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Turing Instability of Equilibrium and Periodic Solution for an Enzyme-Catalyzed System

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Abstract

This paper reports the spatiotemporal dynamics of an enzyme-catalyzed system with no-flux boundary conditions. We demonstrate that the diffusion-free system has a Hopf bifurcation, generating stable periodic solutions in the supercritical case and unstable ones in the subcritical case. For reaction-diffusion system, we explore and address the instability issue of the periodic solution in the Turing sense by employing the regular perturbation method. Our analysis reveals that specific combinations of diffusion coefficients

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can induce Turing instability in periodic solutions. Ultimately, numerical simulations validate these theoretical results and visually illustrate the spatiotemporal oscillatory behavior of the system.

1 Introduction

Enzyme-catalyzed reactions, as one of the most fundamental chemical processes in living organisms, have always been an important topic to study the various kinetic dynamic behaviors. Differential equations can not only qualitatively characterize the kinetic properties of enzyme-catalyzed reactions but also reveal the temporal and spatial evolution patterns induced by the catalytic substrates and products to some extent. As a result, enzyme-catalyzed reaction systems have received widespread attention. Recently, an enzyme-catalyzed reaction system has received extensive attention, and this reaction system takes the form (cf. [1,2]):

$$\begin{cases} \frac{du}{dt} = \alpha - F_1(u,v) - F_3(u), \\ \frac{dv}{dt} = \beta\left(F_1(u,v) - F_2(v)\right), \end{cases} \tag{1}$$

where u and v are the concentrations of the substrate S and product P; $F_1(u,v)$ represents the rate law function of the substrate S and product P, which satisfies $F_1(0,v) = 0, \partial F_1/\partial u > 0$ and $\partial F_1/\partial v > 0$ for $u > 0$ and $v > 0$; moreover, $F_2(v) \geq 0$ and $F_3(u) \geq 0$ are sink rate functions of product P and substrate S, respectively; α and β are two positive reaction control parameters. One can refer to Fig. 1 for the reaction schemes of this enzyme-catalyzed reaction (see also Refs. [3–5] for this schematic diagram).

Now, let us review some existing dynamical results of system (1). When $\beta = 1, F_1(u,v) = u^m v, F_2(v) = v/(v+1)$ and $F_3(u) = 0$, Leng et al. [6] reported the existence and nonexistence of periodic solutions by the aid of Poincaré-Bendixson theorem and integral of divergence. When $F_1(u,v) = uv^m, F_2(v) = v/(v+1)$ and $F_3(u) = u$, Su [7] classified equilibrium types and discussed the existence of saddle-node bifurcation, Bogdanov-Takens bifurcation, and Hopf bifurcation. Taking $F_1(u,v) = uv, F_2(v) = v/(v+1)$ and $F_3(u) = u$, Atabajgi et al. [8] determined the conditions for the occurrence of Hopf bifurcation. If $F_1(u,v) = uv, F_2(v) = v/(v+1)$ and

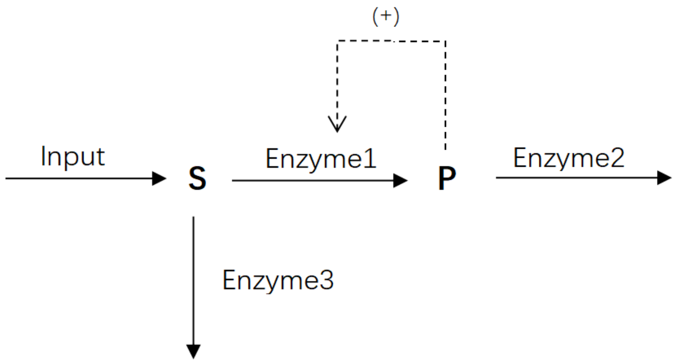


Figure 1. Schematic diagram of enzyme-catalyzed reaction system (1). The enzymatic catalytic reaction process involves two reactants, S and P, and three enzymes (see also Refs. [3–5] for this schematic diagram).

