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Proof of a Conjecture on the Atom-Bond Sum-Connectivity Index of Bicyclic Graphs

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Abstract

Recently, the atom-bond sum-connectivity index (denoted as ABS) was introduced as a novel topological index in chemical graph theory. The ABS index of a graph G is defined as

$$ABS(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i + d_j}},$$

where d_i represents the degree of the vertex v_i in G. An important problem in discrete mathematics is the characterization of extremal structures concerning graph invariants within the class of bicyclic graphs. In this context, Ali et al. [**Extremal results and bounds for the atom-bond sum-connectivity index, MATCH Commun. Math. Comput. Chem. 92 (2024) 271–314**] proposed a conjecture regarding the characterization of bicyclic graphs that minimize the ABS index.

This article fully characterizes the bicyclic graph that achieves the minimum ABS index, thereby resolving the conjecture.

1 Introduction

Throughout this paper, we study simple, unweighted, and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *order* and *size* of G are given by $n = |V(G)|$ and $m = |E(G)|$, respectively. For a vertex $v_i \in V(G)$, the degree of v_i in G is denoted by $d_G(v_i)$ (or simply d_i), and its set of neighbors is represented as $N_G(v_i)$, meaning $d_G(v_i) = |N_G(v_i)|$. A vertex $v_i \in V(G)$ is called a *pendant* vertex if $d_G(v_i) = 1$. Let $d_G(v_i, v_j)$ denote the shortest distance between the vertices v_i and v_j in the graph G. For a subset S of $V(G)$, the subgraph obtained by removing the vertices of S along with their incident edges is denoted by $G-S$. Similarly, for a subset E_1 of $E(G)$, the subgraph formed by deleting the edges of E_1 is represented as $G - E_1$. In particular, if $S = \{v_i\}$ and $E_1 = \{v_k v_\ell\}$, we write $G - v_i$ and $G - v_k v_\ell$ for brevity. For any two nonadjacent vertices v_i and v_j in G, we use $G + v_i v_j$ to denote the graph obtained by adding the edge $v_i v_j$ to G. For other graph-theoretic notations and terminology not explicitly defined here, readers may refer to [10].

In chemical graph theory, molecular structure descriptors that are derived from graphs are known as topological indices. Among the most extensively studied topological indices are the atom-bond connectivity (ABC) index [11, 12, 14–18, 20, 21] and the sum-connectivity (SCI) index [4, 5, 13, 19, 24]. These indices are defined as follows:

$$ABC(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i\, d_j}} \quad \text{and} \quad SCI(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{d_i + d_j}}$$

where d_i represent the degree of the vertex v_i in G. Both indices serve as measures of molecular complexity by analyzing the degrees of atoms (vertices) and bonds (edges) within a chemical graph. Building on the foundational work of the ABC and SCI indices, a new topological index called the atom-bond sum-connectivity (ABS) index was recently proposed in [3]. This novel index seeks to combine the principles of both the ABC and SCI indices while offering a distinct method to analyze molecular structures. The ABS index [6] of a graph G is defined as

$$ABS(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i + d_j}},$$

where d_i represents the degree of the vertex v_i in G. The ABS index has attracted significant attention from mathematicians, leading to a remarkable number of research papers exploring its mathematical properties (see, [1–3, 6–9, 22, 23]). This new ABS index introduces a more refined way to measure the connectivity and branching of molecular structures, as it incorporates a balance between the degrees of the vertices and their mutual relationships within the graph. By combining elements of both the ABC and SCI indices, the ABS index aims to provide more accurate and insightful predictions regarding the properties and behaviors of molecules, especially in fields such as drug design, material science, and molecular chemistry.

Figure 1. The bicyclic graphs referred in Conjecture 1.

Ali et al. [7] proposed a conjecture regarding the characterization of bicyclic graphs that minimize the ABS index.

Conjecture 1. [7] *Only the graphs depicted in Fig. 1, achieve the minimum ABS index over the class of all n-order connected bicyclic graphs for every $n \geq 6$.*

This paper aims to provide a complete solution to the aforementioned conjecture. Our objective is to develop novel methods for identifying the minimal connected bicyclic graph with respect to the ABS index.

2 Main result

In this section, we present a proof of Conjecture 1. To accomplish this, we first define bicyclic graphs.

Definition 1. Let G be a graph of order n. It is called bicyclic if it is a connected graph and has $n+1$ edges.

