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Recent Progress on Lanzhou Index

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Abstract

Let G be a graph. The Lanzhou index, alternatively known as the forgotten coindex, is defined as $Lz(G) = \sum_{u\in V(G)} \overline{d_u} d_u^2$, where d_u (resp. $\overline{d_u}$) represents the degree of vertex u in G (resp. $\overline{G}$).

Research findings substantiate that the Lanzhou index demonstrates enhanced predictive capability compared to both the first Zagreb index and the forgotten index in modeling the logarithmic octanol-water partition coefficient for structural isomers of octane and nonane. This review aims to systematically compiling current extremal results and bounds related to the Lanzhou index. Finally, we outline several open problems as directions for future research.

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$, edge set $E(G)$, order $n = |V(G)|$, and size $m = |E(G)|$. For adjacent vertices u and v in G, we write $uv \in E(G)$. The open neighborhood $N_G(u) = \{v \mid uv \in E(G)\}$ determines the degree $d_u(G) = |N_G(u)|$. The complement graph $\overline{G} = (V, \overline{E})$ satisfies $\overline{E} = \{uv \mid uv \notin E(G)\}$. The graph is called k-regular graph if $d_u(G) = k$ for all $u \in V(G)$. If G is not a regular graph and $d_v(G) \in \{a, b\}$ for all $v \in V(G)$, then we call G is a (a, b)-regular graph. Let K_{n_1,n_2} be a complete 2-partite graph with $n_1 + n_2$ vertices. Let $\Delta(G) = \max\{d_u | u \in V(G)\}$ and $\delta(G) = \min\{d_u | u \in V(G)\}$. The path, star, cycle with n vertices are denoted as P_n, S_n, and C_n, respectively. For a degree sequence $\pi(G) = (d_1, d_2, \ldots, d_n)$, we assume $d_1 \geq d_2 \geq \cdots \geq d_n$ where $d_i = d_{v_i}$ corresponds to vertex $v_i \in V(G) = \{v_1, v_2, \ldots, v_n\}$. For degree sequences with multiplicities, the notation $a^{(b)}$ is used to signify that the integer a is repeated b times throughout the sequence. The join graph $G \vee H$ of two graphs G and H is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(G)\}$. The corona product $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G (with n_1 vertices) and n_1 copies of H (with n_2 vertices), and then joining the i-th vertex of G to every vertex in the i-th copy of H. Let $G \Box H$ be the Cartesian product of two graphs G and H. The vertex set of $G \Box H$ is consisted of all ordered pairs (u, v) where $u \in V(G)$ and $v \in V(H)$. Two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \Box H$ if and only if (1) $u_1 = u_2$ and $v_1v_2 \in E(H)$ or (2) $v_1 = v_2$ and $u_1u_2 \in E(G)$. A vertex subset $S \subseteq V(G)$ is called an independent set if the induced subgraph $G[S]$ contains no edges. The independence number $\alpha(G)$ is defined as the maximum cardinality among all independent sets in G. Any symbols and terms utilized without prior definition are assumed to follow the conventions outlined in Bondy and Murty [10].

Vertex-degree-based topological indices have been extensively studied in both mathematical and chemical literature. The First Zagreb index $M_1(G)$ [22] and second Zagreb index $M_2(G)$ [21] of a graph G are defined as

$$M_1(G) = \sum_{v \in V(G)} d_v^2 = \sum_{uv \in E(G)} (d_u + d_v), \tag{1}$$

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v. \tag{2}$$

The Forgotten index $F(G)$ [17] of a graph G is defined as

$$F(G) = \sum_{v \in V(G)} d_v^3 = \sum_{uv \in E(G)} (d_u^2 + d_v^2). \tag{3}$$

The Lanzhou index[†] was introduced by Vukičević et al. [41]

$$Lz(G) = \sum_{u \in V(G)} \overline{d_u} d_u^2, \tag{4}$$

where $\overline{d_u}$ represents the degree of vertex u in $\overline{G}$.

Since $\overline{d_u} = n - 1 - d_u$, Lanzhou index of G is a linear combination of $M_1(G)$ and $F(G)$. That is

$$Lz(G) = (n-1)M_1(G) + F(G). \tag{5}$$

The forgotten coindex (F-coindex) [15, 26] is defined as

$$\overline{F}(G) = \sum_{uv \notin E(G)} (d_u^2 + d_v^2). \tag{6}$$

Since $\sum\limits_{uv \notin E(G)} (d_u^2 + d_v^2) = \sum\limits_{u \in V(G)} (n - d_u) d_u^2$, then $Lz(G) = \overline{F}(G)$.

The vertex-degree-based topological indices have the following general form

$$TI(G) = \sum_{uv \in E(G)} f(d_u, d_v), \tag{7}$$

where $f(x, y)$ denotes a non-negative real-valued symmetric function of x and y.

[†] The term "Lanzhou" originates from a city in China. It was chosen because the initial research and discovery of this index (see details in [41]) were conducted in Lanzhou, China.

In Table 1, we list some vertex-degree-based topological indices which is used in this paper.

Table 1. Some indices considered in the present review.

Function $f(x, y)$	Equation (7) corresponds to	Symbol
$x + y$	first Zagreb index [22]	M_1
xy	second Zagreb index [21]	M_2
$1/\sqrt{xy}$	Randić index [32]	R
$\sqrt{xy}$	reciprocal Randić index [20]	RR
$\sqrt{x^2 + y^2}$	Sombor index [19]	SO
$x^2 + y^2$	forgotten topological index [17]	F
$2xy/(x + y)$	inverse sum indeg index [40]	ISI
$\|x - y\|$	irregularity index [1]	irr
$y/x + x/y$	symmetric division deg index [39]	SDD
$(x + y)^2$	hyper-Zagreb index [8]	HM
$\frac{1}{\sqrt{x+y}}$	sum-connectivity index [46]	SC
$\sqrt{x + y}$	reciprocal sum-connectivity index [46]	RSC

The structure of this paper is as follows: Section 2 provides a summary of the extremal results and bounds related to the Lanzhou index. Section 3 explores the connections between the Lanzhou index and other indices. Section 4 examines Nordhaus-Gaddum-type results concerning the Lanzhou index. Section 5 proposes several open problems for future research on the Lanzhou index.

2 Extremal results and bounds

In this section, we present the extremal results concerning the Lanzhou index for graphs. Let $\mathcal{G}_n$ be a class of graphs with n vertices. Let $\mathcal{G}_{n,m}$ be a class of graphs with n vertices and m edges.

2.1 Simple graphs

Vukičević [41] gave the upper and lower bounds for $Lz(G)$ of a graph.

Theorem 2.1. [41] *Let $G \in \mathcal{G}_n$. Then*

$$0 \leq Lz(G) \leq \frac{4}{27} n(n-1)^3.$$

The equality on the left is satisfied if and only if $G \cong K_n$ or nK_1. The equality on the right is satisfied if and only if G is k-regular with $k = \frac{2}{3}(n-1)$ and $n \equiv 1 \pmod 3$.

