MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **95** (2026) 777–811
ISSN: 0340–6253
doi: 10.46793/match.95-3.14525

Extremal Trees for Euler Sombor Index with Given Parameters

Parikshit Das[a], Sourav Mondal[b,*], Anita Pal[c]

[a] *Department of Mathematics, Sree Chaitanya College, Habra, Prafullanagar, North 24 Parganas, West Bengal-743268, India*

[b] *Research Institute of Science and Engineering (RISE), MASEP Group, University of Sharjah, Sharjah 27272, UAE*

[b] *Department of Mathematics, SRM IST KTR, Tamil Nadu 603203, India*

[a,c] *Department of Mathematics, National Institute of Technology Durgapur, West Bengal-713209, India*

parikshitdas1010@gmail.com, souravmath94@gmail.com, apal.maths@nitdgp.ac.in

(Received June 13, 2025)

Abstract

The study of bounds for topological indices based on various graph parameters, as well as the identification of extremal graphs where these bounds are achieved, is an active and intriguing area of research in graph theory. The Euler Sombor index ($\mathcal{ESI}$) is a recently introduced degree-based index with a strong linear relationship to several physicochemical properties of octanes. This work explores extremal trees of $\mathcal{ESI}$ for given graph parameters. First, we identify the second smallest and second largest values of the Euler Sombor index in terms of the graph order n. Then, we characterize maximal and minimal quasi-trees for given n employing the majorization principle. Finally, we identify the extremal trees for the $\mathcal{ESI}$ index for given n and domination number.

*Corresponding author.

1 Introduction

A topological index is a numerical parameter that characterizes the topology of a graph and remains invariant under the graph isomorphism. These indices play a key role in correlating the chemical structure of molecules with their chemical reactivity, biological activity, or physical properties [11, 15, 41]. By providing a quantitative measure of a molecule's structure, topological indices are widely used in chemistry, particularly in predicting molecular behavior and properties, aiding in tasks such as drug design and materials science through quantitative structure-activity relationships (QSAR) and cheminformatics. In the year 2021 Gutman introduced a new topological index named as the Sombor index [13], that emerges from the interplay between graph theory and Euclidean geometry. This index has garnered substantial attention in fields such as chemistry and pharmacology due to its effectiveness in characterizing molecular structures and predicting biological activity. For a graph G the Sombor index is described as $SO(G) = \sum\limits_{u_iu_j \in E(G)} \sqrt{d_{u_i}^2(G) + d_{u_j}^2(G)}$, where the notation $d_{u_i}(G)$ indicates the degree of the vertex u_i in G. The Sombor index has rapidly gained significant attention from researchers in a short period. According to Redžepović [35], this index can be used to predict chemical properties in the field of chemistry with favorable findings. We suggest that readers refer to [4, 5, 9, 10, 13, 17, 19, 20, 27, 28, 30] and the references therein for further information. In 2024, Gutman [14] demonstrated that the lengths of the semi-major and semi-minor axes in an ellipse are equal. Leonard Euler determined the ellipse's approximate perimeter as $\pi\sqrt{2(d_{u_i}^2(G) + d_{u_j}^2(G))(d_{u_i}(G) + d_{u_j}(G))^2}$. These relationships led to the proposal of the Euler Sombor index ($\mathcal{ESI}$) of a graph G as $\mathcal{ESI}(G) = \sum\limits_{u_iu_j \in E(G)} \sqrt{d_{u_i}^2(G) + d_{u_j}^2(G) + d_{u_i}(G)d_{u_j}(G)}$. Algebraically, there is a geometric analogy of Sombor and Euler Sombor indices [14]. Tang et al. [40] conducted an initial analysis of the chemical applicability of the Euler Sombor index. They established its mathematical properties and determined the extremal values of this index across all molecular trees. In addition, they characterized the corresponding extremal graphs, providing a comprehensive understanding of the structural implications of the Euler Sombor index in chemical graph theory. Ren et al. [36] investigated the maximum values of the Euler–Sombor index for trees, considering parameters such as matching number, number of pendent vertices, and diameter. Khanra et al. [25] identified trees with the five smallest Euler Sombor indices, extremal unicyclic graphs, and classified extremal chemical and hexagonal graphs based on the Euler Sombor index. Gul Ozkan [32] determined the extremal values of the Euler Sombor index for tricyclic graphs. Zhang et al. [43] established extremal bounds of the Euler Sombor index for molecular trees and explored its correlation and chemical relevance using various regression models on octane isomers. For more works in this index, readers are referred to [1,21,31,33,38]. The remainder of this work is organized as follows. Section 2 presents some preliminary findings that will be useful for deriving the main results. Trees with the second minimum and maximum Euler Sombor index are identified in Section 3. Section 4 demonstrates the sharp bounds of the Euler Sombor index for quasi-trees. In Section 5, the maximal and minimal trees are identified for given n and domination number.

2 Preliminaries

Throughout this paper, we focus only on simple connected graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The set of all neighbours of a vertex u_i in G is denoted as $N_G(u_i) = \{u_j \in V(G) : u_iu_j \in E(G)\}$. The set $N_G(u_i)$ is called the open neighborhood of u_i and $N_G[u_i] = N_G(u_i) \cup \{u_i\}$ is the closed neighbourhood of the vertex u_i. The degree of u_i in a graph G is denoted by $d_{u_i}(G)$ or d_{u_i}, is the the cardinality of $N_G(u_i)$. Furthermore, the total number of vertices with degree i is indicated by $\hbar_i$. Vertex of degree one referred to as a leaf, while other vertex attached to a leaf is referred to as a support. The longest path connecting any two leaves on a tree is referred to as diameter. If $u_1u_2 \cdots u_du_{d+1}$ represents a path where the diameter is achieved, we refer to $u_1u_2 \cdots u_du_{d+1}$ as a diametral path in G. We denote the graph obtained from G by deleting the vertices $u_1, u_2, \cdots, u_i$ as $G - \{u_1, u_2, \cdots, u_i\}$ and by removing the edges

$e_1, e_2, \cdots, e_i$ as $G - \{e_1, e_2, \cdots, e_i\}$, respectively. The maximum vertex degree in the graph G is represented by $\Delta(G)$. As is standard, P_n, C_n and S_n represent the path, cycle and star with n vertices, respectively. Two distinct vertices $u, v \in V(G)$ are dominated each other if they are adjacent in G. Then v is called a dominating vertex of u in G and vice versa. A subset $\mathbb{D} \subseteq V(G)$ is called dominating set if each vertex in $V(G)\backslash\mathbb{D}$ is linked to at least one vertex in $\mathbb{D}$. The minimum dominating set is denoted by $\mathbb{D}_1$. The cardinality of $\mathbb{D}_1$ in G is referred as the domination number, denoted by $\Omega(G)$. For some considerable works on this parameter, readers are refereed to [2,3,12,26,42]. Since each tree T is connected and acyclic, then for any tree T with n vertices, $\Omega(T) = 1$ iff $T \cong S_n$. In 1962 Ore [34] proved that $\Omega(G) \leq \frac{n}{2}$. After that, in 1985, Fink et al. [12] obtained the graphs G of n vertex that satisfy $\Omega(G) = \frac{n}{2}$. Let $\mathscr{T}(n, \Omega)$ be the set of all n-vertex trees with given domination number Ω. A graph G is said to be a quasi-tree if there exist a vertex $u \in V(G)$ such that $G - u$ is a tree. The class of quasi-trees with n vertex is simply denoted as $\mathcal{Q}_n$.
Exploring bounds for topological indices in relation to various graph parameters, along with identifying extremal graphs that achieve these bounds, is an active and fascinating area of research in graph theory [2,7,8,16,18, 24,42]. In this work, our goal is to characterize extremal trees for the Euler Sombor index with respect to various graph parameters.

Let us consider a function $\mho(x, y) = \sqrt{x^2 + y^2 + xy}$, where $x, y \geq 1$.

Lemma 1. *Let $\Psi_a(x) = \mho(x, a+1) - \mho(x, a)$, with $x \geq 1$ and $a > 0$. Then $\Psi_a(x)$ is decreasing in x.*

Proof. Here $\Psi_a(x) = \mho(x, a+1) - \mho(x, a) = \sqrt{x^2 + (a+1)^2 + x(a+1)} - \sqrt{x^2 + a^2 + ax}$. Therefore,

$$\frac{d}{dx}\Psi_a(x) = \Psi_a^{'}(x) = \frac{2x + a + 1}{2\sqrt{x^2 + (a+1)^2 + x(a+1)}} - \frac{2x + a}{2\sqrt{x^2 + a^2 + ax}}.$$

Since $a, x > 0$, we start with an evident relation $3x^2(1+2a) + 3ax(a+1) > 0$. Using this we have, $\big(4x^2 + 4x(a+1) + (a+1)^2\big)(x^2 + a^2 + ax) + \big(3x^2(1+2a) + 3ax(a+1)\big) > \big(4x^2 + 4x(a+1) + (a+1)^2\big)(x^2 + a^2 + ax)$, which implies that

$$(2x+a)^2\big(x^2+(a+1)^2+x(a+1)\big) > (2x+a+1)^2(x^2+a^2+ax)$$

i.e.,

$$\frac{2x+a}{\sqrt{x^2+a^2+ax}} > \frac{2x+a+1}{\sqrt{x^2+(a+1)^2+x(a+1)}},$$

from which it immediately follows that $\Psi_a^{'}(x) < 0,\ \forall x \geq 1$ and for $a > 0$. Hence $\Psi_a(x)$ is decreasing in x. ■

Lemma 2. *If $\xi_a(x) = \mho(x,a) - \mho(x-1,a)$, with $x \geq 2$ and $a > 0$, then $\xi_a(x)$ is increasing in x.*

Proof. We have

$$\xi_a^{'}(x) = \frac{2x+a}{2\sqrt{x^2+a^2+ax}} - \frac{2(x-1)+a}{2\sqrt{(x-1)^2+a^2+a(x-1)}}.$$

Now $x \geq 2$ and $a > 0$ implies $3a^2(2x+a-1) > 0$. Using this, one can easily verify that,

$$\frac{2x+a}{\sqrt{x^2+a^2+ax}} > \frac{2(x-1)+a}{\sqrt{(x-1)^2+a^2+a(x-1)}}.$$

Therefore $\xi_a^{'}(x) > 0$, $\forall x \geq 2$ and $a > 0$. Hence $\xi_a(x)$ is increasing in x. ■

Lemma 3. *Let $\mathscr{J}(s) = (s-1)\sqrt{s^2-s+1} - (3\sqrt{19}+\sqrt{7}-6\sqrt{3})s + 9\sqrt{19}+\sqrt{7}-18\sqrt{3}$. Then $\mathscr{J}(s) > 0$ for $s \geq 5$.*

Proof. Note that $\mathscr{J}^{'}(s) = \sqrt{s^2-s+1} + (s-1)\dfrac{2s-1}{2\sqrt{s^2-s+1}} - (3\sqrt{19}+\sqrt{7}-6\sqrt{3})$. For $s \geq 5$, we obtain that

$$\mathscr{J}^{'}(s) \geq \sqrt{21} + \frac{18}{\sqrt{21}} - 3\sqrt{19} - \sqrt{7} + 6\sqrt{3} \approx 3.1804 > 0.$$

Hence $\mathscr{J}(n) \geq \mathscr{J}(5) = 4\sqrt{21} - 6\sqrt{19} - 4\sqrt{7} + 12\sqrt{3} \approx 2.3785 > 0.$ ■

Lemma 4. *Let us consider a function $\eta(t,t_1)$ as*

$$\eta(t,t_1) = \sqrt{t^2+t+1} + (t_1-1)\big(\sqrt{t^2+t+1} - \sqrt{t^2-t+1}\big) + (t-t_1)\big(\sqrt{t^2+2t+4} - \sqrt{t^2+3}\big),$$

where $t_1 \geq 2$, $t \geq 4$ and $t_1 < t$. Then $\eta(t,t_1)$ is increasing for t and t_1.