$F_3(u) = u/(u+1)$, Zhang et al. [9] studied the saddle-node, transcritical, pitchfork and Hopf bifurcations. For more specific examples of system (1), we recommend referring to papers [10–14].

Unlike the existing literature [6–14], we shall set $F_1(u,v) = u^3 v$, $F_2(v) = v/(v+1)$ and $F_3(u) = 0$. Substituting them into (1), we obtain the following enzyme-catalyzed system:

$$\begin{cases} \frac{du}{dt} = \alpha - u^3 v, \\ \frac{dv}{dt} = \beta \left(u^3 v - \frac{v}{v+1} \right). \end{cases} \tag{2}$$

Clearly, system (2) is a spatially homogeneous enzyme-catalyzed system. It only reflects the temporal dynamical evolutions of the substrate S and its product P. However, in many practical cases (e.g., in living cells), enzyme-catalyzed reaction processes generally interact with each other through diffusion effects, see Refs. [15–18]. In fact, spatial diffusion involved in chemical and biochemical systems has been broadly considered and studied. In [19], Ko investigates the bifurcation and asymptotic behavior of positive steady-states of a diffusive version enzyme-catalyzed system, and they analyzed the global bifurcation structure and limiting profiles under varying parameters. Li et al. in [20] demonstrated that a reaction-diffusion Gierer-

Meinhardt model with saturated activator production exhibits both supercritical and subcritical Hopf bifurcations. By virtue of the cross-diffusion type FitzHugh-Nagumo system, Zhang et al. [21] showed that Turing instability can induce the emergence of spatiotemporal patterns (e.g., stripes and traveling waves). For more dynamical profiles of the biochemical reaction systems with spatial diffusion, please refer to Refs. [22–24]. Therefore, inspired by the aforementioned literature, we can extend system (2) to the following diffusive system:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + \alpha - u^3 v, & x \in \Omega, \quad t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \beta \left(u^3 v - \frac{v}{v+1} \right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{3}$$

where d_1 and d_2 represent the diffusion rates of the substrate S and product P at time t and location x, respectively; Δ is the classical Laplacian operator; $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded region, ν is the outward unit normal vector along the smooth boundary $\partial\Omega$; $u_0(x)$ and $v_0(x)$ describe the initial concentrations of the substrate S and product P. All parameters, d_1, d_2, α and β, involved in system (3) are set to be positive constants.

In light of the diffusion-free system (2) and its reaction-diffusion version (3), let us state our main investigations of these two systems. For system (2), we shall explore the stability and types of the positive equilibrium. In this fashion, we also yield the emergence conditions of the Hopf bifurcation around this positive equilibrium. Of course, the supercritical and subcritical Hopf bifurcations are determined by deducing the first Lyapunov coefficient. We can obtain that there are stable and unstable periodic solutions that are acknowledged by the diffusion-free enzyme-catalyzed system (2). As a consequence, a natural question is that when the periodic solution is stable for the diffusion-free enzyme-catalyzed system (2), how to determine its instability problem in the Turing sense for the reaction-diffusion system (3)? This is typical Turing instability for the periodic solution, which resulted from the Hopf bifurcation. We have to mention that Turing instability of the constant steady states has been broadly investigated, for instance, see Refs. [25–28] and the related references cited therein. However, few existing works provide a complete algorithmic framework for the Turing instability issue of the periodic solution. Clearly, this problem is difficult and creative since it is different from the Turing instability problem of the constant steady states. Fortunately, we can yield this desired result by virtue of the Hopf bifurcation theory and the regular perturbation technique. Our theoretical result illustrates that the periodic solution of the diffusion-free system (2) will suffer from the Turing instability corresponding to reaction-diffusion system (3) as the first derivative of the minimum positive period of the periodic solution is less than zero.

