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Maximum Euler Sombor Index of Unicyclic Graphs with Given Diameter

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Abstract

The Euler Sombor (EU) index of a graph G is defined as

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y) + d_G(x)d_G(y)},$$

where $d_G(x)$ and $d_G(y)$ denote the degrees of vertex x and y in G, respectively. Biswaranjan Khanra, Shibsankar Das [Euler Sombor index of trees, unicyclic and chemical graphs, MATCH Commun. Math. Comput. Chem. 94 (2025) 525-548], posed an open problem about determining the extremal values and extremal graphs for

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the Euler Sombor index among all connected graphs with a given diameter. In this paper, we solve this problem for maximum Euler Sombor index of unicyclic graphs with given diameter. Additionally, we propose a set of open problems for future research.

1 Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we use $N_G(x) = \{y \in V(G)|xy \in E(G)\}$ to denote the neighbors of x in G. Then $d_G(x) = |N_G(x)|$ is the degree of x. The maximum degree of G, denoted by Δ, is $max\{d_G(x) : x \in V(G)\}$. If $d_G(x) = 1$, then x is called a pendent vertex in G. Let $PV(G)$ the set of all pendent vertices in G. The distance between two vertices x and y, denoted $d(x, y)$, is the minimum number of edges in any path connecting x and y. The diameter of a graph G, is define as $D(G) = max\{d_G(x, y)|x, y \in V(G)\}$. A diametral path in G is a shortest path between two vertices u and v such that $d(u, v) = D(G)$.

For a graph G and a vertex $x \in V(G)$ the graph $G - x$ is obtained from G by deleting a vertex x and all edges incident to it. Similarly, for an edge xy, we define: $G + xy$ the graph obtained from G by adding an edge $xy \notin E(G)$ and $G - xy$ the graph obtained from G by deleting an edge $xy \in E(G)$. A graph G is a tree if and only if $m = n - 1$ edges. A graph G is a unicyclic graph if and only if $m = n$ edges. A tree T is called to be a caterpillar if it becomes a path after deleting all pendant vertices. As usual, P_n, S_n and C_n, denote, respectively, the path, the star and the cycle on n vertices. Other undefined notations and terminology in the graph theory can be found in [3].

Graph invariants known as topological indices function as vital components within chemical fields, pharmaceutical sciences, materials science and engineering. The field of science and engineering uses topological indices because they enable researchers to connect them with various properties of molecules. Nowadays scientists use topological indices to model characteristics of both chemical compounds and biological activities within chemistry and nanotechnology. Using geometric principles, Gutman [6] introduced a new method for designing vertex-degree-based topological indices. The research led to creating the Sombor index (SO) through geometric methods. This index is one of the most studied topological indices in recent years (2021-2025). It was defined by

$$SO(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y)}$$

where $E(G)$ stands for the edge set of G and $d_G(x)$ corresponds to the degree of vertex x. For chemical applications of the SO index, see [9, 14]; for a summary of its mathematical aspects, refer to the surveys [5, 11].

Gutman, Furtula, and Oz [8] suggested a new geometric method for constructing vertex-degree-based topological indices which leads to the definition of the elliptic-Sombor (ESO) index. This index is defined as

$$ESO(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y)) \sqrt{d_G^2(x) + d_G^2(y)}.$$

In [8], the authors investigated chemical applications of the ESO index and established various inequalities connecting it to other topological indices. Additionally, they determined extremal graphs for the ESO index among the classes of the following graphs with given orders: (i) trees and (ii) connected graphs. Other studies on the ESO index can be found in [2, 4, 12, 13].

The Euler Sombor index is another topological index constructed following the approach described in [8]. It derives from Euler's approximation formula for the perimeter of an ellipse. Given a graph G, its Euler Sombor index denoted $EU(G)$, and is defined [7, 20] as

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y) + d_G(x) d_G(y)}.$$

Among the topological indices derived from perimeter approximation formulas of an ellipse, Ivan Gutman indicates that the Euler Sombor index represents the most significant one. For more results on EU index one can refer [2, 7, 15–20]. Very recently, Khanra et al. [17] posed the following open problem:

Problem 1. *Determine the extremal values and extremal graphs with respect to Euler Sombor index over the class of all connected graphs with given diameter.*

Concurrently, Ren et al. [15] established sharp upper bounds for the Euler Sombor index in trees, considering parameters such as matching number, pendent vertices, and diameter. In a similar direction, Kizilirmak [19] derived the minimum Euler Sombor index for unicyclic graphs with a fixed diameter, which provides partial progress on the open problem in the context of trees and unicyclic graphs.

Motivated by these studies, as well as by the related works of Alfuraidan [1] and Liu [10], in this article, we determine the maximum Euler Sombor index among all unicyclic graphs with a given diameter.

2 Preliminaries

We start with the following lemmas, which are used frequently in the proofs of main theorems.

Lemma 1. *Let* $g(x,y) = \sqrt{x^2+y^2+xy} - \sqrt{(x-1)^2+y^2+(x-1)y}$, *where* $x>1$ *and* $y>0$, *then the function* $f(x,y)$ *is strictly increasing with* x *and strictly decreasing with* y.