Proof. Here we can write

$$\eta(t,t_1) = t_1\Big(\big(\sqrt{t^2+3} - \sqrt{t^2-t+1}\big) - \big(\sqrt{t^2+2t+4} - \sqrt{t^2+t+1}\big)\Big) + t\Big(\sqrt{t^2+2t+4} - \sqrt{t^2+3}\Big) + \sqrt{t^2-t+1}.$$

Therefore, we have

$$\frac{\partial \eta(t,t_1)}{\partial t_1} = \big(\sqrt{t^2+3} - \sqrt{t^2-t+1}\big) - \big(\sqrt{t^2+2t+4} - \sqrt{t^2+t+1}\big).$$

Using Lemma 1, we obtain $\big(\sqrt{t^2+3} - \sqrt{t^2-t+1}\big) - \big(\sqrt{t^2+2t+4} - \sqrt{t^2+t+1}\big) = \Psi_1(t-1) - \Psi_1(t) > 0$. Hence $\eta(t,t_1)$ is increasing for t_1.
Again, we get

$$\begin{aligned}\frac{\partial \eta(t,t_1)}{\partial t} = & t_1\Bigg(\Big(\frac{2t}{2\sqrt{t^2+3}} - \frac{2t-1}{2\sqrt{t^2-t+1}}\Big) - \Big(\frac{2t+2}{2\sqrt{t^2+2t+4}} - \frac{2t+1}{2\sqrt{t^2+t+1}}\Big)\Bigg) \\ & + \Big(\sqrt{t^2+2t+4} - \sqrt{t^2+3}\Big) + \frac{2t-1}{2\sqrt{t^2-t+1}} \\ & + t\Big(\frac{2t+2}{2\sqrt{t^2+2t+4}} - \frac{2t}{2\sqrt{t^2+3}}\Big) \end{aligned} \tag{1}$$

Let

$$\eta_1(x) = \frac{2x+2}{2\sqrt{x^2+2x+4}} - \frac{2x+1}{2\sqrt{x^2+x+1}}.$$

Then,

$$\eta_1'(x) = \frac{12(x^2+x+1)^{3/2} - 3(x^2+2x+4)^{3/2}}{4(x^2+x+1)^{3/2}(x^2+2x+4)^{3/2}} > 0 \quad for\ x \geq 2.$$

This implies, $\eta_1(x)$ is increasing for $x \geq 2$. Hence, $\eta_1(t-1) - \eta_1(t) < 0$.

Using Lemma 2, we have from (1) that

$$\begin{aligned}\frac{\partial\eta(t,t_1)}{\partial t} =& t_1\Big(\eta_1(t-1)-\eta_1(t)\Big)+\Big(\sqrt{t^2+2t+4}-\sqrt{t^2+3}\Big)\\&+\frac{2t-1}{2\sqrt{t^2-t+1}}+t\Big(\frac{2t+2}{2\sqrt{t^2+2t+4}}-\frac{2t}{2\sqrt{t^2+3}}\Big)\\>& t\Big(\eta_1(t-1)-\eta_1(t)\Big)+\frac{2t-1}{2\sqrt{t^2-t+1}}+t\Big(\frac{2t+2}{2\sqrt{t^2+2t+4}}-\frac{2t}{2\sqrt{t^2+3}}\Big)\\=&\frac{2t^2+t}{2\sqrt{t^2+t+1}}-\frac{2t^2-3t+1}{2\sqrt{t^2-t+1}}.\end{aligned}$$

Let $\eta_2(x)=\dfrac{2x^2+x}{2\sqrt{x^2+x+1}}$. Therefore $\eta_2'(x)=\dfrac{4x^3+6x^2+9x+2}{4(x^2+x+1)^{3/2}}>0$ for $x\geq 1$. This implies that,

$$\frac{\partial\eta(t,t_1)}{\partial t}>\eta_2(t)-\eta_2(t-1)>0.$$

Hence $\eta(t,t_1)$ is incerasing for t. ■

3 Second minimal and maximal trees for the Euler Sombor index

Let $\mathscr{T}_n$ be the class of trees with n vertices. For any tree $T\in\mathscr{T}_n$, an edge $uv\in E(T)$ is called (a,b)-type if $d_u(T)=a$ and $d_v(T)=b$ where $a\geq b$. Now we consider the following subsets of edge set of any tree $T\in\mathscr{T}_n$.
$A_1\equiv$ Set of edges of $(2,1)$ types, $A_2\equiv$ Set of edges of $(2,2)$ types, $A_3\equiv$ Set of edges of $(3,1)$ types, $A_4\equiv$ Set of edges of (a_1,b_1) or (a_2,b_2) types, where $a_1\geq 3$, $b_1\geq 2$, $a_2\geq 4$, $b_2=1$. If $|A_1|=p_1$, $|A_2|=p_2$, $|A_3|=p_3$ and $|A_4|=p_4$, then for any tree $T\in\mathscr{T}_n$, we have $p_1+p_2+p_3+p_4=n-1$. Now

$$\sqrt{d^2_{u_i}(T)+d^2_{u_j}(T)+d_{u_i}(G)d_{u_j}(T)}\begin{cases}=\sqrt{7} & if\ \ u_iu_j\in A_1\\ =2\sqrt{3} & if\ \ u_iu_j\in A_2\\ =\sqrt{13} & if\ \ u_iu_j\in A_3\\ \geq\sqrt{19} & if\ \ u_iu_j\in A_4.\end{cases}$$

Let $\mathscr{T}_{n_1,n_2,n_3}\in\mathscr{T}_n$ be a tree (see Fig. 1) with a maximum-degree vertex

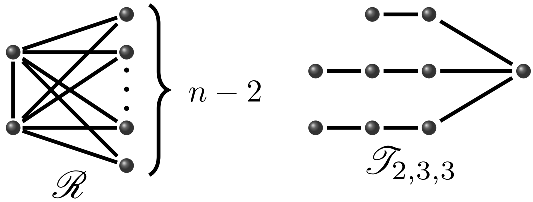

Figure 1. The graphs $\mathscr{R}$ and $\mathscr{T}_{2,3,3}$.

u_i of degree 3, such that removing u_i from $\mathscr{T}_{n_1,n_2,n_3}$ results in three paths P_{n_1}, P_{n_2}, and P_{n_3}, with lengths n_1, n_2, and n_3, respectively, satisfying $n_1+n_2+n_3=n-1$ and $n_1 \geq n_2 \geq n_3 \geq 2$. Now, construct $\mathscr{T}_{n,i} \in \mathscr{T}_n$ for $n \geq 6$ (see Fig. 2), derived from the path $P_{n-1} : u_1u_2 \cdots u_{n-2}u_{n-1}$ by adding a new pendent edge u_iu_n at vertex u_i for $2 \leq i \leq n-3$.

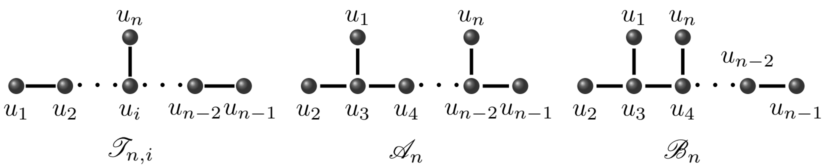

Figure 2. The trees $\mathscr{T}_{n,i}$, $\mathscr{A}_n$ and $\mathscr{B}_n$.

Define $\mathscr{A}_n \in \mathscr{T}_n$ for $n \geq 7$ (see Fig. 2), which is derived from the path $P_{n-2} : u_2u_3 \cdots u_{n-2}u_{n-1}$ by adding a new pendent edge $u_{n-2}u_n$ at vertex u_{n-2} and another pendent edge u_3u_1 at vertex u_3. Similarly, let $\mathscr{B}_n \in \mathscr{T}_n$ for $n \geq 7$ (see Fig. 2), derived from the path $P_{n-2} : u_2u_3 \cdots u_{n-2}u_{n-1}$ by adding a new pendent edge u_3u_1 at vertex u_3 and another pendent edge u_4u_n at vertex u_4.

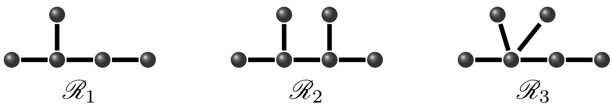

Figure 3. The graphs $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$.

Lemma 5. *[6] If $T \in \mathscr{T}_n$, where $T \ncong P_n$, then $p_4 \geq p_1$ always holds.*

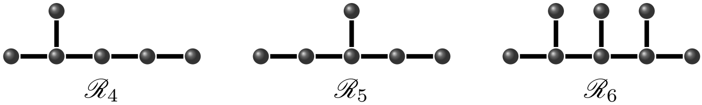

Figure 4. The graphs $\mathscr{R}_4$, $\mathscr{R}_5$ and $\mathscr{R}_6$.

As demonstrated by Tang et al. [40], for any tree $T \in \mathscr{T}_n$, $\mathcal{ESI}(T) \geq \mathcal{ESI}(P_n)$ and $\mathcal{ESI}(T) \leq \mathcal{ESI}(S_n)$, where $\mathcal{ESI}(T) = \mathcal{ESI}(P_n) \Leftrightarrow T \cong P_n$ and $\mathcal{ESI}(T) = \mathcal{ESI}(S_n) \Leftrightarrow T \cong S_n$. In this section, we now identify the trees with the second minimum and maximum Euler Sombor indices.

Theorem 1. *For any $T \in \mathscr{T}_n$, $(n \geq 5)$ with $T \ncong P_n$, we have*

$$\mathcal{ESI}(T) \geq \begin{cases} \sqrt{7} + \sqrt{19} + 2\sqrt{13} & for \quad n = 5 \\ 2\sqrt{7} + 2\sqrt{19} + \sqrt{13} & for \quad n = 6 \\ 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} & for \quad n \geq 7. \end{cases}$$

The equality holds iff $T \cong \mathscr{R}_1$ (see Fig. 3) for $n = 5$, $T \cong \mathscr{R}_5$ (see Fig. 4) for $n = 6$ and $T \cong \mathscr{T}_{n_1,n_2,n_3}$ for $n \geq 7$.

Proof. We have $T \ncong P_n$. For $n = 5$, $T \cong \mathscr{R}_1$ or $T \cong K_{1,4}$. Thus, we have

$$\mathcal{ESI}(\mathscr{R}_1) = \sqrt{7} + \sqrt{19} + 2\sqrt{13}, \quad \mathcal{ESI}(K_{1,4}) = 4\sqrt{21}.$$

Hence one can easily state that $\mathcal{ESI}(T) \geq \sqrt{7} + \sqrt{19} + 2\sqrt{13}$, with equality holds iff $T \cong \mathscr{R}_1$. If $n = 6$, $T \cong \mathscr{R}_2$ (see Fig. 3) or $T \cong \mathscr{R}_3$ (see Fig. 3) or $T \cong \mathscr{R}_4$ (see Fig. 4) or $T \cong \mathscr{R}_5$ (see Fig. 4) or $T \cong K_{1,5}$. Now, we have

$$\mathcal{ESI}(\mathscr{R}_2) = 3\sqrt{3} + 4\sqrt{13},\ \mathcal{ESI}(\mathscr{R}_3) = 3\sqrt{7} + 3\sqrt{21},$$
$$\mathcal{ESI}(\mathscr{R}_4) = \sqrt{7} + 2\sqrt{3} + \sqrt{19} + 2\sqrt{13},\ \mathcal{ESI}(\mathscr{R}_5) = 2\sqrt{7} + 2\sqrt{19} + \sqrt{13}$$
$$\text{and } \mathcal{ESI}(K_{1,5}) = 5\sqrt{31}.$$

Consequently, $\mathcal{ESI}(T) \geq 2\sqrt{7} + 2\sqrt{19} + \sqrt{13}$, where equality holds iff $T \cong \mathscr{R}_5$. Now we consider the case for $n \geq 7$. In this case $\Delta = 3$. Here $T \ncong P_n$, which implies that $p_4 \geq 1$.