Theorem 2.2. [18] *Let $G \in \mathcal{G}_{n,m}$ be a triangle-free graph. Then*

$$Lz(G) \leq (n-1-\delta)nm,$$

with equality if and only if G is a $\frac{n}{2}$-regular graph.

Theorem 2.3. [44] *Let K_{n_1,n_2} be a complete 2-partite graph with $n(= n_1 + n_2)$ vertices. Then*

$$Lz(K_{n_1,n_2}) \leq \begin{cases} n^3(n-1), & if\ n\ is\ even \\ \frac{1}{4}(n^2-1)(n-2), & if\ n\ is\ odd \end{cases},$$

with equality if and only if $n_1 = \lceil \frac{n}{2} \rceil$ and $n_2 = \lfloor \frac{n}{2} \rfloor$.

Yang et al. [44] gave the upper and lower bounds for the Lanzhou index $Lz(G)$ with respect to the number of vertices n, maximum degree Δ and minimum degree δ.

Theorem 2.4. [44] *Let $G \in \mathcal{G}_n$ with maximum degree Δ and minimum degree δ. Then*

$$n\delta^2(n-1-\Delta) \leq Lz(G) \leq n\Delta^2(n-1-\delta),$$

with both equalities if and only if G is a regular graph.

Theorem 2.5. [44] *Let $G \in \mathcal{G}_n$ with maximum degree Δ, minimum degree δ and $n-1 = 2\Delta$. Then*

$$Lz(G) \leq \begin{cases} \frac{n\Delta}{2}(\delta^2 + (n-1-\delta)^2), & if\ \delta + \Delta \leq n-1 \\ \frac{n\Delta}{2}(\Delta^2 + (n-1-\Delta)^2), & if\ \delta + \Delta \geq n-1 \end{cases},$$

with equality if and only if G is a Δ-regular graph.

Theorem 2.6. [44] *Let $G \in \mathcal{G}_{n,m}$ with maximum degree Δ and minimum degree δ. Then*

$$2m(\delta(n-1)-\Delta^2) \leq Lz(G) \leq \frac{m(n-1)^2}{2},$$

with both equalities if and only if n is even and $d_u(\overline{G}) = d_u(G)$ for all $u \in V(G)$.

Theorem 2.7. [44] *Let $G \in \mathcal{G}_n$ and $d_u(G) = \Delta$ or δ for any $u \in V(G)$. Then*

$$n(n-1) - n\Delta^3 \leq Lz(G) \leq n(n-1) - n\delta^3,$$

with both equalities if and only if G is a regular graph.

Theorem 2.8. [9] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then*

$$(n-1)\delta - \Delta^2 \leq \frac{Lz(G)}{2m} \leq (n-1)\Delta - \delta^2,$$

with both equalities if and only if G is a regular graph.

2.2 Trees

Let $\mathcal{T}_n$ be a class of trees with n vertices. Let S_{n_1,n_2} be the double star graph with $n = n_1 + n_2$ vertices.

Theorem 2.9. [41] *Let $n \geq 15$ and $T \in \mathcal{T}_n$. Then*

$$(n-1)(n-2) \leq Lz(T) \leq \begin{cases} \frac{n^3}{4} + \frac{n^2}{2} - 4n + 4, & if\ n\ is\ even \\ \frac{n^3}{4} + \frac{n^2}{2} - \frac{17}{4}n + \frac{7}{2}, & if\ n\ is\ odd \end{cases}.$$

The left equality holds if and only if $T \cong S_n$. The right equality holds if and only if $T \cong \begin{cases} S_{\frac{n}{2},\frac{n}{2}}, & if\ n\ is\ even \\ S_{(n-1)/2,(n+1)/2}, & if\ n\ is\ odd \end{cases}.$

In [41], authors also investigate the extremal values of the Lanzhou index for $\mathcal{T}_n$ when $4 \leq n \leq 14$.

The broom graph $Y_{n,k}$, constructed from the star S_k, is formed by replacing one of its pendent edges with a path of length $n-k$. Subsequent research by Amin et al. [5] determined the trees with the smallest, second-smallest, and third-smallest Lanzhou index.

Theorem 2.10. [5] *Let $T \in \mathcal{T}_n$. Then*

(i) *If $n > 3$, then the star S_n achieves the smallest Lanzhou index.*

(ii) *If $n > 5$, then the broom graph (also called comet) $Y_{n,n-2}$ achieves second smallest Lanzhou index.*

(iii) *If $n > 9$, then the broom graph (also called comet) $Y_{n,n-3}$ achieves third smallest Lanzhou index.*

Liu et al. [31] establish some bounds for Lanzhou index of trees.

Theorem 2.11. [31] *Let $T \in \mathcal{T}_n$. Then*

(i) *If $n \geq 4$, then $F(T) + Lz(T) \geq (n-1)(\Delta^2 + 2 + \frac{(2(n-2)-\Delta)^2}{n-3})$;*

(ii) *If $n \geq 2$, then $F(T) + Lz(T) \leq (n-1)(2(n-1) + (n-2)\Delta)$,*

with equality of (i) if and only if T is a tree such that $\Delta = d_1 \geq d_2 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$, equality of (ii) if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$ for some t, $2 \leq t \leq n-1$.

Theorem 2.12. [31] *Let $T \in \mathcal{T}_n$. Then*

$$2(n-1)(2n-3) \leq F(T) + Lz(T) \leq n(n-1)^2,$$

with left equality if and only if $T \cong P_n$, right equality if and only if $T \cong S_n$.

Theorem 2.13. [31] *Let $T \in \mathcal{T}_n$. Then*

(i) *If $n \geq 4$, then $F(\overline{T}) + Lz(\overline{T}) \geq (n-1)((n-1)^2(n-4) + \Delta^2 + 2 + \frac{(2(n-2)-\Delta)^2}{n-3})$;*

(ii) *If $n \geq 2$, then $F(\overline{T}) + Lz(\overline{T}) \leq (n-1)((n-1)^2(n-4) + 2(n-1) + (n-2)\Delta)$,*

with equality of (i) if and only if T is a tree such that $\Delta = d_1 \geq d_2 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$, equality of (ii) if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$ for some t, $2 \leq t \leq n-1$.

Corollary 2.1. [31] *Let $T \in \mathcal{T}_n$. Then*

$$(n-1)(n-2)(n^2-4n+5) \le F(\overline{T}) + Lz(\overline{T}) \le (n-1)^2(n-2)^2,$$

with left equality if and only if $T \cong P_n$, right equality if and only if $T \cong S_n$.

Wang et al. [42] established exact expressions for the $Lz(G)$ of trees with given some specified diameters.