Proof. Note that $(2x+y)^2[4((x-1)^2+y^2+(x-1)y)]-(2(x-1)+y)^2[4(x^2+y^2+xy)] = 12y^2(2x+y-1) > 0$ for $x>1$ and $y>0$, which means $(2x+y)[2\sqrt{(x-1)^2+y^2+(x-1)y}]-(2(x-1)+y)[2\sqrt{x^2+y^2+xy}] > 0$ for $x>1$ and $y>0$, then we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{2x+y}{2\sqrt{x^2+y^2+xy}} - \frac{2(x-1)+y}{2\sqrt{(x-1)^2+y^2+(x-1)y}} \\ &= \frac{(2x+y)[2\sqrt{(x-1)^2+y^2+(x-1)y}]-(2(x-1)+y)[2\sqrt{x^2+y^2+xy}]}{[2\sqrt{x^2+y^2+xy}][2\sqrt{(x-1)^2+y^2+(x-1)y}]} > 0\end{aligned}$$

$$\frac{\partial g}{\partial y} = \frac{2y+x}{2\sqrt{x^2+y^2+xy}} - \frac{2y+x-1}{2\sqrt{(x-1)^2+y^2+(x-1)y}} < 0$$

for $x>1$ and $y>0$ the lemma holds. ■

Lemma 2. *[17] Let $G \in \mathcal{U}_n$ $(n \geq 3)$. Then $EU(G) \geq 2n\sqrt{3} = EU(C_n)$, with equality if and only if $G \cong C_n$.*

Lemma 3. *[16] Among all unicyclic graphs of order $n \geq 4$, the graph S_n^* has the maximum EU index, equal to*

$$(n-3)\sqrt{n^2-n+1} + 2\sqrt{n^2+3} + \sqrt{12}.$$

3 The maximum Euler-Sombor index of unicyclic graphs with given diameter

Let $\mathcal{U}_{n,D}$ be the set of unicyclic graphs with n vertices and diameter D. For $4 \leq D \leq n-2$, we construct a new unicyclic graph $U_{n,D} \in \mathcal{U}_{n,D}$, as shown in Figure 1.

$$\begin{aligned} EU(U_{n,D}) = &(n-D-1)\sqrt{(n-D+1)^2+1+(n-D+1)} \\ &+ 2\sqrt{(n-D+1)^2+4+2(n-D+1)} + R_1, \end{aligned}$$

where $R_1 = 2\sqrt{19}+\sqrt{13}$ if $D = 4$; $R_1 = 2(D-5)\sqrt{3}+3\sqrt{19}+\sqrt{7}$ if $D \geq 5$.

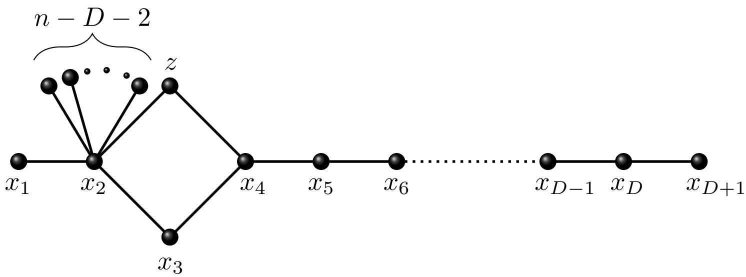

Figure 1. Graph $U_{n,D}$ for $4 \leq D \leq n-2$

In this article, we assume that $C = y_1y_2 \ldots y_{|V(C)|}$ is the unique cycle and $P = x_1x_2 \ldots x_Dx_{D+1}$ is the diametral path of the unicyclic graph under consideration. Let $PV(G)$ represent the set of pendent vertices of G.

We denote by $\mathcal{U}_{n,D}^{max}$ the graph in $\mathcal{U}_{n,D}$ with maximum Euler Sombor index. According to Lemma 2, $\mathcal{U}_{n,D}^{max}$ must contain at least one pendent vertex. We know that $n \geq D+2$, we begin by examining the case where $n = D+2$.

Theorem 3.1. *Let $G \in \mathcal{U}_{n,D}$ with diameter $D = 2$ and $n \geq 4$, then $EU(G) \leq (n-3)\sqrt{(n-1)^2+1+(n-1)} + 2\sqrt{(n-1)^2+2^2+2(n-1)} + 2\sqrt{3}$.*

Proof. Given a graph $G \in \mathcal{U}_{n,D}$ for $D = 2$, there are three structures as follows. If $G = C_4$ or $G = C_5$, then $EU(C_4) = 8\sqrt{3}$ and $EU(C_5) = 10\sqrt{3}$, respectively. For the remaining case, if G be a unicyclic graph obtained from C_3 by attaching $n-3$ pendant edges to one of its vertices, then $EU(G) = (n-3)\sqrt{(n-1)^2+1+(n-1)} + 2\sqrt{(n-1)^2+2^2+2(n-1)} + 2\sqrt{3}$. Above all, the result holds. ■

Theorem 3.2. *Let $G \in \mathcal{U}_{n,D}$ with diameter $D = 3$ and $n \geq 5$, then $EU(G) \leq (n-4)\sqrt{(n-2)^2+1+(n-2)} + \sqrt{(n-2)^2+3^2+3(n-2)} + \sqrt{(n-2)^2+2^2+2(n-2)} + \sqrt{13} + \sqrt{19}$.*

Proof. By Lemma 3, we know the graph S_n^* has the maximum EU index in the class of all unicyclic graphs. When $n_1, n_2 \geq 1$ and $n_3 \geq 0$, we obtain $D(S_n^*) = 3$ and $n = n_1 + n_2 + n_3 + 3 \geq 5$. Thus, the result is true. ■

Theorem 3.3. *Let $G \in \mathcal{U}_{D+2,D}$ with $D \geq 4$, then $EU(G) \leq EU(U_{D+2,D})$, with equality if and only if $G \cong U_{D+2,D}$.*

Proof. Let $G' \in \mathcal{U}_{D+2,D}^{\max}$. Since $n = D+2$, there exists one vertex, denoted by z, that does not lie on the diametral path $P = x_1x_2\ldots x_Dx_{D+1}$. Consequently, the unique cycle C must be either $C = x_ix_{i+1}zx_i$, where $1 \leq i \leq D$ or $C = x_ix_{i+1}x_{i+2}zx_i$ where $1 \leq i \leq D-1$.