Consider a bicyclic graph G, and let B denote its unique bicyclic subgraph that contains no pendant vertices. In other words, G is formed by attaching trees to certain vertices of B. This subgraph B is referred to as the base of the bicyclic graph. It is well known that there are precisely three distinct types of bicyclic graphs that do not contain any pendant vertices (see, Fig. 2).

Type 1. We can consider the bicyclic graph denoted as $B(p,q)$ (see, Fig. 2), which is formed by combining two cycles, namely C_p and C_q, with a single shared vertex.

Type 2. Another variation, denoted as $B(a,\, b,\, c)$ (see, Fig. 2), results from two cycles, C_a and C_c, connected by a unique path P_b that links C_a and C_c.

Type 3. Lastly, the type represented as $B(P_k, P_\ell, P_m)$ (see, Fig. 2) is derived from three pairwise disjoint paths P_k, P_ℓ and P_m, all originating from one vertex x and leading to another vertex y.

Let us define the three distinct classes of bicyclic graphs that have no pendant vertices.

$$\Gamma_1 = \Big\{ B(p,q) \in G \,|\, p+q = n+1,\, p,\, q \geq 3 \Big\},$$

$$\Gamma_2 = \Big\{ B(a,b,c) \in G \,|\, a+b+c = n+2,\, a,\, c \geq 3,\, b \geq 2 \Big\},$$

$$\Gamma_3 = \{ B(P_k, P_\ell, P_m) \in G \,|\, k+\ell+m = n+4,\, k \geq m \geq 3,\, \ell \geq 2 \}.$$

Again let $\Gamma_2^{'}\, (\subseteq \Gamma_2)$ be the class of bicyclic graphs such that

$$\Gamma_2^{'} = \Big\{ B(a,2,c) \in G \,|\, a+c = n,\, a,\, c \geq 3 \Big\},$$

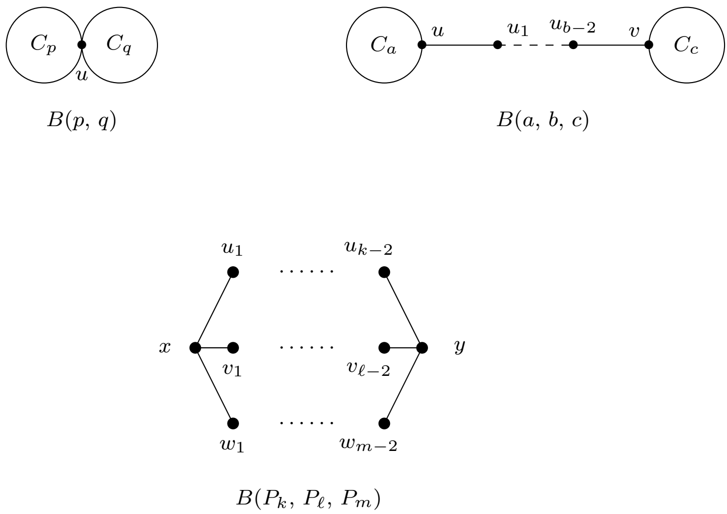

Figure 2. Three types of bicyclic structures $B(p,q)$, $B(a,b,c)$ and $B(P_k,P_\ell,P_m)$.

and let $\Gamma_3^{'}$ ($\subseteq \Gamma_3$) be the class of bicyclic graphs such that

$$\Gamma_3^{'} = \Big\{ B(P_k, P_2, P_m) \in G \,|\, k+m = n+2,\, k \geq m \geq 3 \Big\}.$$

For any graph $G \in \Gamma_2^{'}$ or $G \in \Gamma_3^{'}$, we obtain

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}.$$

We are now ready to find a lower bound on $ABS(G)$ of bicyclic graphs of order n, and characterize the extremal graphs.

Theorem 1. *Let G be a bicyclic graph of order n. Then*

$$ABS(G) \geq \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} \tag{1}$$

with equality if and only if $G \in \Gamma_2'$ or $G \in \Gamma_3'$.

Proof. We consider the following two cases:

Case 1. G has no pendant vertices. Then $G \in \Gamma_1$ or $G \in \Gamma_2$ or $G \in \Gamma_3$. We consider the following three cases:

Case 1.1. $G \in \Gamma_1$. Then there exist two integers $p \geq q \geq 3$ with $p+q = n+1$ such that $G \cong B(p,q)$. Then we obtain

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-3) + 4\sqrt{\frac{2}{3}} > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$\sqrt{\frac{1}{2}} + 3\sqrt{\frac{2}{3}} > 4\sqrt{\frac{3}{5}}.$$

The result in (1) strictly holds.