Theorem 2.14. [42] *Let $P_\ell(m_1, m_2, \cdots, m_\ell)$ be as the graph constructed from the path P_ℓ by attaching m_i pendent edges to the i-th vertex of P_ℓ for $i \in \{1, 2, \cdots, \ell\}$. Then*

$$\begin{aligned} Lz(P_\ell(m_1, m_2, \cdots, m_\ell)) =& m(n-2) + \sum_{i=1,\ell} (n-m_i-2)(m_i+1)^2 \\ &+ \sum_{i=2}^{r-1} (n-m_1-3)(m_1+2)^2, \end{aligned}$$

where $m = \sum_{j=1}^{\ell} m_j$ and $n = m + r$.

By majorization techniques, Wei et al [43] subsequently determined the maximum Lanzhou index among trees with given diameter $d \ge 8$.

Theorem 2.15. [43] *Let $T \in \mathcal{T}_n$ with diameter $d \ge 8$ and maximal Lanzhou index.*

(i) *If $8 \le d \le \frac{n+10}{3}$, then $\pi(T) = (\lceil \frac{n+3-d}{2} \rceil, \lfloor \frac{n+3-d}{2} \rfloor, 2^{(d-3)}, 1^{(n+1-d)})$;*
(ii) *If $d = \frac{n+11}{3}$, then $\pi(T) = (d_1, n+3-d-d_1, 2^{(d-3)}, 1^{(n+1-d)})$;*
(iii) *If $\frac{n+12}{3} \le d \le n-1$, then $\pi(T) = (n+1-d, 2^{(d-2)}, 1^{(n+1-d)})$.*

Let $\mathcal{T}_n^{\Delta}$ denote the family of all n-vertex trees with maximum degree at most Δ.

Theorem 2.16. [41] *Let $n \ge 8$ and $T \in \mathcal{T}_n^3$. Then*

$$4n^2 - 18n + 20 \le Lz(T) \le 5n^2 - 27n + 34 - (n-7)\frac{1-(-1)^n}{2}.$$

The left equality holds if and only if $T \cong P_n$. The right equality holds by any tree that without vertices of degree 2 if n is even, and for any tree featuring exactly one vertex of degree 2 if n is odd.

Theorem 2.17. [41] *Let $n \geq 8$ and $T \in \mathcal{T}_n^4$. Then*

$$4n^2 - 18n + 20 \leq Lz(T) \leq 6n^2 + O(n).$$

The left equality holds if and only if $T \cong P_n$. The greatest value of $Lz(T)$ is attained by trees with the most possible vertices of degree 4.

Let $\mathcal{T}(n, \Delta)$ denote the family of all n-vertex trees with maximum degree Δ. A spider is formally characterized as a tree containing at most one vertex of degree greater than 2, which is designated as the spider's center [14].

Theorem 2.18. [14] *Let $n \geq 11$ and $T \in \mathcal{T}(n, \Delta)$. Then*

$$Lz(T) \geq (n - \Delta - 1)(4n + \Delta^2 - 12) + \Delta(n - 2),$$

with equality if and only if T is a spider.

Theorem 2.19. [37] *Let $n \geq 4$ and $T \in \mathcal{T}(n, \Delta)$. Then*

$$Lz(T) \geq 2(n-2) + (n-1-\Delta)\Delta^2 + \frac{(2(n-2)^2 - (2n-3-\Delta)\Delta)^2}{n^2 - 6n + 7 + \Delta},$$

with equality if and only if $T \cong S_n$ or P_n.

Theorem 2.20. [37] *Let $n \geq 4$ and $T \in \mathcal{T}(n, \Delta)$. Then*

$$Lz(\overline{T}) \geq 2(n-2)^2 + \Delta(n-1-\Delta)^2 + \frac{(2(n-2)^2 - (2n-3-\Delta)\Delta)^2}{2(n-2) - \Delta},$$

with equality if and only if $T \cong S_n$ or P_n.

Li et al. [29] obtained the upper bounds on $Lz(T)$ for $T \in \mathcal{T}(n, \Delta)$.

Theorem 2.21. [29] *Let $T \in \mathcal{T}(n, \Delta)$.*
(I) *For $2 \leq \Delta < \frac{n+2}{3}$ and $n \equiv i + 1 \pmod{\Delta - 1}$, where $1 \leq i \leq \Delta - 1$,*

$$\begin{aligned} Lz(T) \leq & \frac{1}{\Delta - 1}[(-n+1+i)\Delta^3 + (n-1)(n-1-i)\Delta^2 \\ & + (n^2 + ni^2 - i^2 - i^3 - 3n + 2)\Delta - (n-2)(n-1) \\ & - (n-1-i)(i^2 + n - 2)]. \end{aligned}$$

with equality if and only if $\pi(T) = (\underbrace{\Delta, \Delta, \cdots, \Delta}_{\frac{n-1-i}{\Delta-1}}, i, \underbrace{1, 1, \cdots, 1}_{n-\frac{n-1-i}{\Delta-1}-1})$.

(II) *For* $\frac{n+2}{3} \le \Delta \le \frac{n+5}{3}$.

(i) *If* $n \equiv 0 \pmod 3$*, then*

$$Lz(T) \le -3\Delta^2 + (3n+3)\Delta^2 - (4n+2)\Delta + n^2 - 3n + 6,$$

with equality if and only if $\pi(T) = (\Delta, \Delta-1, \Delta-1, \underbrace{1, 1, \cdots, 1}_{n-3})$.

(ii) *If* $n \equiv 1 \pmod 3$ *and* $\Delta = \frac{n+2}{3} (n \ne 7)$*, then*

$$Lz(T) \le -3\Delta^3 + 3n\Delta^2 - (2n+1)\Delta + n^2 - 4n + 6,$$

with equality if and only if $\pi(T) = (\Delta, \Delta, \Delta-1, \underbrace{1, 1, \cdots, 1}_{n-3})$.

If $n = 7$*, then* $Lz(T) \le 90$ *with equality if and only if* $\pi(T) = (3, 3, 2, 1, 1, 1, 1)$ *or* $\pi(T) = (3, 2, 2, 2, 1, 1, 1)$.

(iii) *If* $n \equiv 1 \pmod 3$ *and* $\Delta = \frac{n+5}{3}$*, then*

$$Lz(T) \le -3\Delta^3 + (3n+9)\Delta^2 - (8n+40)\Delta + n^2 + 11n + 54,$$

with equality if and only if $\pi(T) = (\Delta, \Delta, \Delta-4, \underbrace{1, 1, \cdots, 1}_{n-3})$ *or* $\pi(T) = (\Delta, \Delta-1, \Delta-3, \underbrace{1, 1, \cdots, 1}_{n-3})$ *or* $\pi(T) = (\Delta, \Delta-2, \Delta-2, \underbrace{1, 1, \cdots, 1}_{n-3})$.