Claim 1. The vertices x_1 and x_{D+1} are not contained in $V(C)$.

Assume to the contrary that either x_1 or x_{D+1} belongs to $V(C)$. Without loss of generality, assume $x_1 \in V(C)$. Then the cycle C must be either $C = x_1x_2zx_1$ or $C = x_1x_2x_3zx_1$. Define a new graph G'' with vertex set $V(G'') = V(G')$ and edge set $E(G'') = E(P) \cup \{zx_2, zx_3\}$. This construction ensures $G'' \in \mathcal{U}_{D+2,D}$ and

$$\begin{aligned}EU(G') - EU(G'') &= 2\sqrt{2^2+2^2+4} - \sqrt{1+3^2+3} - \sqrt{3^2+3^2+9} \\ &= 4\sqrt{3} - \sqrt{13} - 3\sqrt{3} < 0.\end{aligned}$$

This contradicts the maximality of G'. Therefore, $x_1 \notin V(C)$. By similarly argument, we also conclude that $x_{D+1} \notin V(C)$.

Claim 2. $|E(C)| = 4$.

Suppose $|E(C)| \geq 5$. This leads to a contradiction with the selection of the path $P = x_1x_2 \dots x_Dx_{D+1}$. Thus $|E(C)| = 3$ or $|E(C)| = 4$.

Case 1. $|E(C)| = 3$.

If $|E(C)| = 3$, then $C = x_ix_{i+1}zx_i$. According to Claim 1, we know that $2 \leq i \leq D-1$. Since $D \geq 4$, $x_{i-1} \notin PV(G')$ or $x_{i+2} \notin PV(G')$, say $x_{i-1} \notin PV(G')$. Thus $i \leq D-2$ and $d_{G'}(x_{i-2}) = q_1$ which is 1 or 2. Let $G'' = G' - \{x_iz\} + \{x_{i-1}z\}$, then $G'' \in U_{D+2,D}$, we have

$EU(G') - EU(G'') = \sqrt{q_1^2+2^2+2q_1} - \sqrt{q_1^2+3^2+3q_1} + \sqrt{3^2+3^2+9} - \sqrt{2^2+3^2+6} < 0$.

Case 2. $|E(C)| = 4$.

If $|E(C)| = 4$, then $C = x_ix_{i+1}x_{i+2}zx_i$. According to Claim 1 and Claim 2, we know that $2 \leq i \leq D-2$. If $i \neq 2$ and $i \neq D-2$, we have

$$\begin{aligned}&EU(G') - EU(U_{D+2,D}) \\ &= \sqrt{2^2+3^2+6} - \sqrt{1^2+3^2+3} + \sqrt{1+2^2+2} - \sqrt{2^2+2^2+4} \\ &= \sqrt{19} - \sqrt{13} + \sqrt{7} - 2\sqrt{3} < 0.\end{aligned}$$

Thus $i = 2$ or $i = D-2$, i.e., $G' \cong U_{D+2,D}$. ■

Next, we consider $n \geq D+3$.

Lemma 4. *Let $G' \in \mathcal{U}_{n,D}^{max}$ with $3 \leq D \leq n-3$ and $P^D = x_1x_2 \cdots, x_{D+1}$ be a diametral path. If $x \in PV(G')$ and $xx_2 \in E(G')$ or $xx_D \in E(G')$, then $|V(C) \cap V(P)| > 1$.*

Proof. Suppose, for contradiction that $|V(C) \cap V(P)| \leq 1$.

Case 1. $|V(C) \cap V(P)| = 0$.

Since $|V(C)\cap V(P)|=0$, there exists a path, say $x_iw_1w_2\ldots w_l$ (where $l\geq 1$) joining cycle C and path P in G'. Then $3\leq i\leq d-1$, and $x_{i-1}\notin PV(G')$, $x_{i+1}\notin PV(G')$ (otherwise the diameter of G' would be greater than D).

If $l\geq 2$, let $G''=G'-\{w_1w_2\}+\{x_iw_2\}$; if $l=1$, let G'' be the graph obtained from G' by deleting edge x_iw_1, identifying x_i and w_1, then adding a new pendant edge to vertex x_i. Then, $G''\in U_{n,d}$. Let $d_{G'}(x_{i-1})=q_1\geq 2$ and $d_{G'}(x_{i+1})=q_2\geq 2$. Note that $d_{G'}(x_i)=d_{G'}(w_l)=3$.

Subcase 1.1. $l=1$.

$$\begin{aligned}EU(G')-EU(G'')=&\sqrt{q_1^2+3^2+3q_1}-\sqrt{q_1^2+5^2+5q_1}+\sqrt{q_2^2+3^2+3q_2}\\&-\sqrt{q_2^2+5^2+5q_2}+\sqrt{3^2+3^2+9}+2\sqrt{3^2+2^2+6}\\&-2\sqrt{2^2+5^2+10}-\sqrt{1+5^2+5}<0.\end{aligned}$$

Subcase 1.2. $l=2$.

$$\begin{aligned}EU(G')-EU(G'')=&\sqrt{q_1^2+3^2+3q_1}-\sqrt{q_1^2+4^2+4q_1}+\sqrt{q_2^2+3^2+3q_2}\\&-\sqrt{q_2^2+4^2+4q_2}+2\sqrt{2^2+3^2+6}-\sqrt{3^2+4^2+12}\\&-\sqrt{1^2+4^2+4}<0.\end{aligned}$$

Subcase 1.3. $l\geq 3$.

$$\begin{aligned}EU(G')-EU(G'')=&\sqrt{q_1^2+3^2+3q_1}-\sqrt{q_1^2+4^2+4q_1}+\sqrt{q_2^2+3^2+3q_2}\\&-\sqrt{q_2^2+4^2+4q_2}+\sqrt{2^2+3^2+6}+\sqrt{2^2+2^2+4}\\&-\sqrt{4^2+1+4}-\sqrt{2^2+4^2+8}<0.\end{aligned}$$

Case 2. $|V(C)\cap V(P)|=1$.