Let $p_4 = 1$, since $\Delta = 3$ and $n \geq 7$, then there exist only one non pendent edge of the type $(3,2)$. Hence $T \cong \mathscr{T}_{n,2}$. Therefore,

$$\begin{aligned}\mathcal{ESI}(T) &= \sqrt{7} + 2\sqrt{13} + \sqrt{19} + 2(n-5)\sqrt{3} \\ &> 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\ &= \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),\end{aligned}$$

as $\sqrt{13} + 2\sqrt{3} > \sqrt{7} + \sqrt{19}$. Let $p_4 = 2$. Since $\Delta = 3$, then there exist exactly two non pendent edges of the types $(3,2)$ or $(3,3)$. If both non pendent edges are of $(3,2)$ type, then either $T \cong \mathscr{T}_{n,i}$, $(3 \leq i \leq n-3)$ or $T \cong \mathscr{A}_n$. Therefore

$$\begin{aligned}\mathcal{ESI}(\mathscr{T}_{n,i}) &= 2\sqrt{7} + \sqrt{13} + 2\sqrt{19} + 2(n-6)\sqrt{3} \\ &> 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\ &= \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),\end{aligned}$$

as $\sqrt{13} + 2\sqrt{3} > \sqrt{7} + \sqrt{19}$. We also have

$$\begin{aligned}\mathcal{ESI}(\mathscr{A}_n) &= 2\sqrt{19} + 4\sqrt{13} + 2(n-7)\sqrt{3} \\ &> 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\ &= \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),\end{aligned}$$

as $4\sqrt{13} > 3\sqrt{7} + \sqrt{19}$. If both non pendent edges are of $(3,3)$ type, then only possiblity is that $T \cong \mathscr{R}_6$ (see Fig. 4). Hence $\mathcal{ESI}(\mathscr{R}_6) = 5\sqrt{13} + 6\sqrt{3} > 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} = \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3})$.

If one non pendent edge is of $(3,2)$ type and another is of $(3,3)$ type, then $T \cong \mathscr{B}_n$. Therefore, we obtain

$$\begin{aligned}\mathcal{ESI}(\mathscr{B}_n) &= \sqrt{7} + 3\sqrt{13} + 3\sqrt{3} + \sqrt{19} + 2(n-7)\sqrt{3} \\ &> 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\ &= \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),\end{aligned}$$

as $3\sqrt{13} + 3\sqrt{3} > 2\sqrt{7} + 2\sqrt{19}$. Now, we consider the case only when $p_4 \geq 3$. From Lemma 5, we know that for any tree $T \in \mathscr{T}_n$, where $T \ncong P_n$,

$p_4 \geq p_1$. Therefore, we assume that $p_1 \geq 3$. Hence

$$\begin{aligned}
\mathcal{ESI}(T) =& \sum_{u_iu_j \in E(T)} \sqrt{d^2_{u_i}(T) + d^2_{u_j}(T) + d_{u_i}(T)d_{u_j}(T)} \\
\geq& p_1\sqrt{7} + p_4\sqrt{19} + 2(n-1-p_1-p_4)\sqrt{3} \\
=& p_1(\sqrt{7} - 2\sqrt{3}) + p_4(\sqrt{19} - 2\sqrt{3}) + 2(n-1)\sqrt{3} \\
\geq& p_1(\sqrt{7} - 4\sqrt{3} + \sqrt{19}) + 2(n-1)\sqrt{3} \\
\geq& 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\
=& \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),
\end{aligned}$$

as $\sqrt{7} + \sqrt{19} > 4\sqrt{3}$. Here all the equality holds iff $p_1 = p_4 = 3$ and the remaining $(n-7)$ edges are of the type $(2, 2)$, i.e., $T \cong \mathscr{T}_{n_1,n_2,n_3}$. At last we assume that $p_1 \leq 2$. For this case,

$$\begin{aligned}
\mathcal{ESI}(T) \geq& p_1\sqrt{7} + p_4\sqrt{19} + 2(n-1-p_1-p_4)\sqrt{3} \\
=& p_1(\sqrt{7} - 2\sqrt{3}) + p_4(\sqrt{19} - 2\sqrt{3}) + 2(n-1)\sqrt{3} \\
\geq& p_1(\sqrt{7} - 2\sqrt{3}) + 3(\sqrt{19} - 2\sqrt{3}) + 2(n-1)\sqrt{3} \\
=& 2(\sqrt{7} - 2\sqrt{3}) + 3(\sqrt{19} - 2\sqrt{3}) + 2(n-1)\sqrt{3} \\
>& 3\sqrt{7} + 3\sqrt{19} + 2(n-7)\sqrt{3} \\
=& \mathcal{ESI}(\mathscr{T}_{n_1,n_2,n_3}),
\end{aligned}$$

as $p_4 \geq 3$ and $12\sqrt{3} > 10\sqrt{3} + \sqrt{7}$. Hence proved. ■

Corollary. *If* $T \in \mathscr{T}_n, (n \geq 5)$ *where* $T \ncong P_n$, *then* $\mathcal{ESI}(T) > 2(n-1)\sqrt{3}$.

Proof. Since $\sqrt{7} + \sqrt{19} > 4\sqrt{3}$, from Theorem 1, we can easily derive that $\mathcal{ESI}(T) > 2(n-1)\sqrt{3}$. ■

A Double star graph with n vertices is denoted by $\mathcal{D}_{s,t}$, $(s \geq t \geq 1)$ (see Fig. 5) which is obtained from connecting the center of two stars S_{s+1} and S_{t+1}. Hence $s + t + 2 = n$.

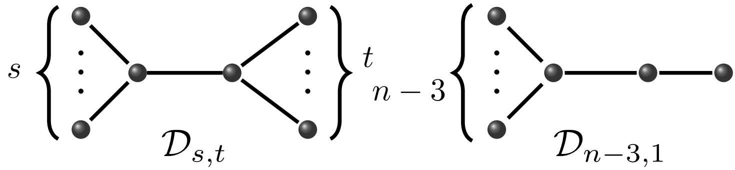

Figure 5. The graphs $\mathcal{D}_{s,t}$ and $\mathcal{D}_{n-3,1}$.

Theorem 2. *If $T \in \mathscr{T}_n$ $(n \geq 5)$ and $T \ncong S_n$. Then*

$$\mathcal{ESI}(T) \leq \sqrt{7} + \sqrt{(n-2)^2 + 2n} + (n-3)\sqrt{(n-2)^2 + n - 1},$$

with equality holds iff $T \cong \mathcal{D}_{n-3,1}$ (see Fig. 5).

Proof. Let d be a diameter of T where $T \ncong S_n$. This implies that $d \geq 3$. First we consider the case when $d = 3$. For that $T \cong \mathcal{D}_{s,t}$ where $s \geq t \geq 1$ and $s + t + 2 = n$. If $T \cong \mathcal{D}_{n-3,1}$, then the equality of the theorem holds. Otherwise, $T \ncong \mathcal{D}_{n-3,1}$. Hence $t \geq 2$. Now we obtain

$$\begin{aligned}\mathcal{ESI}(T) =& s\sqrt{(s+1)^2 + s + 2} + t\sqrt{(t+1)^2 + t + 2} \\ &+ \sqrt{(s+1)^2 + (t+1)^2 + (s+1)(t+1)} \\ =& (n-t-2)\sqrt{(n-t-1)^2 + (n-t)} + t\sqrt{(t+1)^2 + t + 2} \\ &+ \sqrt{(n-t-1)^2 + (t+1)^2 + (n-t-1)(t+1)}.\end{aligned}$$

As $t + 1 \leq n - t - 1$, we have,

$$\begin{aligned}\mathcal{ESI}(T) \leq& (n-2)\sqrt{(n-t-1)^2 + (n-t)} \\ &+ \sqrt{(n-t-1)^2 + (t+1)^2 + (n-t-1)(t+1)}. \qquad (2)\end{aligned}$$

Let $l_1(x) = \sqrt{(n-x-1)^2 + (x+1)^2 + (n-x-1)(x+1)}$, where $2 \leq x \leq \frac{n-2}{2}$. Then $l_1'(x) = \frac{2x-n+2}{2\sqrt{(x+1)^2 - n(x+1) + n^2}}$. So, one can easily verify that $l_1(x)$ is decreasing in $\left[2, \frac{n-2}{2}\right)$. Hence $l_1(x) \leq \sqrt{n^2 - 3n + 9}$. Again, we consider another function $l_2(x) = (n-2)\sqrt{(n-x-1)^2 + (n-x)}$, where

$2 \le x \le \frac{n-2}{2}$, which immediately implies $l_2(x) \le (n-2)\sqrt{n^2-5n+7}$. Hence from (2), we have

$$\mathcal{ESI}(T) \le (n-2)\sqrt{n^2-5n+7} + \sqrt{n^2-3n+9}.$$

Claim I: $\sqrt{7}+\sqrt{n^2-2n+4}+(n-3)\sqrt{n^2-3n+3} > (n-2)\sqrt{n^2-5n+7}$ $+ \sqrt{n^2-3n+9}$.

Proof claim I: Let us construct a function

$$\begin{aligned} l_3(n) =& \sqrt{n^2-2n+4} - \sqrt{n^2-3n+9} + \sqrt{7} + (n-3)\sqrt{n^2-3n+3} \\ & - (n-2)\sqrt{n^2-5n+7}. \end{aligned}$$

This is easy to say that $\sqrt{n^2-2n+4} > \sqrt{n^2-3n+9}$ for $n \ge 6$. Therefore, the expression within the first parenthesis is always positive. Now, we have to prove that

$$\sqrt{7} + (n-3)\sqrt{n^2-3n+3} > (n-2)\sqrt{n^2-5n+7},$$

i.e.,

$$28(n-3)^2(n^2-3n+3) > (n^2-3n-6)^2,$$

i.e.,

$$9n^4 - 82n^3 + 281n^2 - 432n + 240 > 0.$$

Let us consider

$$\begin{aligned} l_4(n) =& 9n^4 - 82n^3 + 281n^2 - 432n + 240 \\ =& 9n^3\left(n - \frac{82}{9}\right) + 281n\left(n - \frac{432}{281}\right) + 240. \end{aligned}$$

Consequently, $l_4(n) > 0$ for $n \ge 10$. Using Mathematica one can easily verify that $l_4(n) > 0$ for $4 \le n \le 9$. This demonstrates that **Claim I** is true.