Case 1.2. $G \in \Gamma_2$. If $G \in \Gamma_2'$, then

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

and hence the equality holds in (1). Otherwise, $G \notin \Gamma_2'$. Since $G \in \Gamma_2$, then there exist three integers a, b, $c \geq 3$ with $a+b+c = n+2$ such that $G \cong B(a,b,c)$. Then we obtain

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-5) + 6\sqrt{\frac{3}{5}} > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$2\sqrt{\frac{3}{5}} > \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}}.$$

The result in (1) strictly holds.

Case 1.3. $G \in \Gamma_3$. If $G \in \Gamma_3'$, then

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

and hence the equality holds in (1). Otherwise, $G \notin \Gamma_3'$. Since $G \in \Gamma_3$, then there exist three integers k, ℓ, $m \geq 3$ with $k+\ell+m = n+4$ such that $G \cong B(P_k, P_\ell, P_m)$. Then we obtain

$$ABS(G) = \sqrt{\frac{1}{2}}\,(n-5) + 6\sqrt{\frac{3}{5}} > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$2\sqrt{\frac{3}{5}} > \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}}.$$

The result in (1) strictly holds.

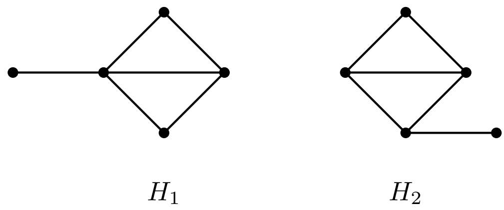

Figure 3. Two bicyclic graphs H_1 and H_2.

Case 2. G contains a pendant vertex. In this case we have to prove that

$$ABS(G) > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}. \tag{2}$$

We prove result (2) by mathematical induction on n. Since G is a bicyclic graph contain pendant vertex, we must have $n \geq 5$. For $n = 5$, $G \cong H_1$ or $G \cong H_2$ (see, Fig. 3). Since $n = 5$. one can easily check that

$$ABS(H_1) = 3\sqrt{\frac{3}{5}} + 2\sqrt{\frac{2}{3}} + \sqrt{\frac{5}{7}} > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}},$$

and

$$ABS(H_2) = 3\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{2}} > \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}.$$

Hence the result (2) holds for $n = 5$, completing the base case. Now, we proceed with the inductive step. Assume that the result (2) holds for any bicyclic graph contains a pendant vertex of order less than n. We will now prove it for n. For this we consider the following cases:

Case 2.1. A pendant vertex is adjacent to a vertex of degree at least 3 in

G.

Let v_k be a pendant vertex with $v_kv_i \in E(G)$, where $d_i \geq 3$. Then $d_i + d_k \geq 4$. For $v_iv_k \in E(G)$, we obtain

$$\sqrt{\frac{d_i + d_k - 2}{d_i + d_k}} = \sqrt{1 - \frac{2}{d_i + d_k}} \geq \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}}$$

as $d_i + d_k \geq 4$. Moreover, the above equality holds if and only if $d_i = 3$. Let $G' = G - v_k$. If G' has no pendant vertices, then by **Case 1**, we have

$$ABS(G') \geq \sqrt{\frac{1}{2}}\,(n-5) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}. \tag{3}$$

Otherwise, G' contains a pendant vertex. Then by the induction hypothesis, we have

$$ABS(G') > \sqrt{\frac{1}{2}}\,(n-5) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}},$$

that is, (3) holds. Now,

$$ABS(G) - ABS(G') = \sum_{\substack{v_j : v_iv_j \in E(G),\\ d_j \geq 2}} \left(\sqrt{1 - \frac{2}{d_i + d_j}} - \sqrt{1 - \frac{2}{d_i + d_j - 1}} \right)$$

$$+ \sum_{\substack{v_j : v_iv_j \in E(G),\\ d_j = 1,\, j \neq k}} \left(\sqrt{1 - \frac{2}{d_i + 1}} - \sqrt{1 - \frac{2}{d_i}} \right)$$

$$+ \sqrt{1 - \frac{2}{d_i + 1}} > \sqrt{1 - \frac{2}{d_i + 1}} \geq \sqrt{\frac{1}{2}}$$

as $d_i \geq 3$. From the above result with (3), we obtain

$$ABS(G) > ABS(G') + \sqrt{\frac{1}{2}} \geq \sqrt{\frac{1}{2}}\,(n-5) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}}$$

$$= \sqrt{\frac{1}{2}}\,(n-4) + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

and (2) holds by induction.