Let $V(C)\cap V(P)=x_i(y_1)$. Let $C=y_1y_2y_3\cdots y_{|V(C)|}y_1$ be the unique cycle.

Subcase 2.1 $|V(C)|=3$.

As $D>2$, then $d_{G'}(x_{i-1})\geq 2$ or $d_{G'}(x_{i+1})\geq 2$. We may assume, without loss of generality, that $d_{G'}(x_{i+1})=q_2\geq 2$. We know that $d_{G'}(x_i)=q_1\geq 4$,

$d_{G'}(x_{i+2}) = q_3 \geq 1$.

Let $G'' = G' - \{y_2y_3\} + \{y_2x_{i+1}\}$, then $G'' \in \mathcal{U}_{n,d}$. Using Lemma 1 and $d_G(x_i) = q_1 \geq 4$, we obtain

$$
\begin{aligned}
&EU(G') - EU(G'')\\
&= \sqrt{2^2+2^2+4} - \sqrt{(q_2+1)^2+2^2+2(q_2+1)} + \sqrt{q_1^2+2^2+2q_1}\\
&\quad - \sqrt{q_1^2+1^2+q_1} + \sqrt{q_1^2+q_2^2+q_1q_2} - \sqrt{q_1^2+(q_2+1)^2+q_1(q_2+1)}\\
&\quad + \sqrt{q_2^2+q_3^3+q_2q_3} - \sqrt{(q_2+1)^2+q_3^2+q_3(q_2+1)}\\
&< 2\sqrt{3} - \sqrt{19} + \sqrt{q_1^2+2^2+2q_1} - \sqrt{q_1^2+1+q_1}\\
&\leq 2\sqrt{3} - \sqrt{19} + 2\sqrt{7} - \sqrt{21} < 0.
\end{aligned}
$$

Subcase 2.2. $|V(C)| \geq 4$.

Then $3 \leq i \leq D-1$. Let $d_{G'}(x_{i-1}) = q_1 \geq 2$ and $d_{G'}(x_{i+1}) = q_2 \geq 2$. Let $C = y_1y_2\dots y_{|V(C)|}$ be the unique cycle, $(y_1 = x_i)$. Let $G'' = G' - \{y_2y_3\} + \{x_iy_3\}$, then $G'' \in \mathcal{U}_{n,D}$.

$$
\begin{aligned}
&EU(G') - EU(G'')\\
&= \left(\sqrt{q_1^2+4^2+4q_1} - \sqrt{q_1^2+5^2+5q_1}\right) + \left(\sqrt{q_2^2+4^2+4q_2}\right.\\
&\quad \left. - \sqrt{q_2^2+5^2+5q_2}\right) + 2\sqrt{2^2+4^2+8} - 2\sqrt{2^2+5^2+10}\\
&\quad + \sqrt{2^2+2^2+4} - \sqrt{1+5^2+5} < 0.
\end{aligned}
$$

All cases above contradict $G' \in \mathcal{U}_{n,D}^{max}$, we conclude that $|V(C) \cap V(P)| > 1$ must hold. ■

Lemma 5. *Let $G' \in \mathcal{U}_{n,D}^{\max}$ with $3 \leq D \leq n-3$. Then there exist a vertex $x_0 \in VP(G')$, such that $G' - x_0 \in \mathcal{U}_{n-1,d}$.*

Proof. Suppose, to the contrary, that for every pendant vertex $x \in V(G')$, we have $G' - x \in \mathcal{U}_{n-1,D-1}$.

Let $P = x_1x_2\dots x_Dx_{D+1}$ be a diameter path of G'. By Lemma 2, G' has atleast one pendant vertex, so we can assume that x_1 is a pendant vertex. Since $G' - x \in \mathcal{U}_{n-1,D-1}$ hold for every pendant vertex $x \in V(G')$, we have G' contains two pendant vertices x_1 and x_{D+1}. By Lemma 4, we have $|V(C) \cap V(P)| > 1$. Let $C = x_i x_{i+1} \cdots\ x_{i+k} y_l y_{l-1} \cdots y_3 y_2 y_1$, where $y_1 = x_i$ and $2 \leq i < i+k \leq D+1$ (so $k \geq 1$). Clearly, $k \leq l$ (otherwise if $k > l$, then the distance between x_1 and x_{D+1} is less than D in G'). Since $n \geq D+3$, then $l > 2$. Note that $d_{G'}(x_{i-1}) = q_1 \geq 1$, $d_{G'}(x_i) = 3$, $d_{G'}(y_2) = d_{G'}(y_3) = 2$. We examine two cases: $l > k$ and $l = k$.

Case 1. $l > k$.