Hence employing the outcome of **Claim I**, one can easily obtain that

$$\begin{aligned}\mathcal{ESI}(T) &\leq (n-2)\sqrt{n^2-5n+7}+\sqrt{n^2-3n+9}\\ &< \sqrt{7}+\sqrt{(n-2)^2+2n}+(n-3)\sqrt{(n-2)^2+n-1}\\ &= \mathcal{ESI}(\mathcal{D}_{n-3,1}).\end{aligned}$$

Now, we consider the case $d \geq 4$. For that case $1 \leq d_{u_i}(T) \leq n-3$ for any $u_i \in V(T)$ and for any $u_r u_s \in E(T)$, $d_{u_r}(T)+d_{u_s}(T) \leq n-1$ always holds. Let $u_1 u_2 \cdots u_d u_{d+1}$ be a diametral path in T. Then $d_{u_1}(T) = d_{u_{d+1}}(T) = 1$ and $d_{u_2}(T) + d_{u_d}(T) \leq n-1$. Therefore

$$\begin{aligned}\mathcal{ESI}(T) =&\sqrt{d^2_{u_1}(T)+d^2_{u_2}(T)+d_{u_1}(T)d_{u_2}(T)}\\ &+\sqrt{d^2_{u_d}(T)+d^2_{u_{d+1}}(T)+d_{u_d}(T)d_{u_{d+1}}(T)}\\ &+\sum_{\substack{u_iu_j\in E(T)\\(i,j)\neq(1,2),(d,d+1)}}\sqrt{d^2_{u_i}(T)+d^2_{u_j}(T)+d_{u_i}(T)d_{u_j}(T)}\\ =&\sqrt{1+d^2_{u_2}(T)+d_{u_2}(T)}+\sqrt{d^2_{u_d}(T)+1+d_{u_d}(T)}\\ &+\sum_{\substack{u_iu_j\in E(T)\\(i,j)\neq(1,2),(d,d+1)}}\sqrt{d^2_{u_i}(T)+d^2_{u_j}(T)+d_{u_i}(T)d_{u_j}(T)}.\end{aligned} \tag{3}$$

Let us consider a function $f_1(x) = \sqrt{1+x^2+x}+\sqrt{(n-x-1)^2+n-x}$, where $1 \leq x \leq n-3$. Then

$$f_1'(x) = \frac{1}{2}\left(\frac{2x+1}{\sqrt{1+x^2+x}}+\frac{1-2n+2x}{(n-x-1)^2+n-x}\right).$$

Therefore $f_1(x)$ is decreasing and increasing in $\left[1, \frac{n-1}{2}\right)$ and $\left(\frac{n-1}{2}, n-3\right]$, respectively. Hence one can easily verify that

$$f_1(x) \leq \sqrt{3}+\sqrt{(n-2)^2+n-1} < \sqrt{7}+\sqrt{(n-2)^2+2n}.$$

Therefore

$$
\begin{aligned}
&\sqrt{1+d_{u_2}^2(T)+d_{u_2}(T)}+\sqrt{d_{u_d}^2(T)+1+d_{u_d}(T)}\\
&\leq \sqrt{1+d_{u_2}^2(T)+d_{u_2}(T)}+\sqrt{(n-d_{u_2}(T)-1)^2+(n-d_{u_2}(T))}\\
&<\sqrt{7}+\sqrt{(n-2)^2+2n}.
\end{aligned}
$$

Let $f_2(x) = \sqrt{x^2+(n-x-1)^2+x(n-x-1)}$, where $1 \leq x \leq n-3$. Therefore $f_2(x)$ is decreasing and increasing in $\left[1, \frac{n-1}{2}\right)$ and $\left(\frac{n-1}{2}, n-3\right]$, respectively. Thus $f_2(x) \leq \sqrt{(n-2)^2+n-1}$. Hence for any $u_r u_s \in E(T)$, $\sqrt{d_{u_r}^2(T)+d_{u_s}^2(T)+d_{u_r}(T)d_{u_s}(T)} \leq \sqrt{(n-2)^2+n-1}$. Using all these results in (3), we have

$$
\begin{aligned}
\mathcal{ESI}(T) <&\sqrt{7}+\sqrt{(n-2)^2+2n}+(n-3)\sqrt{(n-2)^2+n-1}\\
=&\mathcal{ESI}(\mathcal{D}_{n-3,1}).
\end{aligned}
$$

This completes the proof. ■

4 Extremum Euler Sombor index of quasi-trees

Let T be a tree with n vertices, $u_0v_0 \in E(T)$, which is non pendent and $N_T(u_0)$, $N_T(v_0)$ have no common vertex, where $N_T(u_0) = \{v_0, u_1, u_2, \cdots, u_p\}$, $N_T(v_0) = \{u_0, v_1, v_2, \cdots, v_q\}$, with $p, q \geq 1$. Then T' is derived from T by transformation $\mathcal{TA}$ (see Fig. 6) such that $E(T') = E(T) - \{v_0v_i : 1 \leq i \leq q\} + \{u_0v_i : 1 \leq i \leq q\}$ with $V(T') = V(T)$.

Lemma 6. *Let T be a tree with n vertices and u_0v_0 be a non pendent edge in T. Now we obtained a new tree T' from T by $\mathcal{TA}$ rule. Then $\mathcal{ESI}(T') > \mathcal{ESI}(T)$.*

Proof. From the given condition $d_{u_0}(T) = p+1$, $d_{v_0}(T) = q+1$, $d_{u_0}(T') = p+q+1$, $d_{v_0}(T') = 1$, $d_{u_i}(T) = d_{u_i}(T')$, $1 \leq i \leq p$ and $d_{v_j}(T) =$

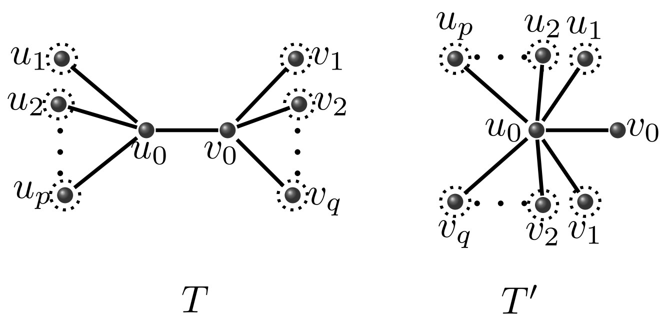

Figure 6. The transformation $\mathcal{TA} : T \to T'$.

$d_{v_j}(T'), \quad 1 \leq j \leq q$. Thus, we obtain

$$\begin{aligned}
\mathcal{ESI}(T') - \mathcal{ESI}(T) = & \sum_{u_iu_j \in E(T')} \sqrt{d_{u_i}^2(T') + d_{u_j}^2(T') + d_{u_i}(T')d_{u_j}(T')} \\
& - \sum_{u_iu_j \in E(T)} \sqrt{d_{u_i}^2(T) + d_{u_j}^2(T) + d_{u_i}(T)d_{u_j}(T)} \\
= & \sum_{i=1}^{p} \Bigg(\sqrt{(p+q+1)^2 + d_{u_i}^2(T') + (p+q+1)d_{u_i}(T')} \\
& - \sqrt{(p+1)^2 + d_{u_i}^2(T) + (p+1)d_{u_i}(T)} \Bigg) \\
& + \sum_{j=1}^{q} \Bigg(\sqrt{(p+q+1)^2 + d_{v_j}^2(T') + (p+q+1)d_{v_j}(T')} \\
& - \sqrt{(q+1)^2 + d_{v_j}^2(T) + (q+1)d_{v_j}(T)} \Bigg) \\
& + \sqrt{(p+q+1)^2 + (p+q+2)} \\
& - \sqrt{(p+q+1)^2 + (p+q+2) - pq}.
\end{aligned}$$

Note that $d_{u_i}(T) = d_{u_i}(T')$ for $1 \leq i \leq p$, and $d_{v_j}(T) = d_{v_j}(T')$ for $1 \leq j \leq q$. Therefore $\mathcal{ESI}(T') > \mathcal{ESI}(T)$. ■

Definition 1. [29] Let $\mathbf{X} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\mathbf{Y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. Then $\mathbf{X} \prec \mathbf{Y}$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $1 \leq k \leq n-1$. If $\mathbf{X} \prec \mathbf{Y}$, then $\mathbf{X}$ is said to be majorized by $\mathbf{Y}$.

Definition 2. [29] Let $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathbf{n}}$. Then $\mathbf{X} \prec_w \mathbf{Y}$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \cdots, n.$$

Here $\mathbf{X}$ is weakly majorized by $\mathbf{Y}$.

Lemma 7. *[29] Let $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathbf{n}}$ and f be any convex function. If $\mathbf{Y}$ majorizes $\mathbf{X}$, then*

$$\big(f(x_1), f(x_2), \cdots, f(x_n)\big) \prec_w \big(f(y_1), f(y_2), \cdots, f(y_n)\big).$$

Theorem 3. *If $G \in \mathcal{Q}_n$, then $\mathcal{ESI}(G) \geq 2\sqrt{3}\, n$, where equality holds iff $G \cong C_n$.*

Proof. If $G \cong C_n$, then $\mathcal{ESI}(G) = 2\sqrt{3}\, n$, and hence the equality holds. Now we consider that $G \ncong C_n$. Since $G \in \mathcal{Q}_n$, then there must be a vertex $u_n \in V(G)$ such that $T = G - u_n$ is a tree with $n-1$ vertices. Hence one can easily verify that $d_{u_i}(G) \geq d_{u_i}(T)$ for $1 \leq i \leq n-1$, also there must be a vertex u_s $(s \neq n)$ in $V(G)$ such that $d_{u_s}(G) > d_{u_s}(T)$. Let the vertex u_n be connected with $u_1, u_2, \cdots, u_k$ in G. Since $G \in \mathcal{Q}_n$, then $k \geq 2$, where $d_{u_n}(G) = k$. Therefore

$$\begin{aligned}
\mathcal{ESI}(G) =& \sum_{i=1}^{k} \sqrt{k^2 + (d_{u_i}(T)+1)^2 + k\,(d_{u_i}(T)+1)} \\
& + \sum_{u_i u_j \in E(T)} \sqrt{d_{u_i}^2(G) + d_{u_j}^2(G) + d_{u_i}(G) d_{u_j}(G)} \\
>& \sum_{i=1}^{k} \sqrt{k^2 + (d_{u_i}(T)+1)^2 + k\,(d_{u_i}(T)+1)} \\
& + \sum_{u_i u_j \in E(T)} \sqrt{d_{u_i}^2(T) + d_{u_j}^2(T) + d_{u_i}(T) d_{u_j}(T)}.
\end{aligned} \tag{4}$$

Now we consider following two cases.

Case 1. T is a path with $n-1$ vertices.

If $k=2$, then u_n is connected with u_r and u_s in P_{n-1} such that $d_{u_r}(T) \geq d_{u_s}(T)$. Let $T_1 = G - u_r u_n$, then $T_1 \in \mathscr{T}_n$. Again $G \ncong C_n$, therefore, $d_{u_r}(T_1) = 3$ and hence $T_1 \ncong P_n$. Using Corollary 3, we have $\mathcal{ESI}(T_1) > 2(n-1)\sqrt{3}$. Hence

$$\begin{aligned}\mathcal{ESI}(G) =& \sqrt{4 + (d_{u_s}(T_1)+1)^2 + 2\,(d_{u_s}(T)+1)} \\ &+ \sum_{u_i u_j \in E(T_1)} \sqrt{d^2_{u_i}(T_1) + d^2_{u_j}(T_1) + d_{u_i}(T_1) d_{u_j}(T_1)} \\ &> 2\sqrt{3} + 2(n-1)\sqrt{3} = 2\sqrt{3}\,n.\end{aligned}$$

Now we consider $k \geq 3$. Thus $\mathcal{ESI}(P_{n-1}) = 2\sqrt{7} + 2(n-4)\sqrt{3}$. From (4), we have

$$\begin{aligned}\mathcal{ESI}(G) >& k\sqrt{19} + 2\sqrt{7} + 2(n-4)\sqrt{3} \\ \geq& 3\sqrt{19} + 2\sqrt{7} + 2(n-4)\sqrt{3} \\ >& 2\sqrt{3}\,n\end{aligned}$$

as $3\sqrt{19} + 2\sqrt{7} > 8\sqrt{3}$.

Case 2. $T \ncong P_{n-1}$.