(iv) *If* $n \equiv 2 \pmod 3$*, then*

$$Lz(T) \le -3\Delta^3 + (3n+6)\Delta^2 - (6n+9)\Delta + n^2 + 10,$$

with equality if and only if $\pi(T) = (\Delta, \Delta-1, \Delta-2, \underbrace{1, 1, \cdots, 1}_{n-3})$.

(III) *If* $\frac{n+5}{3} < \Delta \le \frac{n}{2}$*, then*

$$Lz(T) \le 6\Delta^3 - 6(n+3)\Delta^2 + 2(n^2+6n+5)\Delta - n^2 - 9n + 4,$$

with equality if and only if $\pi(T) = (\Delta, \Delta, n+1-2\Delta, \underbrace{1, 1, \cdots, 1}_{n-3})$.

(IV) *If* $\frac{n+1}{2} \leq \Delta \leq n-1$, *then*

$$Lz(T) \leq -(n+2)\Delta^2 + (n^2+2n)\Delta - 4n + 4,$$

with equality if and only if $\pi(T) = (\Delta, n-\Delta, \underbrace{1, 1, \cdots, 1}_{n-2})$.

Let $\mathcal{T}(n, \Delta, \Delta')$ denote the set of n-vertex trees with maximum degree Δ and second maximum degree Δ'. A vertex v is called as a branching vertex if $d_v \geq 3$. A double spider is formally defined as a tree containing precisely two branching vertices. Saha [33] established an extension of the results originally presented by Dehgardi and Liu [14].

Theorem 2.22. [33] *Let* $n \geq 11$ *and* $T \in \mathcal{T}(n, \Delta, \Delta')$. *Then*

$$Lz(T) \geq (n-1)(\Delta^2+(\Delta')^2)-(\Delta^3+(\Delta')^3)-(3n-10)(\Delta+\Delta')+(4n^2-14n+4).$$

The equality holds if and only if T is a double spider with the degrees of two branching vertices Δ and Δ'.

2.3 Unicyclic graphs and c-cyclic graphs

Let $\mathcal{U}_n$ be a class of unicyclic graphs with n vertices. The authors of [24, 29, 30] independently determined the minimum Lanzhou index among $\mathcal{U}_n$.

Theorem 2.23. [24, 29, 30] *Let* $G \in \mathcal{U}_n$ *with* $n \geq 3$. *Then*

$$Lz(G) \geq n^2 + 3n - 18,$$

with equality if and only if $G \cong S_n^+$, *where* S_n^+ *is the graph obtained from* S_n *by adding one edge between its two pendent vertices.*

Let $U_n(a, b, c)$ denote the unicyclic graph constructed from a triangle C_3 by appending a, b, and c pendent vertices to its three distinct vertices respectively, where $a + b + c + 3 = n$.

Theorem 2.24. [30] *Among the unicyclic graphs with* n *vertices.*

(i) *If* $11 \leq n \leq 26$, *the maximal Lanzhou index is achieved by the unicyclic graph* $U_n(a_1, a_2, a_3)$ *with* $a_1 + a_2 + a_3 + 3 = n$ *and* $\max\limits_{1 \leq i < j \leq 3} |a_i - a_j| \leq 1$.

(ii) *If* $n = 27$, *the maximal Lanzhou index is achieved by* $U_{27}(12, 12, 0)$ *or* $U_{27}(8, 8, 8)$.

(iii) *If* $n \geq 28$, *the maximal Lanzhou index is achieved by the graph* $U_n\left(\lceil \frac{n-2}{2} \rceil - 1,\ \lfloor \frac{n}{2} \rfloor - 1,\ 0\right)$.

Through distinct methodological approaches, Li et al. [29] and Imran et al. [24] independently established rigorous characterizations of maximal Lanzhou index for unicyclic graphs with $n \geq 28$ vertices. Complementing these results, Liu et al. [30] conducted a complete characterization of extremal unicyclic graphs attaining maximal Lanzhou index for $3 \leq n \leq 10$.

Let $G' \in \mathcal{U}_n$ be the unicyclic graphs obtained from C_{n-k} by attaching k pendent vertices to one vertex of C_{n-k}. Let $G'' \in \mathcal{U}_n$ be the unicyclic graphs obtained from C_{n-k} by attaching k pendent vertices to different k vertex of C_{n-k}. In [7], Alrowaili et al. established a complete characterization of maximal and minimal Lanzhou indices among $\mathcal{U}_n$ with k pendent vertices.

Theorem 2.25. [7] *Let* $G \in \mathcal{U}_n$ *with* k *pendent vertices. Then*

$$4n(n-3) - k^2(k+7) - k(6-nk) + nk \leq Lz(G) \leq 2(nk + 2n^2 - 7k - 6n),$$

with left equality if and only if $G \cong G'$, *right equality if and only if* $G \cong G''$.

By majorization techniques, Wei et al. [43] subsequently established the maximal Lanzhou index for unicyclic graphs with given diameter $d \geq 9$ and $n \geq 3d - 8$. Let $C_3(n, d, i)$ be the graph obtained from $C_3 = z_1 z_2 z_3 z_1$ by attaching the path P_i and $\lceil \frac{n-d-2}{2} \rceil$ isolated vertices to vertex z_1, and then attaching the path P_{d-i-1} and $\lfloor \frac{n-d-2}{2} \rfloor$ isolated vertices to vertex z_2. Let $C_4(n, d, i)$ be the graph obtained from $C_4 = z_1 z_2 z_3 z_4 z_1$ by attaching the path P_i and $\lceil \frac{n-d-2}{2} \rceil$ isolated vertices to vertex z_1, and then attaching the path P_{d-i-2} and $\lfloor \frac{n-d-2}{2} \rfloor$ isolated vertices to vertex z_3.

Theorem 2.26. [43] *Let* G *be a unicyclic graph with maximum Lanzhou index among unicyclic graphs with* n *vertices and diameter* $d \geq 8$. *If*

$d \geq 9$ *and* $n \geq 3d - 8$, *then* $G \in \{C_3(n, d, 1), C_3(n, d, 2), \cdots, C_3(n, d, d-2), C_4(n, d, 1), C_4(n, d, 2), \cdots, C_4(n, d, d-3)\}$.

Let $\mathcal{U}_{n,\Delta}$ be the class of unicyclic graphs with maximum Δ. Let $G \in U_{n,\Delta}$ and C be the unique cycle of G. If $G - E(C)$ is some independent vertices and a spider, which center is on cycle C and has $\Delta - 2$ legs, then such graph is denoted by $\mathcal{U}^S_{n,\Delta}$.

Theorem 2.27. [11] *Let* $G \in U_{n,\Delta}$ *with maximum Lanzhou index and* $n \geq 11$. *Then*

$$Lz(G) \geq 4(n-3)(n-\Delta+1) + \Delta^2(n-\Delta-1) + (n-2)(\Delta-2),$$

with equality if and only if $G \in \mathcal{U}^S_{n,\Delta}$.