Let $G'' = G' - \{y_2y_3\} + \{x_iy_3\}$. Then $G'' \in \mathcal{U}_{n,D}$. We have $d_{G'}(x_{i+1}) = d_{G''}(x_{i+1}) = q_2 \geq 2$, $d_{G''}(x_i) = 4$, $d_{G''}(y_2) = 1$ and $d_{G''}(y_3) = 2$. Then

$$\begin{aligned} EU(G') - EU(G'') &= \sqrt{q_1^2 + 3^2 + 3q_1} - \sqrt{q_1^2 + 4^2 + 4q_1} + \sqrt{q_2^2 + 3^2 + 3q_2} \\ &\quad - \sqrt{q_2^2 + 4^2 + 4q_2} + \sqrt{2^2 + 2^2 + 4} + \sqrt{2^2 + 3^2 + 6} \\ &\quad - \sqrt{2^2 + 4^2 + 8} - \sqrt{1 + 4^2 + 4} \\ &< 2\sqrt{3} + \sqrt{19} - \sqrt{28} - \sqrt{21} < 0. \end{aligned}$$

Case 2. $l = k$.

Let $G'' = G' - \{x_iy_2\} - \{y_2y_3\} + \{x_{i+1}y_2\} + \{x_{i+1}y_3\}$. Then $G'' \in \mathcal{U}_{n,D}$, we have $d_{G''}(x_i) = 2$, $d_{G'}(x_{i+1}) = 2, d_{G''}(x_{i+1}) = 4$, $d_{G'}(x_{i+2}) = d_{G''}(x_{i+2}) = 2$, $d_{G''}(y_2) = 1$ and $d_{G''}(y_3) = 2$. We obtain

$$\begin{aligned} EU(G') - EU(G'') &= \sqrt{q_1^2 + 3^2 + 3q_1} - \sqrt{q_1^2 + 2^2 + 2q_1} + 2\sqrt{2^2 + 2^2 + 4} \\ &\quad + 2\sqrt{2^2 + 3^2 + 6} - 3\sqrt{2^2 + 4^2 + 8} - \sqrt{1 + 4^2 + 4}. \end{aligned}$$

Subcase 2.1. $q_1 = 1$.

$$EU(G') - EU(G'') = \sqrt{13} - \sqrt{7} + 4\sqrt{3} + 2\sqrt{19} - 6\sqrt{7} - \sqrt{21} < 0.$$

Subcase 2.2. $q_1 \geq 2$.

$$\begin{aligned} EU(G') - EU(G'') &\leq \sqrt{2^2 + 3^2 + 6} - \sqrt{2^2 + 2^2 + 4} + 2\sqrt{2^2 + 2^2 + 4} \\ &\quad + 2\sqrt{2^2 + 3^2 + 6} - 3\sqrt{2^2 + 4^2 + 8} - \sqrt{1 + 4^2 + 4} \\ &= 3\sqrt{19} + 2\sqrt{3} - 6\sqrt{7} - \sqrt{21} < 0. \end{aligned}$$

This contradicts the assumption that $G' \in \mathcal{U}_{n,D}^{max}$, thus the result holds. ■

Lemma 6. *Let $G' \in \mathcal{U}_{n,D}^{\max}$ with $4 \leq D \leq n-3$. Then there exist a pendant vertex $x_0 \in V(G')$ such that $G' - x_0 \in \mathcal{U}_{n-1,D}$ and the neighbor of x_0 is adjacent to at least two non-pendant vertices.*

Proof. Let $G' \in \mathcal{U}_{n,D}^{\max}$. By Lemma 5, there exists a pendant vertex $x \in V(G')$ such that $G' - x \in \mathfrak{U}_{n-1,D}$. Let

$$\mathfrak{U}^* = \{x \in V(G') | d_{G'}(x) = 1, G' - x \in \mathfrak{U}_{n-1,D}\}.$$

We have $\mathfrak{U}^* \neq \emptyset$. Define $I_{G'}(y) = \{z \in N_{G'}(y) | d_{G'}(z) \geq 2\}$, for all $y \in \bigcup_{x\in\mathfrak{U}^*} N_{G'}(x)$. Note that, $|I_{G'}(y)| \geq 1$. We prove that there exists a vertex $y \in \bigcup_{x\in\mathfrak{U}^*} N_{G'}(x)$, such that $|I_{G'}(y)| \geq 2$. Suppose, for contradiction, that $|I_{G'}(y)| = 1$ for all $y \in \bigcup_{x\in\mathfrak{U}^*} N_{G'}(x)$. Let $C = y_1y_2\ldots y_{|V(C)|}$ be the unique cycle in G' and $P = x_1x_2\ldots x_Dx_{D+1}$ be a diametral path in G'.

Claim 1. $\mathcal{U}^* \subseteq N_{G'}(x_2) \cup N_{G'}(x_D)$.

On the contrary, we suppose that there exist a vertex $y \in \bigcup_{x\in\mathfrak{U}^*} N_{G'}(x)$ such that $y \notin \{x_2, x_D\}$, Note that $y \notin \{x_1, x_{D+1}\}$. Since $|I_{G'}(y)| = 1$, we have $y \notin V(P) \cup V(C)$. Let G'' be the graph obtained from G' by deleting the unique non-pendant edge (say yz) incident with y, identifying y and z, and attaching a new pendant edge incident with y, then $G'' \in \mathcal{U}_{n,D}$. Let $d_{G'}(y) = q_1 \geq 2$, $d_{G'}(z) = q_2 \geq 2$. Then $d_{G''}(y) = q_1 + q_2 - 1$. Since $|I_{G'}(y)| = 1$, any vertex $x^* \in N_{G'}(y)\backslash\{z\}$ has degree 1, we get

$$\begin{aligned}
&\sqrt{d_{G'}(y)^2 + d_{G'}(x^*)^2 + d_{G'}(y)d_{G'}(x^*)} - \sqrt{d_{G''}(y)^2 + d_{G''}(x^*)^2 + d_{G''}(y)d_{G''}(x^*)} \\
&= \sqrt{q_1^2 + 1 + q_1} - \sqrt{(q_1 + q_2 - 1)^2 + 1 + (q_1 + q_2 - 1)} < 0.
\end{aligned}$$