Case 2.2. All the pendant vertices are adjacent to a vertex of degree 2 in G.

We consider the following two cases:

Case 2.2.1. There exists a vertex $v_r \in V(G)\backslash V(B)$ such that $d_r \geq 3$.

For any vertex $v_k \in V(G)\backslash V(B)$ of degree d_k (≥ 3), we define

$$d_G(B, v_k) = \min_{v_j \in V(B)} d_G(v_k, v_j).$$

Let v_s be a vertex in $V(G)\backslash V(B)$ such that $d_s \geq 3$ and the distance from v_s to the base B is maximized, that is,

$$d_G(B, v_s) = \max_{\substack{v_k \in V(G)\backslash V(B),\\ d_k \geq 3}} d_G(B, v_k).$$

In this case we have $d_G(B, v_s) \geq 1$. Since $d_G(B, v_s)$ is maximum, the vertex v_s has $d_s - 1$ pendant paths of length at least 2. Assume that one of the pendant path $P_1 : v_s u_1 u_2 \dots u_a$, where $d_G(u_1) = d_G(u_2) = \cdots = d_G(u_{a-1}) = 2$, $d_G(u_a) = 1$ $(a \geq 2)$. Define the graph G' as $G' = G - \{u_1, u_2, \dots, u_a\}$. Then the number of vertices in G' is $|V(G')| = n - a$. Since the graph G' contains a pendant vertex and $|V(G')| = n - a < n$, applying the induction hypothesis, we obtain the following:

$$ABS(G') > (n-a-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}. \tag{4}$$

Now,

$$ABS(G) - ABS(G') = \sum_{\substack{v_i : v_i v_s \in E(G),\\ v_i \notin V(P_1)}} \left(\sqrt{1 - \frac{2}{d_s + d_i}} - \sqrt{1 - \frac{2}{d_s + d_i - 1}} \right)$$
$$+ \sqrt{1 - \frac{2}{d_s + 2}} + (a-2)\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}.$$

From the above result with (4), we obtain

$$ABS(G) \geq \sum_{\substack{v_i: v_i v_s \in E(G),\\ v_i \notin V(P_1)}} \left(\sqrt{1 - \frac{2}{d_s + d_i}} - \sqrt{1 - \frac{2}{d_s + d_i - 1}} \right) \tag{5}$$

$$+ \sqrt{1 - \frac{2}{d_s + 2}} + (n-6)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}. \tag{6}$$

If $d_s \geq 5$, then from the above, we obtain

$$ABS(G) > \sqrt{\frac{5}{7}} + (n-6)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}$$

$$> (n-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$\sqrt{1 - \frac{2}{d_s + d_i}} > \sqrt{1 - \frac{2}{d_s + d_i - 1}}, \ \sqrt{1 - \frac{2}{d_s + 2}} \geq \sqrt{\frac{5}{7}}, \ \sqrt{\frac{5}{7}} + \sqrt{\frac{1}{3}} > 2\sqrt{\frac{1}{2}}.$$

Hence the result (2) holds. Otherwise, $3 \leq d_s \leq 4$. From (6), we obtain

$$ABS(G) > \left(\sqrt{1 - \frac{2}{d_s + 2}} - \sqrt{1 - \frac{2}{d_s + 1}} \right) (d_s - 2) + \sqrt{1 - \frac{2}{d_s + 2}}$$

$$+ (n-6)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}. \tag{7}$$

For $d_s = 3$, from (7), we obtain

$$ABS(G) > (n-7)\sqrt{\frac{1}{2}} + 6\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}$$

$$> (n-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$2\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{3}} > 3\sqrt{\frac{1}{2}}.$$

Thus (2) holds. For $d_s = 4$, from (7), we obtain

$$ABS(G) > (n-6)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}} + 4\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}$$

$$> (n-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}$$

as

$$3\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} > 2\sqrt{\frac{3}{5}} + 2\sqrt{\frac{1}{2}}.$$

Again (2) holds.

Case 2.2.2. All the vertices in $V(G)\backslash V(B)$ are of degree 1 or 2.