Theorem 2.28. [9] *Let* $G \in U_{n,\Delta}$ *with* $\Delta(G) = 4$ *and* $n \geq 8$. *Then*

$$Lz(G) \leq 6n^2 + O(n).$$

Let F_c denote the graph constructed by merging c triangles at a common vertex, known as Dutch windmill graphs. Lan et al. [28] subsequently characterized maximal and minimal Lanzhou indices for c-cyclic graphs with n vertices.

Theorem 2.29. [28] *Let* G *be a c-cyclic graph with* n *vertices. If* $n \geq 3c+4$ *and* $c \geq 1$, *then*

$$Lz(G) \geq (n-1)(n-2) + 2c(3n-10),$$

with equality if and only if $G \cong H_0$, *where* H_0 *denotes the graph constructed from* F_c *by appending* $n - 2c - 1$ *pendent vertices to the unique vertex of degree* $2c$.

Theorem 2.30. [28] *Let* G *be a c-cyclic graph with* n *vertices. If* $3 \leq c \leq \frac{n}{13}$, *then* $Lz(G) \leq Lz(G_0)$, *with equality if and only if* $G \cong G_0$, *where* G_0 *is the graph constructed from* W_0 *(see Figure 1) by appending* $\left\lfloor \frac{n-c-2}{2} \right\rfloor$ *pendent vertices to vertex* v_1 *and* $\left\lceil \frac{n-c-2}{2} \right\rceil$ *pendent vertices to vertex* v_2.

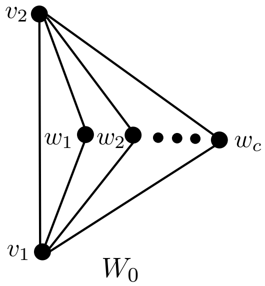

Figure 1. The graph W_0 [28].

2.4 Chemical graphs

Let $GH(m, k_1, k_2, k_3, k_4)$ (see Figure 2) be a general hexagonal system [27], where $m \geq 1$ is the number of benzenoids in the lowest layer, $0 \leq k_1 \leq k_3 \leq m$, $0 \leq k_4 \leq k_2 \leq m$ and $k_1 + k_2 = k_3 + k_4$. In [42], Wang et al. calculated Lanzhou index of $GH(m, k_1, k_2, k_3, k_4)$.

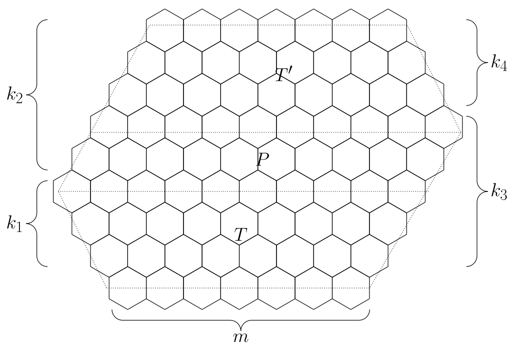

Figure 2. A general hexagonal system $GH(m, k_1, k_2, k_3, k_4)$ [27].

Theorem 2.31. [42] *Let $m \geq 1$, $0 \leq k_1 \leq k_3 \leq m$, $0 \leq k_4 \leq k_2 \leq m$ and $k_1 + k_2 = k_3 + k_4$. Then*

$$Lz(GH(m, k_1, k_2, k_3, k_4)) = 9n_3(n_2 + n_3 - 4) + 4n_2(n_2 + n_3 - 3),$$

where $n_3 = 2m(k_1 + k_2 + 1) + k_1(k_1 + 2k_2 + 1) - k_4(k_4 + 1) - 2$ and $n_2 = 2m + 3k_1 + 2k_2 - k_4 + 4$.

In hexagonal systems, a vertex is called an internal vertex if the vertex shared by three hexagons. The hexagonal system without internal vertices is called a catacondensed hexagonal system.

Theorem 2.32. [42] *Let HS_n be a catacondensed hexagonal system with n hexagons. Then $Lz(HS_n) = 104n^2 - 52n + 20$.*

In [18], Ghalavand et al. ordered the chemical graphs with respect to Lanzhou index. They determined the first twenty-six chemical trees for $n \geq 22$, the first thirty-five chemical unicyclic graphs for $n \geq 24$, the first thirty-three chemical bicyclic graphs for $n \geq 22$, the first thirty-nine chemical tricyclic graphs for $n \geq 24$, the first thirty-three chemical tetracyclic graphs for $n \geq 22$, the first thirty-seven chemical pentacyclic graphs for $n \geq 24$. Further, they determined the first forty-two chemical graphs for $n \geq 22$.

Ali et al. [3] determined the graphs with minimum Lanzhou index among connected chemical (n, m)-graphs, where n and m satisfy the conditions $3n \geq 2m, n \geq 4, m \geq n + 1$.

Theorem 2.33. [3] *Let n and m be fixed integers satisfying the conditions $3n \geq 2m, n \geq 4, m \geq n + 1$. Then among all connected chemical (n, m)-graphs, those with maximum degree 3 and minimum degree at least 2 attain the minimum Lanzhou index.*

2.5 Graph operations

Bera and Das [9] computed the Lanzhou index of corona and joined graphs.

Theorem 2.34. [9] *Let G and H be two graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$ and $|E(H)| = m_2$. Then*

$$\begin{aligned} Lz(G \vee H) = & Lz(G) + Lz(H) + 4n_2(n_1 - 1)m_1 + 4n_1(n_1 - 1)m_2 \\ & - 2n_2M_1(G) - 2n_1M_1(H) + n_1n_2(2n_1n_2 - n_1 - n_2) \\ & - 2n_2^2m_1 - 2n_1^2m_2. \end{aligned}$$

Theorem 2.35. [9] *Let G and H be two graphs with $|V(G)| = n_1$, $|V(H)| = n_2$, $|E(G)| = m_1$ and $|E(H)| = m_2$. Then*

$$\begin{aligned}Lz(G \circ H) =& Lz(G) + n_1 Lz(H) + 2n_2(2n_1n_2 + 2n_1 - 3n_2 - 2)m_1 \\ & - 3n_2 M_1(G) + n_1 n_2^2 (n_1 - 1)(n_2 + 1) - 3n_1 M_1(H) \\ & + 2n_1(2n_1n_2 + 2n_1 - 5)m_2 + n_1n_2(n_1n_2 + n_1 - 2).\end{aligned}$$

Denote by $R(G)$ the graph obtained from G by adding a new vertex corresponding to each edge of G and connecting every new vertex to the end vertices of corresponding edge [12]. The balanced double star with n vertices is either $S_{\frac{n}{2},\frac{n}{2}}$ when n is even or $S_{\frac{n+1}{2},\frac{n-1}{2}}$ when n is odd. Let BS_n be the balanced double star with n vertices. Let n be odd and $T_{\frac{n+1}{2}}$ be a tree with $\frac{n+1}{2}$ vertices. Zeng et al. [45] obtained the sharp upper bound of Lanzhou index of $R\left(T_{\frac{n+1}{2}}\right)$.