For any vertex $z^* \in N_{G'}(z)\backslash\{y\}$, we have

$$\begin{aligned}
&\sqrt{d_{G'}(z)^2 + d_{G'}(z^*)^2 + d_{G'}(z)d_{G''}(z^*)} - \sqrt{d_{G''}(z)^2 + d_{G''}(z^*)^2 + d_{G''}(z)d_{G''}(z^*)} \\
&= \sqrt{q_2^2 + d_{G'}(z^*)^2 + q_2 d_{G'}(z^*)} \\
&\quad - \sqrt{(q_1 + q_2 - 1)^2 + d_{G'}(z^*)^2 + (q_1 + q_2 - 1)d_{G'}(z^*)} < 0.
\end{aligned}$$

Using Equations above, we have

$$EU(G') - EU(G'')$$

$$= (q_2-1)\Bigg(\sqrt{q_2^2+d_{G'}^2(z^*)+q_2d_{G'}(z^*)}$$
$$-\sqrt{(q_1+q_2-1)^2+d_{G'}^2(z^*)+(q_1+q_2-1)d_{G'}(z^*)}\Bigg)$$
$$+(q_1-1)\Bigg(\sqrt{q_1^2+1+q_1}-\sqrt{(q_1+q_2-1)^2+1+(q_1+q_2-1)}\Bigg)$$
$$+\Bigg(\sqrt{q_2^2+q_1^2+q_1q_2}-\sqrt{(q_1+q_2-1)^2+1+(q_1+q_2-1)}\Bigg)<0.$$

Thus, Claim 1 holds.

As $\mathfrak{U}^* \neq \emptyset$ and by Claim 1, $\mathfrak{U}^* \subseteq N_{G'}(x_2)\bigcup N_{G'}(x_D)$, we assume that there exists a vertex $x \in \mathfrak{U}^*$ and $x \in N_{G'}(x_2)$, then $|I_{G'}(x_2)| = 1$. Thus, x_1 and x_2 is not in C (otherwise $|I_{G'}(x_2)| \geq 2$), and by Claim 1, we have $d_{G'}(x_2) = q \geq 3$ (otherwise $G' - x_1 \in \mathfrak{U}_{n-1,D-1}$). By Lemma 4, $|V(C)\cap V(P)| > 1$. Let $C = x_ix_{i+1}\cdots x_{i+k}y_ly_{l-1}\cdots y_3y_2y_1$ where $y_1 = x_i$, $3 \leq i < i+k \leq D+1$ (so $k \geq 1$), $l \geq 2$ and the vertices $y_2y_3y_4\cdots y_{l-1}y_l$ are not in P. Clearly, $k \leq l$ (otherwise if $k > l$, then the distance between x_1 and x_{D+1} is less than D in G').

Claim 2. $|V(C)\backslash V(P)| = 1$

On the contrary, we suppose that $|V(C)\backslash V(P)| \geq 2$, then $l \geq 3$. we have $d_{G'}(x_{i-1}) = q_1 \geq 2$. Note that $d_{G'}(x_i) = 3$ and $d_{G'}(y_2) = d_{G'}(y_3) = 2$. We examine two cases: $l > k$ and $l = k$.

Case 1. $l > k$.

Let $G'' = G' - \{y_2y_3\} + \{x_iy_3\}$. Then $G'' \in \mathcal{U}_{n,D}$. We have $d_{G'}(x_{i+1}) = d_{G''}(x_{i+1}) = q_2 \geq 2$, $d_{G''}(x_i) = 4$ and $d_{G''}(y_2) = 1$ and $d_{G''}(y_3) = 2$. We obtain

$$EU(G') - EU(G'')$$
$$= \Bigg(\sqrt{q_1^2+3^2+3q_1}-\sqrt{q_1^2+4^2+4q_1}\Bigg)+\Bigg(\sqrt{q_2^2+3^2+3q_2}-\sqrt{q_2^2+4^2+4q_2}\Bigg)$$
$$+\sqrt{2^2+3^2+6}+\sqrt{2^2+2^2+4}-\sqrt{2^2+4^2+8}-\sqrt{1+4^2+4}<0.$$

Case 2. $l = k$.

Let $G'' = G' - \{x_iy_2\} - \{y_2y_3\} + \{x_{i+1}y_2\} + \{x_{i+1}y_3\}$. Then $G'' \in \mathcal{U}_{n,D}$. We have $d_{G''}(x_i) = 2$, $d_{G'}(x_{i+1}) = 2$, $d_{G''}(x_{i+1}) = 4$, $d_{G'}(x_{i+2}) =$

$d_{G''}(x_{i+2}) = q_3 \geq 2$, $d_{G''}(y_2) = 1$ and $d_{G''}(y_2) = 2$. Then

$$\begin{aligned}
&EU(G') - EU(G'') \\
&= \left(\sqrt{q_3^2 + 2^2 + 2q_3} - \sqrt{q_3^2 + 4^2 + 4q_3}\right) + \left(\sqrt{q_1^2 + 3^2 + 3q_1} - \sqrt{q_1^2 + 2^2 + 2q_1}\right) \\
&\quad + 2\sqrt{2^2 + 3^2 + 6} + \sqrt{2^2 + 2^2 + 4} - 2\sqrt{2^2 + 4^2 + 8} - \sqrt{1 + 4^2 + 4} \\
&< \sqrt{2^2 + 3^2 + 6} - \sqrt{2^2 + 2^2 + 4} + 2\sqrt{2^2 + 3^2 + 6} + \sqrt{2^2 + 2^2 + 4} \\
&\quad - 2\sqrt{2^2 + 4^2 + 8} - \sqrt{1 + 4^2 + 4} = 3\sqrt{19} - 4\sqrt{7} - \sqrt{21} < 0.
\end{aligned}$$

Thus, Claim 2 holds.