Using Corollary 3 in (4), we have

$$\begin{aligned}\mathcal{ESI}(G) >& 2k\sqrt{3} + 2(n-2)\sqrt{3} \\ \geq& 4\sqrt{3} + 2(n-2)\sqrt{3} \\ =& 2\sqrt{3}\,n.\end{aligned}$$

Hence the proof is completed. ■

Theorem 4. *If $G \in \mathcal{Q}_n$, then $\mathcal{ESI}(G) \leq \sqrt{3}(n-1) + 2(n-2)\sqrt{n^2+3}$, where equality holds iff $G \cong \mathscr{R}$ (see Fig. 1).*

Proof. Since $G \in \mathcal{Q}_n$, then there must be a vertex $u_n \in V(G)$ with $d_{u_n}(G) = k$ such that $T = G - u_n$ is a tree with $n-1$ vertices. It is clear that $T \in \mathscr{T}_{n-1}$. Let $u_n u_i \in E(G)$ for $i = 1, 2, \cdots, k$. Note that

$2 \leq k \leq n-1$. Using Lemma 6, we have

$$\mathcal{ESI}(G) \leq \sum_{i=1}^{n-1} \sqrt{(n-1)^2 + (d_{u_i}(T)+1)^2 + (n-1)(d_{u_i}(T)+1)} + \sum_{u_iu_j \in E(T)} \sqrt{(d_{u_i}(T)+1)^2 + (d_{u_j}(T)+1)^2 + (d_{u_i}(T)+1)(d_{u_j}(T)+1)} \tag{5}$$

where equality holds iff $d_{u_n}(G) = n-1$. Since $T \in \mathscr{T}_{n-1}$, for any $u_ru_s \in E(T)$, $d_{u_r}(T) + d_{u_s}(T) \leq n-1$ always holds. Hence $1 \leq d_{u_r}, d_{u_s} \leq n-2$. Again

$$\sqrt{(d_{u_r}(T)+1)^2 + (d_{u_s}(T)+1)^2 + (d_{u_i}(T)+1)(d_{u_j}(T)+1)}$$
$$\leq \sqrt{(d_{u_r}(T)+1)^2 + (n-d_{u_r}(T))^2 + (d_{u_r}(T)+1)(n-d_{u_r}(T))}.$$

Let us consider a function $g_1(t) = t^2 + (n-t+1)^2 + t(n-t+1)$, where $2 \leq t \leq n-1$. Then $g_1(t)$ is decreasing and increasing in $\left[2, \frac{n+1}{2}\right)$ and $\left(\frac{n+1}{2}, n-1\right]$, respectively. Hence for any $u_ru_s \in E(T)$, we derive

$$\sqrt{(d_{u_r}(T)+1)^2 + (d_{u_s}(T)+1)^2 + (d_{u_i}(T)+1)(d_{u_j}(T)+1)}$$
$$\leq \sqrt{(d_{u_r}(T)+1)^2 + (n-d_{u_r}(T))^2 + (d_{u_r}(T)+1)(n-d_{u_r}(T))}$$
$$\leq \sqrt{n^2+3}.$$

Therefore, we obtain

$$\sum_{u_iu_j \in E(T)} \sqrt{(d_{u_i}(T)+1)^2 + (d_{u_j}(T)+1)^2 + (d_{u_i}(T)+1)(d_{u_j}(T)+1)}$$
$$\leq (n-2)\sqrt{n^2+3}.$$

Let us consider another function $g_2(t) = \sqrt{t^2 + (n-1)^2 + (n-1)t}$, where $2 \leq t \leq n-1$.

One can easily verify that $g_2''(t) = \dfrac{3(n-1)^2}{4\sqrt{t^2 + (n-1)^2 + (n-1)t}} > 0$ for

$2 \leq t \leq n-1$. This implies that $g_2(t)$ is a convex function on the defined interval $[2, n-1]$. Since $T \in \mathscr{T}_{n-1}$, we have $\sum_{i=1}^{n-1}(d_{u_i}(T)+1) = 2(n-2)+n-1$. Hence using Definition 1, we get

$$\big(d_{u_1}(T)+1, d_{u_2}(T)+1, \cdots, d_{u_{n-1}}(T)+1\big) \prec \big(n-1, \underbrace{2, 2, \ldots, 2}_{n-2}\big).$$

Then using Lemma 7, one can easily obtain that

$$\begin{aligned}&\big(g_2(d_{u_1}(T)+1), g_2(d_{u_2}(T)+1), \cdots, g_2(d_{u_{n-1}}(T)+1)\big)\\ &\prec_w \big(g_2(n-1), \underbrace{g_2(2), g_2(2), \ldots, g_2(2)}_{n-2}\big).\end{aligned}$$

Finally by Definition 2, we get

$$\begin{aligned}\sum_{i=1}^{n-1} g_2(d_{u_i}(T)+1) &\leq g_2(n-1) + \underbrace{g_2(2) + g_2(2), \ldots + g_2(2)}_{n-2}\\ &= g_2(n-1) + (n-2)g_2(2)\\ &= \sqrt{3}(n-1) + (n-2)\sqrt{n^2+3}.\end{aligned}$$

Hence from (5), we have

$$\begin{aligned}\mathcal{ESI}(G) &\leq \sqrt{3}(n-1) + (n-2)\sqrt{n^2+3} + (n-2)\sqrt{n^2+3}\\ &= \sqrt{3}(n-1) + 2(n-2)\sqrt{n^2+3}\end{aligned}$$

with equality holds iff $T \cong S_{n-1}$ and $d_{u_n}(G) = n-1$, i.e., iff $G \cong \mathscr{R}$ (see Fig. 1). ■

5 Extremal trees for Euler Sombor index with fixed domination number

This section contains sharp lower and upper bounds of the Euler Sombor index for the class of all trees in terms of graph order and domination number. The extremal trees are also characterized. In figures, dotted circle surrounding a vertex indicates that the degree of the corresponding vertex is unknown.

5.1 Lower bound

Let $\mathscr{T}$ be a collection of trees with $P_{3k} \in \mathscr{T}$, where $k \geq 1$. Now we construct the set $\mathscr{T}$ by recursion as follows.

(1) If $T' \in \mathscr{T}$, with $u \in V(T')$ such that $d_u(T') = 1$, then attaching a path $P = u_1u_2 \cdots u_{3p}$ to u, we get a new tree $T \in \mathscr{T}$.

(2) If $T' \in \mathscr{T}$ with $v \in \mathbb{D}_1$ of T' such that $N_{T'}(v) = \{v_1, v_2\}$ where $d_{v_1}(T^{'}) = d_{v_2}(T^{'}) = 2$, then attaching a path $P = u_1u_2 \cdots u_{3p+1}$ to v, we obtain a new tree $T \in \mathscr{T}$.

Let $\mathscr{T}_n^{\Omega}$ represent the set of all trees in $\mathscr{T}$ with n vertex and domination number Ω. For simplicity in the calculations, we take

$$g(n, \Omega) = (3\sqrt{19} + \sqrt{7} - 6\sqrt{3})n + (24\sqrt{3} - 9\sqrt{19} - 3\sqrt{7})\Omega + 2\sqrt{7} - 6\sqrt{3}.$$

Lemma 8. *If $T \in \mathscr{T}_n^{\Omega}$, then $\mathcal{ESI}(T) = g(n, \Omega)$.*

Proof. Since $T \in \mathscr{T}_n^{\Omega}$, one can easily obtain that $\hbar_1 + \hbar_2 + \hbar_3 = n$, $\hbar_1 + 2\hbar_2 + 3\hbar_3 = 2(n-1)$ and $\hbar_3 + \dfrac{n - 4\hbar_3}{3} = \Omega$. Hence $\hbar_1 = n - 3\Omega + 2$, $\hbar_2 = 6\Omega - n - 2$ and $\hbar_3 = n - 3\Omega$. By the definition of Euler Sombor index and $\mathscr{T}$, we have

$$\begin{aligned}\mathcal{ESI}(T) =& 3\hbar_3\mho(2,3) + \hbar_1\mho(1,2) + (n - 1 - 3\hbar_3 - \hbar_1)\mho(2,2)\\ =& (3\sqrt{19} + \sqrt{7} - 6\sqrt{3})n + (24\sqrt{3} - 9\sqrt{19} - 3\sqrt{7})\Omega + 2\sqrt{7} - 6\sqrt{3}\\ =& g(n, \Omega).\end{aligned}$$

■

Theorem 5. *If $T \in \mathscr{T}(n,\Omega)$, then $\mathcal{ESI}(T) \geq g(n,\Omega)$. The equality holds iff $T \cong \mathscr{T}_n^{\Omega}$.*

Proof. To prove the result, we will use mathematical induction on $n \geq 2$. When $n = 2$ or $n = 3$, we have $\mathcal{ESI}(P_2) = \sqrt{3} > g(2,1) = 6\sqrt{3} - 3\sqrt{19} + \sqrt{7}$ and $\mathcal{ESI}(P_3) = 2\sqrt{7} = g(3,1)$. If $n = 4$, either T is a path or a star. If T is a path, then $\mathcal{ESI}(P_4) = 2(\sqrt{7} + \sqrt{3}) > g(4,2) = 18\sqrt{3} - 6\sqrt{19}$. If T is a star, then $\mathcal{ESI}(S_4) = 3\sqrt{13} > g(4,1) = 3\sqrt{19} + 3\sqrt{7} - 6\sqrt{3}$. Now, we have to prove this result, when $n \geq 5$. Let $u_1 u_2 \cdots u_{d+1}$ be a diametral path in T and the result is true for any tree with $n-1$ vertices. If $d = 2$, the only possibility is $T \cong S_n$ and $\Omega(S_n) = 1$. Hence $\mathcal{ESI}(S_n) - g(n,1) = (n-1)\sqrt{(n-1)^2 + n} - (3\sqrt{19} + \sqrt{7} - 6\sqrt{3})n - 18\sqrt{3} + 9\sqrt{19} + \sqrt{7}$. Using Lemma 3, for $n \geq 5$, we have $\mathcal{ESI}(S_n) > g(n,1)$ and the theorem holds. Now we take $d \geq 3$. For this, suppose $d_{u_2}(T) = h$, $N_T(u_2) = \{u_1, u_3, x_1, x_2, \cdots, x_{h-2}\}$, $d_{u_3}(T) = k$, $N_T(u_3) = \{u_2, u_4, v_1, v_2, \cdots, v_{k-2}\}$, where $h, k \geq 2$. Here we consider two cases.

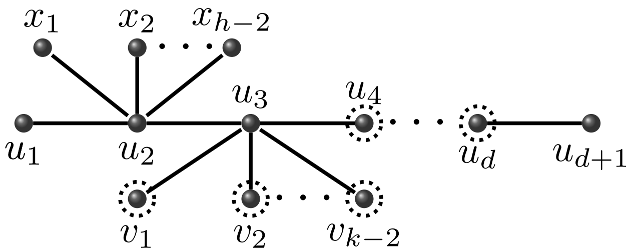

Figure 7. Tree for **Case 1**.

Case 1. $d_{u_2}(T) = h \geq 3$ (see Fig. 7). Note that, $d_{x_i}(T) = 1$, $\forall i \in \{1, 2, \cdots, h-2\}$. We consider $T_1 = T - u_1$. Therefore $d_{u_2}(T_1) = h - 1$ and $T_1 \in \mathscr{T}(n-1, \Omega)$. From induction hypothesis, we have

$$\mathcal{ESI}(T) = \mathcal{ESI}(T_1) + \mho(h,1) + \sum_{i=1}^{h-2}\Big(\mho(d_{u_2}(T),1) - \mho(d_{u_2}(T_1),1)\Big)$$
$$+ \mho(h,k) - \mho(h-1,k)$$

$$\geq \mathcal{ESI}(T_1) + \mho(h,1) + (h-2)\big(\mho(h,1) - \mho(h-1,1)\big)$$
$$+ \mho(h,k) - \mho(h-1,k)$$
$$\geq g(n-1,\Omega) + \sqrt{h^2+1+h} + (h-2)\big(\sqrt{h^2+1+h}$$
$$- \sqrt{h^2-h+1}\big) + \sqrt{h^2+k^2+hk} - \sqrt{(h-1)^2+k^2+(h-1)k}.$$

Case 1.1. $h \geq 4$.

Using Lemma 2, we have

$$\begin{aligned}\mathcal{ESI}(T) >& g(n-1,\Omega) + \sqrt{h^2+1+h}\\ &+ (h-2)\big(\sqrt{h^2+1+h} - \sqrt{h^2-h+1}\big)\\ \geq& g(n-1,\Omega) + \sqrt{21} + 2(\sqrt{21} - \sqrt{13})\\ =& g(n,\Omega) - (3\sqrt{19} + \sqrt{7} - 6\sqrt{3}) + 3\sqrt{21} - 2\sqrt{13}\\ \approx& g(n,\Omega) + 1.2065\\ >& g(n,\Omega).\end{aligned}$$

Case 1.2. $h = 3$, $k \leq 4$.