In this case, all pendant paths in the bicyclic graph G are attached to vertices belonging to the base B. Let $S_1 \subseteq V(B)$ denote the set of vertices in the base B such that each vertex in S_1 is incident to at least one pendant path. Let v_k be a vertex in S_1 such that

$$d_k = \max_{v_r \in S_1} d_r.$$

Suppose one of the pendant paths attached to v_k is given by $P_x : v_k w_1 w_2 \dots w_b$ with $d_G(w_1) = d_G(w_2) = \cdots = d_G(w_{b-1}) = 2$, and $d_G(w_b) = 1$, where $b \geq 2$. Define the graph $G'' = G - \{w_1, w_2, \dots, w_b\}$. Then the number of vertices in G'' is $|V(G'')| = n - b$. First we assume that G'' contains no pendant vertex. By **Case 1**, we obtain the following:

$$ABS(G'') \geq (n-b-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}. \tag{8}$$

Next we assume that the graph G'' contains a pendant vertex. Since $|V(G'')| = n - b < n$, by the induction hypothesis, the result (8) is strictly valid. As in **Case 2.2.1**, we obtain

$$ABS(G) \geq \sum_{\substack{v_i : v_i v_k \in E(G), \\ v_i \notin V(P_x)}} \left(\sqrt{1 - \frac{2}{d_k + d_i}} - \sqrt{1 - \frac{2}{d_k + d_i - 1}} \right)$$

$$+\sqrt{1-\frac{2}{d_k+2}}+(n-6)\sqrt{\frac{1}{2}}+4\sqrt{\frac{3}{5}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{3}}. \qquad (9)$$

If $d_k \geq 5$, then as in **Case 2.2.1**, the result (2) holds. Otherwise, $3 \leq d_k \leq 4$. One can easily see that

$$\sqrt{1-\frac{2}{d_k+d_i}}-\sqrt{1-\frac{2}{d_k+d_i-1}}$$
$$=\frac{2}{\sqrt{\sqrt{(d_k+d_i)\,(d_k+d_i-1)}\left(\sqrt{d_k+d_i-2}+\sqrt{d_k+d_i-3}\right)}}. \qquad (10)$$

We consider the following two cases:

Case 2.2.2.1. $d_k = 3$. Let u and w be the vertices in $V(B)$ such that $N_G(v_k) = \{u,\, w,\, w_1\}$. Thus we have $2 \leq d_u \leq 3$ and $2 \leq d_w \leq 3$. Using these, from (10), we obtain

$$\sum_{\substack{v_i : v_i v_k \in E(G),\\ v_i \notin V(P_x)}} \left(\sqrt{1-\frac{2}{d_k+d_i}}-\sqrt{1-\frac{2}{d_k+d_i-1}}\right)$$
$$=\frac{2}{\sqrt{\sqrt{(d_u+3)\,(d_u+2)}\left(\sqrt{d_u+1}+\sqrt{d_u}\right)}}$$
$$+\frac{2}{\sqrt{\sqrt{(d_w+3)\,(d_w+2)}\left(\sqrt{d_w+1}+\sqrt{d_w}\right)}}$$
$$\geq \frac{4}{\sqrt{30}\,(2\sqrt{5}+3\sqrt{2})} > \sqrt{\frac{3}{5}}-\sqrt{\frac{1}{2}}.$$

Using this result in (9), again the result (2) holds as

$$2\sqrt{\frac{3}{5}}+\sqrt{\frac{1}{3}} > 3\sqrt{\frac{1}{2}}.$$

Case 2.2.2.2. $d_k = 4$. Similarly to **Case 2.2.2.1**, we obtain

$$\sum_{\substack{v_i : v_i v_k \in E(G), \\ v_i \notin V(P_x)}} \left(\sqrt{1 - \frac{2}{d_k + d_i}} - \sqrt{1 - \frac{2}{d_k + d_i - 1}} \right)$$

$$= \sum_{\substack{v_i : v_i v_k \in E(G), \\ v_i \notin V(P_x)}} \frac{2}{\sqrt{(d_i + 4)(d_i + 3)} \left(\sqrt{(d_i + 2)(d_i + 3)} + \sqrt{(d_i + 4)(d_i + 1)} \right)}$$

$$\geq \frac{2 \times 3}{\sqrt{56}\,(\sqrt{42} + \sqrt{40})} > 0.062$$

as $2 \leq d_i \leq 4$. From (9), we obtain

$$ABS(G) \geq 0.062 + (n-6)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + 2\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}$$

$$> (n-4)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{3}}.$$

Again, the result (2) holds. This completes the proof of the theorem. ■

3 Conclusions and future work

In conclusion, the article provides a comprehensive characterization of the bicyclic graph that minimizes the atom-bond sum-connectivity (ABS) index, thus resolving the conjecture proposed by Ali et al. in their recent work [6]. This result offers significant insight into the extremal structures of bicyclic graphs in the context of the ABS index.

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