Theorem 2.36. [45] *Let n be odd and $R\left(T_{\frac{n+1}{2}}\right)$ a graph with $n \geq 27$ vertices. Then*

$$Lz\left(R\left(S_{\frac{n+1}{2}}\right)\right) \leq Lz\left(R\left(T_{\frac{n+1}{2}}\right)\right) \leq Lz\left(R\left(BS_{\frac{n+1}{2}}\right)\right).$$

It was also shown that the result of Theorem 2.36 is not true for $n \leq 26$. The Cartesian product [10] is a widely studied operation in graph theory. Wang et al. [42] calculated Lanzhou index of Cartesian product graphs.

Theorem 2.37. [42] *Let $\prod_{j=1}^{k} P_{m_j+2} = P_{m_1+2} \square P_{m_2+2} \square \cdots \square P_{m_k+2}$ be the Cartesian product of paths $\{P_{m_j+2}\}_{j=1}^{k}$, where k is a positive integer and m_j is a non-negative integer for $j = 1, 2, \cdots, k$. Then*

$$Lz\left(\prod_{j=1}^{k} P_{m_j+2}\right) = \sum_{d=0}^{k} (n-1-k-d)(k+d)^2 2^{k-d} \sum_{J \in [k]^d} \prod_{j \in J} m_j,$$

where $[k]^d = \{S : S \subseteq \{1, 2, \cdots, k\}$ and $|S| = d\}$ and $n = \prod_{j=1}^{k} (m_j + 2)$.

Especially, if $m_j = 0$ for $j = 1, 2, \cdots, k$, hence $\Delta = k$, then the Cartesian product is a k-dimension cube, and Lanzhou index is $2^k k^2 (2^k - k - 1)$;

if $m_j = m > 0$ *for* $j = 1, 2, \cdots, k$*, hence* $\Delta = 2k$*, then Lanzhou index is*

$$\sum_{d=0}^{k}((m+2)^k - 1 - k - d)(k+d)^2 2^{k-d}\binom{k}{d}m^d.$$

De et al. [15] investigated Lanzhou index under several graph operations such as union, join, Cartesian product, composition, tensor product, strong product, corona product, symmetric difference of graphs. Other results about Lanzhou index of the operations on graphs can be found in [6, 36]. Jahanbani et al. [25] also obtained some bounds for Lanzhou index, most of the bounds are not sharp. The chemical applications of Lanzhou index can be found in [2, 15, 30].

3 Relations between Lanzhou index and other indices

In this section, we present the relations between Lanzhou index and other indices. Hua et al. [23] systematically investigated the interrelations among three degree-based topological indices, namely, the Lanzhou index $Lz(G)$, the second Zagreb index $M_2(G)$, and the forgotten index $F(G)$.

Theorem 3.1. [23] *Let* $G \in \mathcal{G}_n$ *with independence number* $\alpha(G)$*. Then*

$$Lz(G) \geq F(G) - 2\alpha M_2(G),$$

with equality if and only if $G \cong K_n$ *or* $\overline{K_n}$*.*

Theorem 3.2. [23] *Let* $G \in \mathcal{G}_n$*. Then*

$$Lz(G) \geq 2M_2(G) - F(G),$$

with equality if and only if $G \cong K_n$ *or* $\overline{K_n}$*.*

Theorem 3.3. [23] *Let* $T \in \mathcal{T}_n$ *with* $n \geq 3$*. Then*

$$Lz(T) \geq 3M_2(T) - F(T),$$

with equality if and only if $T \cong P_3$.

The eccentricity of a vertex v is defined as $ecc_G(v) = \max\{d_G(u,v)|u \in V(G)\}$. The radius of a graph is defined as $r(G) = \min\{ecc_G(v)|v \in V(G)\}$. The eccentric connectivity index [35] of a graph G is defined as $\xi^c(G) = \sum\limits_{u \in V(G)} d_u ecc_G(u)$. The Schultz index [34] of a graph G is defined as $SI(G) = \frac{1}{2} \sum\limits_{\{u,v\} \subseteq V(G)} (d_u + d_v) d_G(u,v)$. Bera et al. [9] gave some bounds for Lanzhou index by $M_1(G)$, $M_2(G)$, $\xi^c(G)$, $irr(G)$, $SI(G)$, $F(G)$, $ISI(G)$ and $SDD(G)$.

Theorem 3.4. [9] *Let* $G \in \mathcal{G}_n$. *Then*

$$Lz(G) \geq (r(G) - 1)M_1(G),$$

with equality if and only if $G \cong K_n$ *or* $G \cong K_n - \frac{n}{2}K_2$ (n is even).

Theorem 3.5. [9] *Let* $G \in \mathcal{G}_n$. *If* $d_u + d_v \geq n$ *for any* $uv \in E(G)$, *then*

$$Lz(G) \leq 2M_2(G) - M_1(G).$$

Theorem 3.6. [9] *Let* $G \in \mathcal{G}_n$ *with maximum degree* Δ *and minimum degree* δ. *Then*

$$Lz(G) \leq (n-1)M_1(G) - 2M_2(G),$$

with equality if and only if every connected component of G *is regular.*

Moreover,

$$(n-1-\Delta)M_1(G) \leq Lz(G) \leq (n-1-\delta)M_1(G),$$

with equality if and only if G *is a regular graph.*

Theorem 3.7. [9] *Let* $G \in \mathcal{G}_n$ *with minimum degree* δ. *Then*

$$Lz(G) \geq \delta\xi^c(G) - M_1(G),$$

with equality if and only if $G \cong K_n$ *or* $G \cong K_n - \frac{n}{2}K_2$ (n is even).

Theorem 3.8. [9] *Let $G \in \mathcal{G}_n$ be a connected graph with minimum degree δ. Then*

$$\frac{2(n-2)}{\Delta}M_2(G) \leq Lz(G) \leq \frac{2(n-2)}{\delta}M_2(G),$$

with equality if and only if G is a regular graph.

Theorem 3.9. [9] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then*

$$(n-1)M_1(G) \leq irr(G)^2 + Lz(G) + 2M_2(G) \leq (n-1)M_1(G) + 2\binom{m}{2}(\Delta-\delta)^2,$$

with left equality if and only if G is a regular graph, right equality if and only if G is a regular graph or a bipartite semiregular graph.