The structure of this article is arranged as follows: In Sec. 2, we determine the types, stability, and Hopf bifurcation of (2). In Sec. 3, we explore the Hopf bifurcation and the Turing instability of (3). In Sec. 4, we conduct numerical results to verify the theoretical conclusions. Finally, we end this article with a discussion in Sec. 5.

2 Hopf bifurcation of system (2)

Clearly, system (2) enjoys a unique positive equilibrium, say E_*, where $E_* = (u_*, v_*) = \left((1-\alpha)^{\frac{1}{3}}, \frac{\alpha}{1-\alpha}\right)$ for $0 < \alpha < 1$. Then the Jacobian matrix at E_* is given by

$$J_0(\beta) = \begin{pmatrix} -\frac{3\alpha}{u_*} & \alpha - 1 \\ \frac{3\alpha\beta}{u_*} & \alpha(1-\alpha)\beta \end{pmatrix}.$$

As such, we can get the following characteristic equation:

$$\lambda^2 - T_0(\beta)\lambda + D_0(\beta) = 0, \tag{4}$$

where $T_0(\beta) = \alpha(1-\alpha)\beta - \frac{3\alpha}{u_*}$ and $D_0(\beta) = \frac{3\alpha(1-\alpha)^2\beta}{u_*}$. From (4), one has

$$\lambda_{1,2} = \frac{T_0(\beta) \pm \sqrt{T_0^2(\beta) - 4D_0(\beta)}}{2} \in \mathbb{C}.$$

Based on the above analysis, we can establish the following.

Theorem 1. *Suppose that* $0<\alpha<1$ *holds.*

(i) If $0<\beta\le\beta_1<\beta_0<\beta_2$*, then* (u_*,v_*) *is a stable node;*

(ii) If $0<\beta_1<\beta<\beta_0<\beta_2$*, then* (u_*,v_*) *is a stable focus;*

(iii) If $0<\beta_1<\beta_0<\beta<\beta_2$*, then* (u_*,v_*) *is a unstable focus;*

(iv) If $0<\beta_1<\beta_0<\beta_2\le\beta$*, then* (u_*,v_*) *is a unstable node;*

(v) If $\beta=\beta_0$*, then* (u_*,v_*) *is a center, where*

$$\beta_0=\frac{3}{(1-\alpha)u_*},\ \beta_1=\frac{3(2-\alpha-2\sqrt{1-\alpha})}{\alpha(1-\alpha)u_*},\ \beta_2=\frac{3(2-\alpha+2\sqrt{1-\alpha})}{\alpha(1-\alpha)u_*}.$$

Proof. By direct computation, we know that

$$T_0^2(\beta)-4D_0(\beta)=\alpha^2(1-\alpha)^2\beta^2-\frac{6\alpha(1-\alpha)(2-\alpha)}{u_*}\beta+\frac{9\alpha^2}{u_*^2}=0$$

has two positive roots β_1 and β_2. Accordingly, when $0<\beta\le\beta_1$ or $\beta\ge\beta_2$ is presented, one has $T_0^2(\beta)-4D_0(\beta)\ge 0$. However, if $\beta_1<\beta<\beta_2$ is satisfied, there holds $T_0^2(\beta)-4D_0(\beta)<0$. Additionally, if $\beta_0=\frac{3}{(1-\alpha)u_*}$ is valid, we can have $T_0(\beta)=0$. So, when $\beta<\beta_0$, we have $T_0(\beta)<T_0(\beta_0)=0$; conversely, we have $T_0(\beta)>T_0(\beta_0)=0$. Also, note that when $0<\alpha<1$, we find $D_0(\beta)>0$. Therefore, for (i), if $0<\beta\le\beta_1<\beta_0<\beta_2$ is valid, one can deduce that $T_0(\beta)<0$ and $T_0^2(\beta)-4D_0(\beta)\ge 0$. These two conclusions imply that (4) possesses a pair of negative real roots. Hence, (u_*,v_*) is a stable node. Utilizing similar protocols, we can confirm the validity of (ii)-(v). This ends the proof.