Claim 3. $d_{G'}(x_{D+1}) = 1$.

On the contrary, we suppose $d_{G'}(x_{D+1}) = 2$. Then x_D is not adjacent to a pendant vertex, otherwise $|I(x_D)| \geq 2$. As $d \geq 4$, then $d_{G'}(x_{D-2}) = q_1 \geq 2$. Since $|V(C)\backslash V(P)| = 1$ (by Claim 2), then $|V(C)| = 3$ or 4.

Case 1. $|V(C)| = 3$.

So $C = x_D x_{D+1} z x_D$. Let $G'' = G' - \{x_{D+1}z\} + \{x_{D-1}z\}$, then $G'' \in \mathcal{U}_{n,D}$.

$$\begin{aligned}
&EU(G') - EU(G'') \\
&= \sqrt{q_1^2 + 2^2 + 2q_1} - \sqrt{q_1^2 + 3^2 + 3q_1} + 2\sqrt{2^2 + 3^2 + 6} + \sqrt{2^2 + 2^2 + 4} \\
&\quad - \sqrt{1^2 + 3^2 + 3} - \sqrt{2^2 + 3^3 + 6} - \sqrt{3^2 + 3^2 + 9} \\
&< \sqrt{19} + 2\sqrt{3} - \sqrt{13} - 3\sqrt{3} < 0.
\end{aligned}$$

Case 2. $|V(C)| = 4$.

So $C = x_{D-1} x_D x_{D+1} z x_{D-1}$. Let $G'' = G' - \{x_{D+1}z\} + \{x_D z\}$, then $G'' \in \mathcal{U}_{n,D}$.

$$\begin{aligned}
EU(G') - EU(G'') &= 2\sqrt{2^2 + 2^2 + 4} - \sqrt{1^2 + 3^2 + 3} - \sqrt{3^2 + 3^2 + 9} \\
&= 4\sqrt{3} - \sqrt{13} - 3\sqrt{3} < 0.
\end{aligned}$$

Thus, Claim 3 holds.

Claim 4. $|V(C)| = 4$.

On the contrary, we suppose that $|V(C)| = 3$, the unique cycle $C = x_i x_{i+1} z x_i (3 \leq i \leq D - 1)$. Since $\mathfrak{U}^* = \{x \in V(G') | d_{G'}(x) = 1, G' - x \in$

$\mathcal{U}_{n-1,D}\} \neq \emptyset$, then $d_{G'}(x_2) = q \geq 3$.

Case 1. $i = 3$.

Let $G'' = G' - \{x_3 z\} + \{x_2 z\}$, then $G'' \in \mathcal{U}_{n,D}$.

$$\begin{aligned}
&EU(G') - EU(G'') \\
&= (q-1)\left(\sqrt{q^2+1^2+q} - \sqrt{(q+1)^2+1^2+(q+1)}\right) \\
&\quad + \sqrt{q^2+3^2+3q} - 2\sqrt{(q+1)^2+2^2+2(q+1)} + \sqrt{3^2+3^2+9} \\
&< 0.
\end{aligned}$$

Case 2. $i \geq 4$.

Let $G'' = G' - \{x_i z\} - \{x_{i+1} z\} + \{x_2 z\} + \{x_4 z\}$. We have $d_{G'}(x_{i+2}) = q_1 \geq 1$. Then

$$\begin{aligned}
&EU(G') - EU(G'') \\
&= (q-1)\left(\sqrt{q^2+1^2+q} - \sqrt{(q+1)^2+1^2+(q+1)}\right) + \sqrt{q^2+2^2+2q} \\
&\quad + \sqrt{q_1^2+3^2+3q_1} + \sqrt{3^2+3^2+9} - 2\sqrt{(q+1)^2+2^2+2(q+1)} \\
&\quad - \sqrt{q_1^2+2^2+2q_1} + \sqrt{2^2+3^2+6} - \sqrt{2^2+2^2+4} \\
&\leq 2(\sqrt{3^2+1^2+3} - \sqrt{4^2+1^2+4}) + \sqrt{q^2+2^2+2q} \\
&\quad - 2\sqrt{(q+1)^2+2^2+2(q+1)} + (\sqrt{1^2+3^2+3} - \sqrt{1^2+2^2+2}) + 3\sqrt{3} \\
&\quad + \sqrt{19} - 2\sqrt{3} \\
&\leq 3\sqrt{13} - 2\sqrt{21} + 2\sqrt{19} - 5\sqrt{7} + \sqrt{3} < 0.
\end{aligned}$$

Thus, Claim 4 holds, then the unique cycle $C = x_i x_{i+1} x_{i+2} z x_i$ ($3 \leq i \leq D-2$), $d_{G'}(x_{d+1}) = 1$, $d_{G'}(x_2) = q \geq 3$ and $d_{G'}(x_{i+3}) = q_2 \geq 1$.