By Lemma 1, we have

$$\begin{aligned}\mathcal{ESI}(T) \geq& g(n-1,\Omega) + \sqrt{h^2+1+h}\\ &+ (h-2)\big(\sqrt{h^2+1+h} - \sqrt{h^2-h+1}\big)\\ &+ \sqrt{h^2+16+4h} - \sqrt{h^2+2h+13}.\end{aligned}$$

Since $h = 3$, we obtain

$$\begin{aligned}\mathcal{ESI}(T) \geq& g(n-1,\Omega) + \sqrt{13} + (\sqrt{13} - \sqrt{7}) + \sqrt{37} - \sqrt{28}\\ =& g(n,\Omega) + 2\sqrt{13} - 4\sqrt{7} + \sqrt{37} - 3\sqrt{19} + 6\sqrt{3}\\ \approx& g(n,\Omega) + 0.0265\\ >& g(n,\Omega).\end{aligned}$$

Case 1.3. $h = 3$ and $k \geq 5$.

In this case, we consider a tree $T_1' = T - \{u_1, u_2, x_1\}$. Let there exist $\mathbb{D}_1$ in

T s.t $u_2 \in \mathbb{D}_1$ and $u_3 \in \mathbb{D}_1$ or $u_3 \in N_T[\mathbb{D}_1 \setminus \{u_2\}]$, then $\Omega(T) = \Omega(T_1') + 1$, $d_{u_3}(T_1') = k - 1$ and $T_1' \in \mathscr{T}(n-3, \Omega - 1)$. Hence using Lemma 2, we get

$$\begin{aligned}
\mathcal{ESI}(T) =& \mathcal{ESI}(T_1') + \mho(3,k) + 2\mho(3,1) + \mho(k, d_{u_4}) - \mho(k-1, d_{u_4}) \\
&+ \sum_{i=1}^{k-2} \Big(\mho(k, d_{v_i}(T)) - \mho(k-1, d_{v_i}(T_1')) \Big) \\
>& \mathcal{ESI}(T_1') + \sqrt{9 + k^2 + 3k} + 2\sqrt{13} \\
=& g(n-3, \Omega - 1) + \sqrt{9 + k^2 + 3k} + 2\sqrt{13} \\
=& g(n, \Omega) - 6\sqrt{3} + 2\sqrt{13} + \sqrt{9 + k^2 + 3k} \\
>& g(n, \Omega) - 6\sqrt{3} + 2\sqrt{13} + 7 \\
\approx& g(n, \Omega) + 3.8188 \\
>& g(n, \Omega).
\end{aligned}$$

Case 2. $d_{u_2}(T) = 2$.
Case 2.1 $d_{u_3}(T) = k \geq 3$.
Suppose, $d_{u_4}(T) = t$. Now, by case 1, we take $d_{v_i}(T) \leq 2$, $1 \leq i \leq k-2$. Consider $T_2 = T - \{u_1, u_2\}$. Since there exist $\mathbb{D}_1$ in T such that $u_2 \in \mathbb{D}_1$ and $u_3 \in \mathbb{D}_1$ or $u_3 \in N_T[\mathbb{D}_1 \setminus \{u_2\}]$, then $\Omega(T) = \Omega(T_2) + 1$, $d_{u_3}(T_2) = k-1$ and $T_2 \in \mathscr{T}(n-2, \Omega - 1)$. Using Lemmas 1 and 2, we have

$$\begin{aligned}
&\mathcal{ESI}(T) = \mathcal{ESI}(T_2) + \mho(2,k) + \mho(2,1) + \mho(k, d_{u_4}(T)) \\
&\quad - \mho(k-1, d_{u_4}(T_2)) + \sum_{i=1}^{k-2} \Big(\mho(k, d_{v_i}(T)) - \mho(k-1, d_{v_i}(T_2)) \Big) \\
&\geq g(n-2, \Omega - 1) + \sqrt{4 + k^2 + 2k} + \sqrt{7} + \sqrt{k^2 + t^2 + kt} \\
&\quad - \sqrt{(k-1)^2 + t^2 + (k-1)t} + (k-2)\big(\sqrt{k^2 + 4 + 2k} - \sqrt{k^2 + 3}\big) \\
&> g(n-2, \Omega - 1) + \sqrt{4 + k^2 + 2k} + \sqrt{7} \\
&\quad + (k-2)\big(\sqrt{k^2 + 4 + 2k} - \sqrt{k^2 + 3}\big) \\
&\geq g(n-2, \Omega - 1) + \sqrt{19} + \sqrt{7} + (\sqrt{19} - 2\sqrt{3})
\end{aligned}$$

$$=g(n,\Omega)+5\sqrt{19}+2\sqrt{7}-14\sqrt{3}$$

$$\approx g(n,\Omega)+2.8372>g(n,\Omega).$$

Case 2.2. $d_{u_3}(T)=k=2$.
Denote $N_T(u_4)=\{u_3,u_5,z_1,z_2,\cdots,z_{t-2}\}$ and $d_{u_5}(T)=s$. For $1\leq i\leq t-2$, if $z',z''\in V(T)$ such a way that $z'\in N_T(z_i)$ and $z''\in N_T(z')$, therefore $z''z'z_iu_4u_5\cdots u_{d+1}$ is a diameter of T. If z_i is a support vertex with $d_{z_i}(T)\geq 3$, as in Case 1, we can demonstrate that $\mathcal{ESI}(T)>g(n,\Omega)$. Now we assume that $d_{z_i}(T)\leq 2$ for $1\leq i\leq t-2$.
Case 2.2.1 $d_{u_4}(T)=t\geq 3$. Suppose $T_3=T-\{u_1,u_2,u_3\}$. Then $T_3\in\mathscr{T}(n-3,\Omega-1)$ and $d_{u_4}(T_3)=t-1$. Using Lemmas 1 and 2, we have

$$\begin{aligned}\mathcal{ESI}(T)=&\mathcal{ESI}(T_3)+\mho(2,t)+\mho(2,2)+\mho(2,1)+\mho(t,s)-\mho(t-1,s)\\&+\sum_{i=1}^{t-2}\Big(\mho(t,d_{z_i}(T))-\mho(t-1,d_{z_i}(T_3))\Big)\\\geq&g(n-3,\Omega-1)+\sqrt{4+t^2+2t}+2\sqrt{3}+\sqrt{7}+\sqrt{t^2+s^2+ts}\\&-\sqrt{(t-1)^2+s^2+(t-1)s}+(t-2)\big(\sqrt{t^2+4+2t}-\sqrt{t^2+3}\big)\\>&g(n-3,\Omega-1)+\sqrt{4+t^2+2t}+2\sqrt{3}+\sqrt{7}\\&+(t-2)\big(\sqrt{t^2+4+2t}-\sqrt{t^2+3}\big)\\\geq&g(n-3,\Omega-1)+\geq\sqrt{19}+2\sqrt{3}+\sqrt{7}+(\sqrt{19}-2\sqrt{3})\\=&g(n,\Omega)+2\sqrt{19}+\sqrt{7}-6\sqrt{3}\\\approx&g(n,\Omega)+0.9712\\>&g(n,\Omega).\end{aligned}$$

Case 2.2.2. $d_{u_4}(T)=t=2$. We now wish to demonstrate this outcome in light of weather $u_4\in\mathbb{D}_1$.
Case 2.2.2.1 There exist $\mathbb{D}_1$ with $u_4\in\mathbb{D}_1$ in T.
Let $T_4=T-\{u_1,u_2\}$. Then $T_4\in\mathscr{T}(n-2,\Omega-1)$. Hence

$$\mathcal{ESI}(T)=\mathcal{ESI}(T_4)+\mho(2,2)-\mho(2,1)+\mho(2,2)+\mho(2,1)$$

$$
\begin{aligned}
&\geq g(n-2,\Omega-1)+4\sqrt{3}=g(n,\Omega)+3\sqrt{19}-8\sqrt{3}+\sqrt{7}\\
&\approx g(n,\Omega)+1.866\\
&> g(n,\Omega).
\end{aligned}
$$

Case 2.2.2.2. Suppose $u_4 \notin \mathbb{D}_1$ in T.
Consider $N_T(u_5)=\{u_4,w_1,w_2,\cdots,w_{s-1}\}$, where $s\geq 2$.

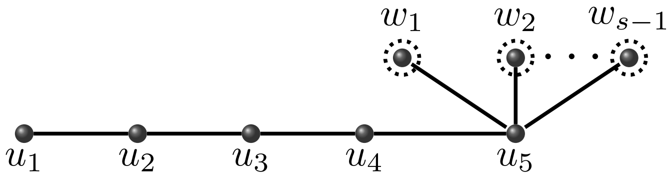

Figure 8. Tree for **Case 2.2.2.2.**.

Case 2.2.2.2.1 $d_{u_5}(T)=s=2$ (see Fig. 8). In this case, we set $T_5 = T-\{u_1,u_2,u_3\}$. It is clear that $T_5\in\mathscr{T}(n-3,\Omega-1)$. Thus

$$
\begin{aligned}
\mathcal{ESI}(T)=&\mathcal{ESI}(T_5)+\mho(2,2)-\mho(2,1)+2\mho(2,2)+\mho(2,1)\\
\geq& g(n-3,\Omega-1)+3\mho(2,2)\\
=& g(n-3,\Omega-1)+6\sqrt{3}\\
=& g(n,\Omega).
\end{aligned}
$$

Where equality holds only if $T_5\in\mathscr{T}_{n-3}^{\Omega-1}$, which gives $T\in\mathscr{T}_n^{\Omega}$.
Case 2.2.2.2.2 $d_{u_5}(T)=s\geq 3$.
Case 2.2.2.2.2.1. For each $i\in\{1,2,\cdots,s-1\}$, $d_{w_i}(T)\leq 2$.
Set $T_6=T-\{u_1,u_2,u_3,u_4\}$. Hence $T_6\in\mathscr{T}(n-4,\Omega-1)$ and $d_{u_5}(T_6)=s-1$. Therefore, using Lemma 2, we have

$$
\begin{aligned}
\mathcal{ESI}(T)=&\mathcal{ESI}(T_6)+\mho(2,s)+2\mho(2,2)+\mho(2,1)\\
&+\sum_{i=1}^{s-1}\Big(\mho(s,d_{w_i}(T))-\mho(s-1,d_{w_i}(T_6))\Big)\\
\geq& g(n-4,\Omega-1)+\sqrt{4+s^2+st}+4\sqrt{3}+\sqrt{7}\\
&+(s-1)\big(\sqrt{s^2+4+2s}-\sqrt{s^2+3}\big)
\end{aligned}
$$

$$\geq g(n-4,\Omega-1)+\sqrt{19}+4\sqrt{3}+\sqrt{7}+2(\sqrt{19}-2\sqrt{3})$$
$$=g(n,\Omega).$$

Equality occurs iff $T_6 \in \mathscr{T}_{n-4}^{\Omega-1}$ and $d_{u_5}(T) = s = 3$, $d_{w_i}(T) = 2$ for each $i \in \{1, 2, \cdots, s-1\}$. This implies $T \cong \mathscr{T}_n^{\Omega}$.

Case 2.2.2.2.2.2 Let $\ell = \max\{d_{w_i}(T) : 1 \leq i \leq s-1\} \geq 3$.