Theorem 3.10. [9] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with minimum degree δ. Then*

$$SI(G) \geq \frac{1}{2(n-1)}(Lz(G) + F(G)) + (n(n-1) - 2m)\delta,$$

with equality if and only if $G \cong K_n$ or $G \cong K_\delta \vee (n-\delta)K_1 (\delta < n-1)$ or G is a regular graph with diameter 2.

Theorem 3.11. [9] *Let $G \in \mathcal{G}_n$ with maximum degree Δ. Then*

$$Lz(G) \geq 4(n-1-\Delta)ISI(G),$$

with equality if and only if G is a regular graph.

Theorem 3.12. [9] *Let $G \in \mathcal{G}_n$ with maximum degree Δ. Then*

$$Lz(G) \leq \Delta^2(n(n-1) - SDD(G)),$$

with equality if and only if G is a regular graph.

Ghalavand et al. [18] gave some bounds for Lanzhou index by $M_1(G)$, $F(G)$, minimum degree δ and maximum degree Δ.

Theorem 3.13. [18] *Let $G \in \mathcal{G}_{n,m}$. Then*

$$Lz(G) \leq \sqrt{[(n-1)(n^2-n-4m) + M_1(G)]M_1^4(G)},$$

with equality if and only if G is a regular graph.

Theorem 3.14. [18] *Let $G \in \mathcal{G}_n$. Then*

$$Lz(G) \leq \frac{[(n-1)M_1(G)]^2}{4F(G)},$$

with equality if and only if G is a $\frac{n-1}{2}$-regular graph.

Theorem 3.15. [18] *Let $G \in \mathcal{G}_{n,m}$ and $G \ncong K_n$. Then*

$$Lz(G) \geq \frac{(n-1)^2 M_1(G) + M_1^4(G) - 2(n-1)F(G) + 2m(n-1-\Delta)(n-1-\delta)}{2(n-1)-(\Delta+\delta)}.$$

If G is not a regular graph and $d_v(G) \in \{a, b\}$ for all $v \in V(G)$, then we call G is a (a, b)-regular graph.

Theorem 3.16. [18] *Let $G \in \mathcal{G}_{n,m}$. Then*

(i) $Lz(G) \geq 2m((n-1)(2\delta+1) + \delta(\delta+1)) - (2\delta+1)M_1(G) - n(n-1)\delta(\delta+1)$;

(ii) $Lz(G) \geq 2m((n-1)(2\Delta-1) + \Delta(\Delta-1)) - (2\Delta-1)M_1(G) - n(n-1)\Delta(\Delta+1)$;

(iii) $Lz(G) \leq 2m((n-1)(\delta+\Delta) + \delta\Delta) - (\delta+\Delta)M_1(G) - n(n-1)\delta\Delta$.

With equalities if and only if G is a regular graph or (a, b)-regular graph.

Yang et al. [44] gave some bounds for Lanzhou index by $HM(G)$, $F(G)$, $SO(G)$, $R(G)$, $RR(G)$, $SDD(G)$, $SC(G)$ and $RSC(G)$.

Theorem 3.17. [44] *Let $G \in \mathcal{G}_{n,m}$. Then*

$$\frac{4(n-1)m^2}{n} - HM(G) \leq Lz(G) \leq \frac{(n-1)m}{2} + \frac{n-3}{2}HM(G),$$

with equalities if and only if $G \cong K_n$.

Theorem 3.18. [44] *Let $G \in \mathcal{G}_{n,m}$ with maximum degree Δ and minimum degree δ. Then*

$$Lz(G) \geq (n-1)m\left(\frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2} + 4\sqrt{2}\Delta}\right) + (n-1)(1-\sqrt{2})SO(G),$$

with equality if and only if G is a regular graph.

Theorem 3.19. [44] *Let $G \in \mathcal{G}_n$ with minimum degree δ. Then*

$$Lz(G) \leq (n-1-\sqrt{\delta})\sqrt{2}SO(G),$$

with equality if and only if G is an empty (edgeless) graph.

Theorem 3.20. [44] *Let $G \in \mathcal{G}_{n,m}$ with maximum degree Δ and minimum degree δ. Then*

$$2(\delta^2(n-1)-\Delta^3)R(G) \leq Lz(G) \leq (\Delta(m-1)(n-1)-\delta^3)R(G),$$

with equalities if and only if G is a regular graph and $d_u(G)+d_v(G) = m+1$ for $uv \in E(G)$.

Theorem 3.21. [44] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then*

$$(2(n-1)-\frac{2\Delta^2}{\delta})RR(G) \leq Lz(G) \leq (\sqrt{2}(n-1)(\frac{\Delta}{\delta}+\frac{\delta}{\Delta})-2)RR(G),$$

with equalities if and only if G is a regular graph.

Theorem 3.22. [44] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then*

$$(\frac{2(n-1)\Delta\delta^2}{\Delta^2+\delta^2}-\Delta^2)SDD(G) \leq Lz(G) \leq (\frac{(n-1)(m+1)\Delta}{2\delta}-\delta^2)SDD(G),$$

with equalities if and only if G is a regular graph and $d_u(G)+d_v(G) = m+1$ for $uv \in E(G)$.

Theorem 3.23. [44] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then*

$$(\sqrt{2}(n-1)\delta-\frac{\sqrt{2}\Delta^2}{\delta})RSC(G) \leq Lz(G) \leq (\sqrt{2}(n-1)\Delta-\frac{\sqrt{2}\delta^2}{\Delta})RSC(G),$$

with equalities if and only if G is a regular graph.

Theorem 3.24. [44] *Let $G \in \mathcal{G}_{n,m}$ be a connected graph with maximum degree Δ and minimum degree δ. Then $(2\sqrt{2}(n-1)\delta^{\frac{5}{2}}-2\Delta^2\sqrt{2(m+1)})SC$*

$(G) \leq Lz(G) \leq ((n-1)(m+1)^{\frac{3}{2}} - 2\sqrt{2}\delta^{\frac{5}{2}})SC(G)$, *with equalities if and only if* G *is a regular graph and* $d_u(G) + d_v(G) = m+1$ *for* $uv \in E(G)$.

Let G be a graph with n vertices and $d_{v_i} = d_i$, then $d_1 \geq d_2 \geq \cdots \geq d_n$ denotes the degree sequence of G. The first Zagreb coindex $\overline{M_1}(G)$ [13] of a graph G is defined as $\overline{M_1}(G) = \sum\limits_{uv \notin E(G)} (d_u + d_v)$. Inverse degree index $ID(G)$ [16] of a graph G is defined as $ID(G) = \sum\limits_{v \in V(G)} \frac{1}{d_v}$. Inverse degree coindex $\overline{ID}(G)$ of a graph G is defined as $\overline{ID}(G) = \sum\limits_{uv \notin V(G)} (\frac{1}{d_u^2} + \frac{1}{d_v^2})$.