Case 1. $i = 3$.

$$\begin{aligned}
&EU(G') - EU(U_{n,D}) \\
&= (q-1)\left(\sqrt{q^2+1+q} - \sqrt{(q+1)^2+1+(q+1)}\right) + \sqrt{q^2+3^2+3q} \\
&\quad + \sqrt{q_2^2+3^2+3q_2} - \sqrt{q_2^2+2^2+2q_2} - 2\sqrt{(q+1)^2+2^2+2(q+1)} \\
&\quad + \sqrt{2^2+3^2+6}
\end{aligned}$$

$$\leq 2(\sqrt{13}-\sqrt{21})+3\sqrt{3}+\sqrt{13}-\sqrt{7}-2\sqrt{28}+\sqrt{19}<0.$$

Case 2. $4 \leq i \leq D-2$.

$$\begin{aligned}
&EU(G')-EU(U_{n,D})\\
&=(q-1)\Big(\sqrt{q^2+1^2+q}-\sqrt{(q+1)^2+1+(q+1)}\Big)\\
&-2\sqrt{(q+1)^2+2^2+2(q+1)}+\sqrt{q^2+2^2+2q}+2\sqrt{2^2+3^2+6}\\
&-\sqrt{2^2+2^2+4}+\sqrt{q_2^2+3^2+3q_2}-\sqrt{q_2^2+2^2+2q_2}\\
&<\sqrt{19}-2\sqrt{28}+2\sqrt{19}-2\sqrt{3}+\sqrt{13}-\sqrt{7}<0.
\end{aligned}$$

This is a contradiction with the assumption $G' \in \mathcal{U}_{n,D}^{max}$. Thus, there must exists a pendant vertex $x_0 \in V(G')$ such that $G'-x_0 \in \mathcal{U}_{n-1,D}$ and $|I_{G'}(N_{G'}(x_0))| \geq 2$. This completes the proof. ■

Theorem 3.4. *Let $G \in \mathcal{U}_{n,D}$ with $4 \leq D \leq n-2$. Then $EU(G) \leq EU(U_{n,D})$, with equality if and only if $G \cong U_{n,D}$.*

Proof. We prove the theorem by induction on n. If $n = D+2$, conclusion hold (Theorem 3.3). Suppose the conclusion holds for $n-1$. Let $G' \in \mathcal{U}_{n,D}^{max}$, $4 \leq D \leq n-3$. By Lemma 6, there exist a pendant vertex $x \in V(G')$ such that $G'-x \in \mathcal{U}_{n-1,D}$ and $y = N_{G'}(x)$ attached to at least two non-pendent vertices, say z_1, z_2. Let $N_{G'}(y) = \{x, y_1, y_2, \ldots, y_{l-1}\}$. Then $|N_{G'}(y)| = l$, and $3 \leq l \leq n-D+1$. Denote $d_{G'}(y_i) = l_i$ for $1 \leq i \leq l-1$. Let $G'' = G'-x$, then $G'' \in \mathcal{U}_{n-1,D}$.

$$\begin{aligned}
EU(G') &= EU(G'')+\sqrt{l^2+1^2+l}+\sum_{i=1}^{l-1}\Big(\sqrt{l^2+l_i^2+ll_i}-\sqrt{(l-1)^2+l_i^2+(l-1)l_i}\Big)\\
&\leq EU(U_{n-1,D})+\sqrt{l^2+1^2+l}+2(\sqrt{l^2+2^2+2l}-\sqrt{(l-1)^2+2^2+2(l-1)})\\
&\quad+(l-3)(\sqrt{l^2+1^2+l}-\sqrt{(l-1)^2+1^2+(l-1)})\\
&=(n-D-2)\sqrt{(n-D)^2+1+(n-D)}+2\sqrt{(n-D)^2+4+2(n-D)}+R_1\\
&\quad+2\sqrt{l^2+2^2+2l}+(l-2)\sqrt{l^2+1^2+l}-2\sqrt{(l-1)^2+2^2+2(l-1)}\\
&\quad-(l-3)\sqrt{(l-1)^2+1^2+(l-1)}\\
&\leq(n-D-2)\sqrt{(n-D)^2+1+(n-D)}+2\sqrt{(n-D)^2+4+2(n-D)}+R_1
\end{aligned}$$

$$+ \sqrt{(n-D+1)^2+1+(n-D+1)}$$
$$+ 2\left(\sqrt{(n-D+1)^2+2^2+2(n-D+1)} - \sqrt{(n-D)^2+2^2+2(n-D)}\right)$$
$$+ (n-D-2)\left(\sqrt{(n-D+1)^2+1+(n-D+1)} - \sqrt{(n-D)^2+1+(n-D)}\right)$$
$$= (n-D-1)\sqrt{(n-D+1)^2+1+(n-D+1)}$$
$$+ 2\sqrt{(n-D+1)^2+4+2(n-D+1)} + R_1 = EU(U_{n,D})$$

with equalities if and only if $G'' \cong U_{n-1,D}$, $l = n-D+1$, exactly two vertices in $N_{G'}(y)$ have degree 2 and the remaining $l-2$ vertices in $N_{G'}(y)$ have degree 1, which implies that $G' \cong U_{n,D}$. ■

4 Conclusion

In this paper, we determined the maximum Euler-Sombor index of unicyclic graphs with a given diameter. This work raises several open problems, which are outlined below:

Open problem 1. Determine the first, second, third, and fourth minimum values of the Euler Sombor index among trees with a given diameter, as well as the second, third, and fourth maximum values.

Open problem 2. Determine the second, third, and fourth maximum as well as the second, third, and fourth minimum values of the Euler Sombor index among unicyclic graphs of a given diameter.

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