Theorem 2. *Suppose* $0<\alpha<1$*, then system* (2) *undergoes a Hopf bifurcation when* $\beta=\beta_0$*. Besides, if* $Re(c(\beta_0))<0$ *(resp.* $Re(c(\beta_0))>0$*), the Hopf bifurcation is supercritical (resp. subcritical) and produces stable (resp. unstable) periodic solutions, where* $c(\beta_0)$ *has been defined in* (6)*.*

Proof. When $\beta=\beta_0$ and $0<\alpha<1$, we have $T_0(\beta_0)=0$ and $D_0(\beta_0)>0$. Let $\lambda=l(\beta)\pm iw(\beta)$ be the roots of (4), where $l(\beta)=\frac{T_0(\beta)}{2}$ and $w(\beta)=\frac{\sqrt{4D_0(\beta)-T_0^2(\beta)}}{2}$ satisfying $l(\beta_0)=0$ and $w(\beta_0)=\sqrt{D_0(\beta_0)}:=w_0$. This implies that equation (4) has a pair of purely imaginary eigenvalues $\lambda_{1,2}=\pm iw_0$. In addition, one obtains $\frac{d}{d\beta}\mathrm{Re}(\lambda)\Big|_{\beta=\beta_0}=\frac{1}{2}\alpha(1-\alpha)>0$.

Then, the diffusion-free (2) undergoes the spatially homogeneous Hopf bifurcation at E_* as $\beta = \beta_0$ with $0 < \alpha < 1$.

In the sequel, to analyze the detailed properties of Hopf bifurcation, we set $\widetilde{u} = u - u_*$ and $\widetilde{v} = v - v_*$. If we still u and v to designate the original variables, then system (2) becomes

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = J_0(\beta) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f^1(u,v,\beta) \\ g^1(u,v,\beta) \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} f^1(u,v,\beta) =& -3\alpha(1-\alpha)^{-\frac{2}{3}}u^2 - 3(1-\alpha)^{\frac{2}{3}}uv - \frac{\alpha}{1-\alpha}u^3 \\ & - 3(1-\alpha)^{\frac{1}{3}}u^2v + O(4), \\ g^1(u,v,\beta) =& 3\alpha(1-\alpha)^{-\frac{2}{3}}\beta u^2 + (1-\alpha)^3\beta v^2 + 3(1-\alpha)\beta uv \\ & + \frac{\alpha\beta}{1-\alpha}u^3 - (1-\alpha)^4\beta v^3 + 3(1-\alpha)^{\frac{1}{3}}\beta u^2 v + O(4). \end{aligned}$$

Let us introduce a matrix $T = \begin{pmatrix} 1 & 0 \\ P(\beta) & Q(\beta) \end{pmatrix}$, where we set $P(\beta) = -\frac{1}{2}\alpha\left(\beta + \frac{3}{(1-\alpha)u_*}\right)$ and $Q(\beta) = \frac{w(\beta)}{1-\alpha}$. Let $(u,v)^T = T(x_1,y_1)^T$, then

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \end{pmatrix} = \begin{pmatrix} l(\beta) & -w(\beta) \\ w(\beta) & l(\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} f^2(x_1,y_1,\beta) \\ g^2(x_1,y_1,\beta) \end{pmatrix},$$

where

$$\begin{aligned} f^2(x_1,y_1,\beta) =& m_{20}x_1^2 + m_{11}x_1y_1 + m_{30}x_1^3 + m_{21}x_1^2y_1 + O(4), \\ g^2(x_1,y_1,\beta) =& n_{20}x_1^2 + n_{11}x_1y_1 + n_{02}y_1^2 + n_{30}x_1^3 + n_{21}x_1^2y_1 + n_{12}x_1y_1^2 \\ & + n_{03}y_1^3 + O(4), \end{aligned}$$

