.e. the sum of the absolute values of its adjacency eigenvalues) [9,13] is shown by $\mathcal{E}(G)$. According to [6], the *Sombor index* is defined as $SO(G) = \sum_{xy\in E(G)} \sqrt{d_x^2 + d_y^2}$, where d_x and d_y are the degree of vertices x and y,

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respectively. The reader can refer to [5], [7], [10], [11] and [12] for more information on the Sombor index of graphs.

By [3], the energy of the vertex v_i of a graph G is given by

$$\mathcal{E}(v_i) = \sum_{i=1}^{n} |A(G)|_{ii} \qquad \text{for } i = 1, \ldots, n,$$

where $|A| = (AA^*)^{1/2}$ and A is the adjacency matrix of G. In the mentioned paper, the authors proved that

$$\mathcal{E}(G) = \sum_{v \in V(G)} \mathcal{E}(v).$$

So, one can easely see that:

$$\mathcal{E}(G) = \sum_{xy \in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y} \right).$$

2 Preliminaries

We start this section by stating the following lower bounds for the energy of a vertex which will be useful in proving the results of the article

Theorem 1. *[3, Pro. 4.9] Let G be a regular graph with at least one edge. Then $\mathcal{E}_G(v) \geq 1$.*

Theorem 2. *[3, Pro. 3.3] Let G be a connected graph with at least one edge. Then $\mathcal{E}_G(v) \geq \frac{d_v}{\Delta(G)}$, for all $v \in V(G)$. Equality holds if and only if G is isomorphic to complete bipartite graph $K_{d,d}$.*

Theorem 3. *[3, Thm. 3.6] Let G be a graph with at least one edge. Then $\mathcal{E}_G(v) \geq \sqrt{\frac{d_v}{\Delta(G)}}$, for all $v \in V(G)$.*

3 Main results

In recent years, various inequalities between the energy and Sombor index of graphs have been introduced (see [1], [2], [14], [15], [16], [17] and [18]). Among the results, Ülker et al. in Theorem 3.1 of [17] proved that

$$\mathcal{E}(G)\Delta^3(G) \geq SO(G),$$

for any graph G. Also, in Theorems 3.2, they showed that

$$\mathcal{E}(G)\Delta^2(G) \geq SO(G),$$

if G is regular. In the proof of these theorems, it is used that the square root of any positive real number is always smaller than that number. However, it should be noted that this statement holds if and only if the given number is greater than one. In the following example, the incorrectness of the proof of the aforementioned theorems is noticeable.

Example 1. (i) Let $G = K_{1,2}$. Then we have:

$$\sum_{xy\in E(G)} \frac{\frac{d_x}{\Delta(G)}d_y^2 + \frac{d_y}{\Delta(G)}d_x^2}{d_x^2 d_y^2} = \frac{3}{2} < \sqrt{3} = \sum_{xy\in E(G)} \frac{\sqrt{\frac{d_x}{\Delta(G)}d_y^2 + \frac{d_y}{\Delta(G)}d_x^2}}{d_x d_y}$$

(ii) Let $G = K_3$. Then we get:

$$\sum_{xy\in E(G)} \frac{d_x^2 + d_y^2}{d_x^2 d_y^2} = \frac{3}{2} < \frac{3\sqrt{2}}{2} = \sum_{xy\in E(G)} \frac{\sqrt{d_x^2 + d_y^2}}{d_x d_y}$$

Now, we present the modified proof of the mentioned theorems.

Theorem 4. *Let G be an arbitrary graph. Then $SO(G) < \mathcal{E}(G)\Delta^3(G)$.*

Proof. It is enough to prove the theorem only for connected graphs. Clearly, if $G = K_2$, then $SO(G) = \sqrt{2} < 2 = \mathcal{E}(G)\Delta^3(G)$. Now, let $G \neq K_2$. Therefore, G has at least one vertex with degree greater than one. By Theorem 2 and considering the fact that $r + s > \sqrt{r^2 + s^2}$ for any positive integer r and s, we have:

$$\begin{aligned}
\mathcal{E}(G) &= \textstyle\sum_{e=xy\in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y}\right) \\
&\geq \textstyle\sum_{e=xy\in E(G)} \left(\frac{\frac{d_x}{\Delta(G)}}{d_x} + \frac{\frac{d_y}{\Delta(G)}}{d_y}\right) \\
&> \frac{1}{\Delta(G)} \textstyle\sum_{e=xy\in E(G)} \left(\frac{1}{d_x} + \frac{1}{d_y}\right)
\end{aligned}$$