Theorem 3.25. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $n \geq 2$, $\delta \geq 1$. *Then*

$$Lz(G) \leq (\Delta+2\delta)\overline{M_1}(G) + \Delta\delta^2[(n-1)ID(G) - n] - \delta(2\Delta+\delta)[n(n-1) - 2m],$$

with equality if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta$ *for some* t, $1 \leq t \leq n-1$.

Theorem 3.26. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $\delta \geq 1$. *Then*

$$Lz(\overline{G}) \leq (\Delta + \delta)\overline{M_1}(G) - \Delta\delta[(n-1)^2 ID(G) - 2n(n-1) + 2m],$$

with equality if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta$ *for some* t, $1 \leq t \leq n-1$.

Theorem 3.27. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $n \geq 3$ *vertices,* $\delta \geq 1$. *Then*

$$\begin{aligned}&((n-1)(ID(G) - \frac{\Delta+\delta}{\Delta\delta}) - n + 2)^2(Lz(G) - (n-1)(\Delta^2+\delta^2) + \Delta^3 + \delta^3)\\ &\geq ((n-1)(n-2) - 2m + \delta + \Delta)^3,\end{aligned}$$

with equality if and only if $\Delta = d_1 \geq d_2 = \cdots = d_{n-1} \geq d_n = \delta$ *or* $n - 1 = \Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_{n-1} \geq d_n = \delta$ *for some* t, $1 \leq t \leq n-2$.

Theorem 3.28. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $n \geq 3$, $\delta \geq 1$. *The complement graph* $\overline{G}$ *has* $\overline{m}$ *edges. Then*

$$(\overline{ID}(G) - \frac{n-1-\Delta}{\Delta^2} - \frac{n-1-\delta}{\delta^2})(Lz(G) - (n-1-\Delta)\Delta^2 - (n-1-\delta)\delta^2) \geq (2\overline{m} - 2(n-1) + \Delta + \delta)^2,$$

with equality if and only if $\Delta = d_1 \geq d_2 = \cdots = d_{n-1} \geq d_n = \delta$ *or* $n-1 = \Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_{n-1} \geq d_n = \delta$ *for some* t, $1 \leq t \leq n-1$.

Note that $Lz(G) = \overline{F}(G)$.

Theorem 3.29. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $\delta \geq 1$. *Then*

$$\sqrt{ID(G)F(G)} + \sqrt{\overline{ID}(G)Lz(G)} \geq n(n-1),$$

with equality if and only if G *is a regular graph.*

Let $\overline{\delta}$ be the minimum degree of $\overline{G}$.

Theorem 3.30. [31] *Let* $G \in \mathcal{G}_{n,m}$ *with* $\delta \geq 1$, $\overline{\delta} \geq 1$. *Then*

$$\sqrt{ID(G)F(G)} + \sqrt{ID(\overline{G})F(\overline{G})} \geq n(n-1),$$

$$\sqrt{ID(G)Lz(\overline{G})} + \sqrt{ID(\overline{G})Lz(G)} \geq n(n-1),$$

with equalities if and only if G *is a regular graph.*

4 Nordhaus-Gaddum-type results

In this section, we collect some Nordhaus-Gaddum-type results of Lanzhou index.

Theorem 4.1. [42] *Let* $G \in \mathcal{G}_n$. *Then*

$$0 \leq Lz(G) + Lz(\overline{G}) \leq \frac{1}{4}n(n-1)^3,$$

with left equality if and only if $G \cong K_n$ *or* $\overline{K_n}$, *with right equality if and only if* n *is odd and* G *is a* $\frac{n-1}{2}$*-regular graph.*

Theorem 4.2. [44] *Let* $G \in \mathcal{G}_{n,m}$. *Then*

$$0 \leq Lz(G) + Lz(\overline{G}) \leq 2m(n-1)^2 - \frac{4m^2(n-1)}{n},$$

with left equality if and only if $G \cong K_n$*, with right equality if and only if* G *is a regular graph.*

Corollary 4.1. [44] *Let* $G \in \mathcal{G}_{n,m}$*. Then*

$$0 \leq Lz(G)Lz(\overline{G}) \leq \frac{m^2(n-1)^2(n-1-2m)^2}{n^2}.$$

Liu et al. [31] obtained the Nordhaus-Gaddum type results [38] for Lanzhou index of trees.

Theorem 4.3. [31] *Let* $T \in \mathcal{T}_n$*. Then*

(i) *If* $n \geq 2$*, then* $Lz(T) + Lz(\overline{T}) \geq (n-1)(n-2)(2n-2-\Delta)$*;*

(ii) *If* $n \geq 4$*, then* $Lz(T) + Lz(\overline{T}) \leq (n-1)(2(n-1)^2 - \Delta^2 - 2 - \frac{(2(n-2)-\Delta)^2}{n-3})$*,*

with equality of (i) if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$ *for some* t*,* $1 \leq t \leq n-2$*, equality of (ii) if and only if* T *is a tree such that* $\Delta = d_1 \geq d_2 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$*.*

Corollary 4.2. [31] *Let* $T \in \mathcal{T}_n$ *with* $n \geq 2$*. Then*

$$(n-1)^2(n-2) \leq Lz(T) + Lz(\overline{T}) \leq 2(n-1)(n-2)^2,$$

with left equality if and only if $T \cong S_n$*, right equality if and only if* $T \cong P_n$*.*

Further, Milovanović et al. [37] also gave the Nordhaus-Gaddum results of trees with respect to Lanzhou index.

Theorem 4.4. [37] *Let* $T \in \mathcal{T}(n, \Delta)$ *with* $n \geq 4$*. Then*

$$Lz(T) + Lz(\overline{T}) \geq$$
$$(n-1)(2(n-2) + \Delta(n-1-\Delta)^2 + \frac{(n-3)(2(n-2)^2 - (2n-3-\Delta)\Delta)^2}{(2(n-2)-\Delta)(n^2-6n+7+\Delta)}),$$

with equality if and only if $T \cong P_n$ *or* S_n*.*

5 Open problems

In [7], Alrowaili et al. determined the maximum and minimum the Lanzhou index among all unicyclic graphs n vertices and k pendent vertices. This naturally motivates the investigation of extremal graphs with other given parameters, which leads to the following open problems.

Problem 5.1. Characterize the extremal graphs with respect to the Lanzhou index among all unicyclic graphs with several given parameters, such as matching number, domination number, and other graph invariants.

Problem 5.2. Determine the extremal values of the Lanzhou index among all trees with several given parameters, such as matching number, domination number, branching number, and other graph invariants.

If $TI(G)$ is the topological index of graph G, then $TI(\overline{G})$ is called the ad-hoc version of $TI(G)$. Based on this , Ali et al. [4] defined the ad-hoc Lanzhou index, $\widetilde{Lz}(G) = \sum_{u \in V(G)} d_u(\overline{d_u})^2 = Lz(\overline{G})$. It is also interesting to consider the properties of the ad-hoc Lanzhou index.

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