Without losing generality, we assume that $d_{w_1}(T) = \ell$. Suppose $N_T(w_1) = \{u_5, t_1, t_2, \cdots, t_{\ell-1}\}$. Now $d_{t_i}(T) \leq 2$ for $1 \leq i \leq \ell - 1$. We take two distinct components T_7 with n_1 vertices and T_8 with n_2 vertices in $T - u_5w_1$, containing the vertices u_5 and w_1, respectively. Then $n_1 + n_2 = n$ and $\Omega(T_7) + \Omega(T_8) = \Omega(T)$, where $d_{u_5}(T_7) = s - 1$ and $d_{w_1}(T_8) = \ell - 1$. Using Lemmas 1 and 2, we have

$$\begin{aligned}
\mathcal{ESI}(T) =&\mathcal{ESI}(T_7)+\mathcal{ESI}(T_8)+\mho(s,2)+\mho(s-1,2)+\mho(\ell,s)\\
&+\sum_{i=2}^{s-1}\Big(\mho(s,d_{w_i}(T))-\mho(s-1,d_{w_i}(T_7))\Big)\\
&+\sum_{j=1}^{\ell-1}\Big(\mho(\ell,d_{t_j}(T))-\mho(\ell-1,d_{t_j}(T_8))\Big)\\
\geq& g(n,\Omega)+2\sqrt{7}-6\sqrt{3}+\sqrt{s^2+4+2s}-\sqrt{s^2+3}\\
&+\sqrt{\ell^2+s^2+\ell s}+(s-2)\big(\sqrt{s^2+\ell^2}-\sqrt{(s-1)^2+\ell^2}\big)\\
&+(\ell-1)\big(\sqrt{\ell^2+4+2\ell}-\sqrt{\ell^2+3}\big)\\
>& g(n,\Omega)+2\sqrt{7}-6\sqrt{3}+\sqrt{s^2+4+2s}-\sqrt{s^2+3}\\
&+\sqrt{\ell^2+s^2+\ell s}+(\ell-1)\big(\sqrt{\ell^2+4+2\ell}-\sqrt{\ell^2+3}\big)\\
\geq& g(n,\Omega)+2\sqrt{7}-6\sqrt{3}+\sqrt{19}-2\sqrt{3}+3\sqrt{3}+2(\sqrt{19}-2\sqrt{3})\\
\approx& g(n,\Omega)+2.7797\\
>& g(n,\Omega).
\end{aligned}$$

Hence the proof is done. ■

5.2 Upper bound

Let $\mathscr{T}_{n,\Omega} \in \mathscr{T}(n,\Omega)$ be the collection of trees derived from $S_{n-\Omega+1}$ by connecting $\Omega-1$ isolated vertices to it's $\Omega-1$ pendent vertices (see Fig. 9).

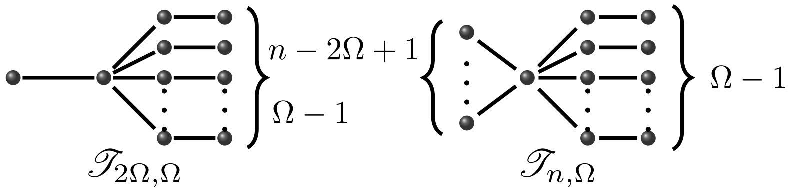

Figure 9. The graphs $\mathscr{T}_{2\Omega,\Omega}$ and $\mathscr{T}_{n,\Omega}$.

For simplification, we consider

$$\begin{aligned}\mathscr{H}(n,\Omega) =&(n-2\Omega+1)\sqrt{(n-\Omega)^2+1+(n-\Omega)}\\ &+(\Omega-1)\sqrt{(n-\Omega)^2+4+2(n-\Omega)}+(\Omega-1)\sqrt{7}.\end{aligned}$$

Theorem 6. *If $T \in \mathscr{T}(n,\Omega)$, then $\mathcal{ESI}(T) \leq \mathscr{H}(n,\Omega)$. The equality holds iff $T \cong \mathscr{T}_{n,\Omega}$.*

Proof. If $n=2$ or $n=3$, we have $\mathcal{ESI}(P_2)=\sqrt{3}=\mathscr{H}(2,1)$ and $\mathcal{ESI}(P_3)=2\sqrt{7}=\mathscr{H}(3,1)$. If $n=4$, then either T is a path or a star. If T is a path, then $\mathcal{ESI}(P_4)=2(\sqrt{7}+\sqrt{3})=\mathscr{H}(4,2)$. If T is a star, then $\mathcal{ESI}(S_4)=3\sqrt{13}=\mathscr{H}(4,1)$. Now, we have to prove this result, when $n \geq 5$. Let $u_1u_2\cdots u_{d+1}$ be a diametral path in T and the result is true for any tree with $n-1$ vertices. If $d=2$, the only possibility is $T \cong S_n$ and $\Omega(S_n)=1$. Hence $\mathcal{ESI}(S_n)=(n-1)\sqrt{n^2-n+1}=\mathscr{H}(n,1)$. So, we consider $d \geq 3$ and $\Omega(T) \geq 2$. Let us consider $d_{u_2}(T)=h$, $N_T(u_2)=\{u_1,u_3,x_1,x_2,\cdots,x_{h-2}\}$, $d_{u_3}(T)=k$, $N_T(u_3)=\{u_2,u_4,v_1,v_2,\cdots,v_{k-2}\}$, where $h,k \geq 2$. If $T_1=T-u_1$, then $d_{u_2}(T_1)=h-1$. Now we take into consideration the next two cases.
Case 1. $\Omega(T_1)=\Omega(T)$.

By Lemmas 1, 2, it follows th

$$
\begin{aligned}
\mathcal{ESI}(T) =&\mathcal{ESI}(T_1) + \mho(h,1) + \mho(h,k) - \mho(h-1,k) + \sum_{i=1}^{h-2}\Big(\mho(h,1) - \mho(h-1,1)\Big)\\
\leq&\mathscr{H}(n-1,\Omega) + \sqrt{h^2+1+h} + \sqrt{h^2+k^2+hk} - \sqrt{(h-1)^2+k^2+k(h-1)}\\
&+ (h-2)\big(\sqrt{h^2+1+h} - \sqrt{h^2-h+1}\big)\\
\leq&\mathscr{H}(n,\Omega) - \sqrt{h^2+3} + (h-2)\big(\sqrt{h^2+1+h} - \sqrt{h^2-h+1}\big)\\
&- (n-2\Omega+1)\big(\sqrt{(n-\Omega)^2+1+(n-\Omega)} - \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)}\big)\\
&- (\Omega-1)\big(\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)}\big)\\
&- \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)} + \sqrt{h^2+1+h} + \sqrt{h^2+4+2h}.
\end{aligned}
$$

Since, $\Omega \leq \dfrac{n-h+2}{2}$, i.e., $h \leq n-2\Omega+2$, we have

$$
\begin{aligned}
\mathcal{ESI}(T) \leq&\mathscr{H}(n,\Omega) - (n-2\Omega+1)(\sqrt{(n-\Omega)^2+1+(n-\Omega)} - \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)})\\
&- (\Omega-1)(\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)})\\
&- \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)} + \sqrt{(n-2\Omega+2)^2+1+(n-2\Omega+2)}\\
&+ \sqrt{(n-2\Omega+2)^2+4+2(n-2\Omega+2)} - \sqrt{(n-2\Omega+1)^2+4+2(n-2\Omega+1)}\\
&+ (n-2\Omega)(\sqrt{(n-2\Omega+2)^2+1+(n-2\Omega+2)} - \sqrt{(n-2\Omega+1)^2+(n-2\Omega+2)}).
\end{aligned}
$$

If $\Omega = 2$, $\mathcal{ESI}(T) \leq \mathscr{H}(n,\Omega)$. The equalities hold iff $k=2$ and $n=h+2$, which implies that $T \cong \mathscr{T}_{n,2}$.
If $\Omega \geq 3$, we have $n-\Omega > n-2\Omega+2$. Using Lemma 2, from above we have

$$
\begin{aligned}
\mathcal{ESI}(T) \leq&\mathscr{H}(n,\Omega) + (n-2\Omega+1)\Big((\sqrt{(n-2\Omega+2)^2+1+(n-2\Omega+2)}\\
&- \sqrt{(n-2\Omega+1)^2+1+(n-2\Omega+1)}) - (\sqrt{(n-\Omega)^2+1+(n-\Omega)}\\
&- \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)})\Big)\\
&+ (\Omega-2)(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)})\\
&+ \sqrt{(n-2\Omega+1)^2+1+(n-2\Omega+1)} - \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)}\\
&+ (\sqrt{(n-2\Omega+2)^2+4+2(n-2\Omega+2)} - \sqrt{(n-2\Omega+1)^2+4+2(n-2\Omega+1)})\\
&- (\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)})\\
=&\mathscr{H}(n,\Omega) + (n-2\Omega+1)\Big(\xi_1(n-2\Omega+2) - \xi_1(n-\Omega)\Big)\\
&+ (\Omega-2)(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)})\\
&+ \sqrt{(n-2\Omega+1)^2+1+(n-2\Omega+1)} - \sqrt{(n-\Omega-1)^2+1+(n-\Omega-1)}\\
&+ \Big(\xi_2(n-2\Omega+2) - \xi_2(n-\Omega)\Big)\\
<&\mathscr{H}(n,\Omega).
\end{aligned}
$$

Case 2. $\Omega(T_1) = \Omega(T) - 1$.

In this case, we take $h = 2$. Otherwise $u_2 \in \mathbb{D}_1$ and which gives that $\Omega(T_1) = \Omega(T)$. If $k = n - \Omega$, then $T \cong \mathscr{T}_{n,\Omega}$ and the result follows. So, we consider that $k \leq n - \Omega - 1$. By Case 1, we presume that $d_{v_i}(T) \leq 2$, $1 \leq i \leq k-2$. If $d_{u_4}(T) = 1$ or u_4 is a support vertex with $d_{u_4}(T) = 2$, then $T \cong \mathscr{T}_{n,\Omega}$. In other cases, we consider that $d_{v_1}(T) = \cdots = d_{v_{k_1}}(T) = 1$, $d_{v_{k_1+1}}(T) = \cdots = d_{v_{k_1+k_2}}(T) = 2$, where $k_1 + k_2 = k - 2$.

Case 2.1. $k_1 \geq 2$.

Set $T_2 = T - v_1$, which gives $\Omega(T_2) = \Omega(T)$ and $d_{u_3}(T_2) = k - 1$. Using Lemmas 1 and 2, we obtain that

$$\begin{aligned}
\mathcal{ESI}(T) =& \mathcal{ESI}(T_2) + \mho(k,1) + \mho(k,2) - \mho(k-1,2) + \mho(d_{u_4}(T),k) - \mho(d_{u_4}(T_2),k-1) \\
& + (k_1 - 1)(\mho(k,1) - \mho(k-1,1)) + k_2(\mho(k,2) - \mho(k-1,2)) \\
\leq& \mathscr{H}(n-1,\Omega) + \sqrt{k^2+k+1} + (k_1-1)(\sqrt{k^2+k+1} - \sqrt{k^2-k+1}) \\
& + (k-k_1)(\sqrt{k^2+2k+4} - \sqrt{k^2+3}) \\
\leq& \mathscr{H}(n,\Omega) - (n-2\Omega+1)(\sqrt{(n-\Omega)^2+1+(n-\Omega)} - (\sqrt{(n-\Omega-1)^2+(n-\Omega)})) \\
& - (\Omega-1)(\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)}) \\
& - \sqrt{(n-\Omega-1)^2+(n-\Omega)} + \sqrt{k^2+k+1} + (k_1-1)(\sqrt{k^2+k+1} - \sqrt{k^2-k+1}) \\
& + (k-k_1)(\sqrt{k^2+2k+4} - \sqrt{k^2+3}).
\end{aligned}$$

Since, $\Omega \leq \dfrac{n-k_1+1}{2}$, i.e., $k_1 \leq n - 2\Omega + 1$ and $k \leq n - \Omega - 1$, using Lemmas 2 and 4, we get

$$\begin{aligned}
\mathcal{ESI}(T) \leq& \mathscr{H}(n,\Omega) - (n-2\Omega+1)(\sqrt{(n-\Omega)^2+1+(n-\Omega)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)}) \\
& - (\Omega-1)(\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)}) \\
& - \sqrt{(n-\Omega-1)^2+(n-\Omega)} + \sqrt{(n-\Omega-1)^2+(n-\Omega)} \\
& + (n-2\Omega)(\sqrt{(n-\Omega-1)^2+(n-\Omega)} - \sqrt{(n-\Omega-2)^2+(n-\Omega-1)}) \\
& + (\Omega-2)(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega-2)^2+4+2(n-\Omega-2)}) \\
=& \mathscr{H}(n,\Omega) + (n-2\Omega+1)\Big(\sqrt{(n-\Omega-1)^2+(n-\Omega)} - \sqrt{(n-\Omega-2)^2+(n-\Omega-1)} \\
& - (\sqrt{(n-\Omega)^2+1+(n-\Omega)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)})\Big) \\
& + (\sqrt{(n-\Omega-2)^2+(n-\Omega-1)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)}) \\
& + (\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)}) \\
& + (\Omega-2)\Big(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega-2)^2+4+2(n-\Omega-2)} \\
& - (\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)})\Big) \\
=& \mathscr{H}(n,\Omega) + (n-2\Omega+1)(\xi_1(n-\Omega-1) - \xi_1(n-\Omega)) \\
& + (\sqrt{(n-\Omega-2)^2+(n-\Omega-1)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)}) \\
& + (\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)}) \\
& + (\Omega-2)(\xi_2(n-\Omega-1) - \xi_2(n-\Omega)).
\end{aligned}$$

This implies that $\mathcal{ESI}(T) < \mathscr{H}(n,\Omega)$.