$$= \frac{1}{\Delta(G)} \sum\nolimits_{e=xy\in E(G)} \left(\frac{d_x + d_y}{d_x d_y} \right)$$
$$> \frac{1}{\Delta^3(G)} \sum\nolimits_{e=xy\in E(G)} \left(\sqrt{d_x^2 + d_y^2} \right) = \frac{1}{\Delta^3(G)} SO(G),$$

and the result follows. ■

Theorem 5. *Let G be a regular graph. Then $SO(G) < \mathcal{E}(G)\Delta^2(G)$.*

Proof. By Theorem 1, we have $\mathcal{E}(G) \geq \sum_{e=xy\in E(G)} \left(\frac{1}{d_x} + \frac{1}{d_y} \right)$ and the desired result is obtained similar to the method stated in the proof of Theorem 4. ■

In the following, we will improve the stated bound for regular graphs and give a result for arbitrary graphs.

Theorem 6. *Let $G \neq K_2$ be a connected graph of size m. Then*

$$SO(G) < \mathcal{E}(G)\Delta^2(G) - \frac{2}{3} m \sqrt{\Delta(G)}.$$

Proof. Note that $\sqrt{r} + \sqrt{s} \geq \sqrt{r+s} + \frac{2}{3}$ for any positive integers r and s with $r + s \geq 3$. Thus, by Theorem 3 we have:

$$\mathcal{E}(G) = \sum\nolimits_{e=xy\in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y} \right)$$
$$\geq \sum\nolimits_{e=xy\in E(G)} \left(\frac{\sqrt{\frac{d_x}{\Delta(G)}}}{d_x} + \frac{\sqrt{\frac{d_y}{\Delta(G)}}}{d_y} \right)$$
$$= \frac{1}{\sqrt{\Delta(G)}} \sum\nolimits_{e=xy\in E(G)} \left(\frac{1}{\sqrt{d_x}} + \frac{1}{\sqrt{d_y}} \right)$$
$$= \frac{1}{\sqrt{\Delta(G)}} \sum\nolimits_{e=xy\in E(G)} \left(\frac{\sqrt{d_x} + \sqrt{d_y}}{\sqrt{d_x d_y}} \right)$$
$$> \frac{1}{\Delta(G)\sqrt{\Delta(G)}} \sum\nolimits_{e=xy\in E(G)} \left(\sqrt{d_x} + \sqrt{d_y} \right)$$
$$> \frac{1}{\Delta(G)\sqrt{\Delta(G)}} \sum\nolimits_{e=xy\in E(G)} \left(\sqrt{d_x + d_y} + \frac{2}{3} \right)$$
$$= \frac{1}{\Delta(G)\sqrt{\Delta(G)}} \sum\nolimits_{e=xy\in E(G)} \sqrt{\frac{d_x^2}{d_x} + \frac{d_y^2}{d_y}} + \frac{2m}{3\Delta(G)\sqrt{\Delta(G)}}$$

$$> \frac{1}{\Delta^2(G)} \sum_{e=xy\in E(G)} \sqrt{d_x^2 + d_y^2} + \frac{2m}{3\Delta(G)\sqrt{\Delta(G)}}$$
$$= \frac{1}{\Delta^2(G)} SO(G) + \frac{2m}{3\Delta(G)\sqrt{\Delta(G)}}.$$

Hence,

$$SO(G) < \mathcal{E}(G)\Delta^2(G) - \frac{2}{3} m\sqrt{\Delta(G)}$$

and the proof is complete. ■

Recall that the Randić index [4] is defined as $R(G) = \sum_{xy\in E(G)} \frac{1}{\sqrt{d_x d_y}}$. We finish this paper, by stating the following inequalities between the energy and the Randić index of graphs.

Theorem 7. *Let G be a graph. Then*

$$2\sqrt{\frac{\delta(G)}{\Delta(G)}} R(G) \le \mathcal{E}(G) \le 2\sqrt{\Delta(G)} R(G)\ .$$

Proof. The second inequality holds by [8, Pro. 4]. Also, by the proof of Theorem 6 we have:

$$\mathcal{E}(G) \ge \frac{1}{\sqrt{\Delta(G)}} \sum_{e=xy\in E(G)} \left(\frac{\sqrt{d_x} + \sqrt{d_y}}{\sqrt{d_x d_y}} \right) \ge 2\sqrt{\frac{\delta(G)}{\Delta(G)}} R(G),$$

and the proof is complete. ■

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