Case 2.2. $k_1 = 1$.

In this case, we consider a tree $T_3 = T - \{u_1, u_2\}$.
Therefore $T_3 \cong \mathscr{T}(n-2, \Omega-1)$ and $d_{u_3}(T_3) = k-1$. Then using Lemmas 1 and 2, we get

$$
\begin{aligned}
\mathcal{ESI}(T) =&\mathcal{ESI}(T_3) + \mho(k, d_{u_4}(T)) - \mho(k-1, d_{u_4}(T_3)) + \mho(k, d_{v_1}(T)) - \mho(k-1, d_{v_1}(T_3)) \\
&+ \sum_{i=2}^{k-2}\Big(\mho(k, d_{v_i}(T)) - \mho(k-1, d_{v_i}(T_3))\Big) + \mho(k,2) + \mho(1,2) \\
\leq&\mathscr{H}(n-2, \Omega-1) + (k-2)(\sqrt{k^2+4+2k} - \sqrt{k^2+3}) \\
&+ \sqrt{k^2+k+1} - \sqrt{k^2-k+1} + \sqrt{k^2+4+2k} + \sqrt{7} \\
\leq&\mathscr{H}(n, \Omega) - (n-2\Omega+1)(\sqrt{(n-\Omega)^2+1+(n-\Omega)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)}) \\
&- (\Omega-2)(\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)}) \\
&- \sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{7} + \sqrt{(n-\Omega-1)^2+(n-\Omega)} + \sqrt{7} \\
&+ (n-\Omega-3)(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega-2)^2+4+2(n-\Omega-2)}) \\
&- \sqrt{(n-\Omega-2)^2+1+(n-\Omega-2)} + \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} \\
=&\mathscr{H}(n, \Omega) + (n-2\Omega)\Big((\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega)^2+1+(n-\Omega)}) \\
&- (\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega-1)^2+(n-\Omega)})\Big) \\
&+ 2(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)}) \\
&+ \Big((\sqrt{(n-\Omega-1)^2+(n-\Omega)} - \sqrt{(n-\Omega-2)^2+1+(n-\Omega-2)}) \\
&- (\sqrt{(n-\Omega)^2+1+(n-\Omega)}) - \sqrt{(n-\Omega-1)^2+(n-\Omega)}\Big) \\
&+ (n-\Omega-3)\Big((\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} \\
&- \sqrt{(n-\Omega-2)^2+4+2(n-\Omega-2)}) \\
&- (\sqrt{(n-\Omega)^2+4+2(n-\Omega)} - \sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)})\Big) \\
=&\mathscr{H}(n, \Omega) + (n-2\Omega)(\Psi_1(n-\Omega) - \Psi_1(n-\Omega-1)) \\
&+ 2(\sqrt{(n-\Omega-1)^2+4+2(n-\Omega-1)} - \sqrt{(n-\Omega)^2+4+2(n-\Omega)}) \\
&+ (\xi_1(n-\Omega-1) - \xi_1(n-\Omega)) + (n-\Omega-3)(\xi_2(n-\Omega-1) - \xi_2(n-\Omega)).
\end{aligned}
$$

Since $k \leq n-\Omega-1$, i.e., $n-\Omega-3 \geq k-2 \geq 0$, then $\mathcal{ESI}(T) < \mathscr{H}(n, \Omega)$. Hence the result is proved. ■

6 Conclusions

This work explores the extremal properties of the Euler–Sombor index ($\mathcal{ESI}$) in trees. We determined the second smallest and largest $\mathcal{ESI}$ values for a given order n, characterized maximal and minimal quasi-trees using majorization, and identified extremal trees with fixed domination number. We also established sharp bounds for in such classes, specifying the structures that attain them.

***Acknowledgment*:** We express our sincere gratitude to the referee for the insightful suggestions.

References

[1] A. M. Albalahi, A. M. Alanazi, A. M. Alotaibi, A. E. Hamza, A. Ali, Optimizing the Euler-Sombor index of (molecular) tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 549–560.

[2] B. Borovićanin, B. Furtula, On extremal Zagreb indices of trees with given domination number, *Appl. Math Comput.* **279** (2016) 208–218.

[3] C. Bujtás, S. Klavžar, Improved upper bounds on the domination number of graphs with minimum degree at least five, *Graphs Comb.* **32** (2016) 511–519.

[4] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, *Appl. Math. Comput.* **399** (2021) #126018.

[5] K. C. Das, Open problems on Sombor index of unicyclic and bicyclic graphs, *Appl. Math. Comput.* **473** (2024) #128647.

[6] K. C. Das, I. Gutman, On Sombor index of trees, *Appl. Math. Comput.* **412** (2022) #126575.

[7] K. C. Das, S. Mondal, D.Y. Huh, Open problem on the maximum exponential augmented Zagreb index of unicyclic graphs, *Comput. Appl. Math.* **43** (2024) #317.

[8] K. C. Das, S. Mondal, On exponential geometric-arithmetic index of graphs, *J. Math. Chem.* **62** (2024) 2740–2760.

[9] P. Das, S. Mondal, A. Pal, On second Zagreb energy of graphs, *MATCH Commun. Math. Comput. Chem.* **92** (2024) 105–131.

[10] P. Das, S. Mondal, Z. Raza, A. Pal, On exponential inverse symmetric division deg index of graphs, *J. Appl. Math. Comput.* (2025) doi: https://doi.org/10.1007/s12190-025-02461-9.

[11] J. Devillers, A. T. Balaban, *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon & Breach, Amsterdam, 1999.

[12] J. F. Fink, M. S. Jacobson, L. F. Kinch, J. Roberts, On graphs having domination number half their order, *Period. Math. Hungar* **16** (1985) 287–293.

[13] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 11–16.

[14] I. Gutman, Relating Sombor and Euler indices, *Vojnotehnički glasnik* **72** (2024) 1–12.

[15] I. Gutman, B. Furtula, *Novel Molecular Structure Descriptors – Theory and Applications I*, Univ. Kragujevac, Kragujevac, 2010.

[16] I. Gutman, B. Furtula, K. C. Das, On some degree-and-distance-based graph invariants of trees, *Appl. Math. Comput.* **289** (2016) 1–6.

[17] I. Gutman, B. Furtula, M. S. Oz, Geometric approach to vertex-degree-based topological indices — Elliptic Sombor index, theory and application, *Int. J. Quantum Chem.* **124** (2024) #e27346.

[18] I. Gutman, B. Furtula, I. Redžepović, On topological indices and their reciprocals, *MATCH Commun. Math. Comput. Chem.* **91** (2024) 287–297.

[19] I. Gutman, N. K. Gürsoy, A. Gürsoy, A. Ülker, New bounds on Sombor index, *Commun. Comb. Optim.* **8** (2023) 305–311.

[20] I. Gutman, I. Redžepović, B. Furtula, On the product of Sombor and modified Sombor indices, *Open J. Discr. Appl. Math.* **6** (2023) 1–6.

[21] I. Gutman, I. Redžepović, G. Ö. Kizilirmak, V. R. Kulli, Euler-Sombor index and its congeners, *Open J. Math. Sci.* **9** (2025) 141–148.

[22] Y. Hou, J. Li, Bounds on the largest eigenvalues of trees with a given size of matching, *Lin. Algebra Appl.* **342** (2002) 203–217.

[23] Y. Hu, J. Fang, Y. Liu, Z. Lin, Bounds on the Euler Sombor index of maximal outerplanar graphs, *El. J. Math.* **8** (2024) 39–47.

[24] X. Hu, L. Zhong, The harmonic index for trees with given domination number, *Discr. Math. Lett.* **9** (2022) 31–37.

[25] B. Khanra , S. Das, Euler Sombor index of trees, unicyclic and chemical graphs, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 525–548.

[26] S. Klavžar, M. Ma, The domination number of exchanged hypercubes, *Inf. Process. Lett.* **114** (2014) 159–162.

[27] V. Maitreyi, S. Elumalai, S. Balachandran, On the extremal general Sombor index of trees with given pendent vertices, *MATCH Commun. Math. Comput. Chem.* **92** (2024) 225–248.

[28] V. Maitreyi, S. Elumalai, S. Balachandran, H. Liu, The minimum Sombor index of trees with given number of pendent vertices, *Comp. Appl. Math.* **42** (2023) #331.

[29] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer, New York, 2011.

[30] P. Nithya, S. Elumalai, S. Balachandran, M. Masre, Ordering unicyclic graphs with a fixed girth by Sombor indices, *MATCH Commun. Math. Comput. Chem.* **92** (2024) 205–224.

[31] G. Özkun Kızılırmak, Extremal Euler Sombor index of unicyclic graphs with fixed diameter, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 725–738.

[32] G. Özkan Kızılırmak, On Euler Sombor index of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 247–262.

[33] G. Özkan Kızılırmak, I. Gutman, Kimyasal graflarin Euler Sombor indeksi (Euler Sombor index of chemical graphs), *Gazi Üniversitesi Fen Fakültesi Dergisi* **6** (2025) 146–160 (in Turkish).

[34] O. Ore, *Theory of Graphs*, AMS, Providence, 1962.

[35] I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.* **86** (2021) 445–457.

[36] X. Ren, G. Cao, F. Wang, M. Zhou, The Euler-Sombor index of trees, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 739–760.

[37] Z. Su, Z. Tang, Extremal unicyclic and bicyclic graphs of the Euler Sombor index, *AIMS Math.* **10** (2025) 6338–6354.

[38] Z. Su, Z. Tang, Extremal unicyclic graphs for the Euler Sombor index: Applications to benzenoid hydrocarbons and drug molecules, *Axioms* **14** (2025) #249.

[39] A. P. Tache, R. M. Tache, I. Stroe, Extremal unicyclic graphs for the Euler Sombor index, *MATCH Commun. Math. Comput. Chem.* **94** (2025) 561–578.

[40] Z. Tang, Y. Li, H. Deng, The Euler Sombor index of a graph, *Int. J. Quantum Chem.* **124** (2024) #e27387.

[41] R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.

[42] S. Wang, C. Wang, J. B. Liu, On extremal multiplicative Zagreb indices of trees with given domination number, *Appl. Math Comput.* **332** (2018) 338–350.

[43] Q. Zhang, F. Tchier, Z. Hussain, A. Aslam, X. Zuo, On the bounds for the Euler Sombor index of molecular trees and its applications, *J. Appl. Math. Comput.* (2025) doi: https://doi.org/10.1007/s12190-025-02506-z.