with

$$\begin{aligned} m_{20} =& -3\alpha(1-\alpha)^{-\frac{2}{3}} - 3P(\beta)(1-\alpha)^{\frac{2}{3}},\ m_{11} = -3Q(\beta)(1-\alpha)^{\frac{2}{3}}, \\ m_{30} =& -\frac{\alpha}{1-\alpha} - 3P(\beta)(1-\alpha)^{\frac{1}{3}},\ m_{21} = -3Q(\beta)(1-\alpha)^{\frac{1}{3}}, \end{aligned}$$

$$
\begin{aligned}
n_{11} =& 3(1-\alpha)\beta + 2P(\beta)(1-\alpha)^3\beta + 3P(\beta)(1-\alpha)^{\frac{2}{3}},\\
n_{21} =& 3(1-\alpha)^{\frac{1}{3}}\beta + 3P(\beta)(1-\alpha)^{\frac{1}{3}} - 3P(\beta)^2(1-\alpha)^4\beta,
\end{aligned}
$$

$$
\begin{aligned}
n_{20} =& \frac{1}{Q(\beta)}[3\alpha(1-\alpha)^{-\frac{2}{3}}\beta + 3P(\beta)(1-\alpha)\beta + 3P(\beta)(1-\alpha)^{-\frac{2}{3}}\\
&+ P(\beta)^2(1-\alpha)^3\beta + 3P(\beta)^2(1-\alpha)^{\frac{2}{3}}],\\
n_{30} =& \frac{1}{Q(\beta)}[\frac{\alpha\beta}{1-\alpha} + P(\beta)\frac{\alpha}{1-\alpha} + 3P(\beta)(1-\alpha)^{\frac{1}{3}}\beta\\
&+ 3P(\beta)^2(1-\alpha)^{\frac{1}{3}} - P(\beta)^3(1-\alpha)^4\beta],\ n_{02} = Q(\beta)(1-\alpha)^3\beta,\\
n_{12} =& -3P(\beta)Q(\beta)(1-\alpha)^4\beta,\ n_{03} = -Q(\beta)^2(1-\alpha)^4\beta.
\end{aligned}
$$

Benefiting from Refs. [29–31], the first Lyapunov coefficient is given by:

$$
c(\beta_0) = \frac{i}{2w_0}(r_{20}r_{11} - 2|r_{11}|^2 - \frac{1}{3}|r_{02}|^2) + \frac{r_{21}}{2}, \tag{6}
$$

where

$$
\begin{aligned}
r_{11} =& \frac{1}{4}\big(f^2_{x_1x_1} + i(g^2_{x_1x_1} + g^2_{y_1y_1})\big),\\
r_{02} =& \frac{1}{4}\left(f^2_{x_1x_1} - 2g^2_{x_1y_1} + i(g^2_{x_1x_1} - g^2_{y_1y_1} + 2f^2_{x_1y_1})\right),\\
r_{20} =& \frac{1}{4}\big(f^2_{x_1x_1} + 2g^2_{x_1y_1} + i(g^2_{x_1x_1} - g^2_{y_1y_1} - 2f^2_{x_1y_1})\big),\\
r_{21} =& \frac{1}{8}(f^2_{x_1x_1x_1} + g^2_{x_1x_1y_1} + g^2_{y_1y_1y_1}\\
&+ i(g^2_{x_1x_1x_1} + g^2_{x_1y_1y_1} - f^2_{x_1x_1y_1} - f^2_{y_1y_1y_1})).
\end{aligned}
$$

All the quantities are to be evaluated at $(0, 0, \beta_0)$. We have

$$
\begin{aligned}
\mathrm{Re}(c(\beta_0)) =& \frac{1}{16}(f^2_{x_1x_1x_1} + g^2_{x_1x_1y_1} + g^2_{y_1y_1y_1})\\
&+ \frac{1}{16w_0}[f^2_{x_1y_1}f^2_{x_1x_1} - g^2_{x_1y_1}(g^2_{x_1x_1} + g^2_{y_1y_1}) - f^2_{x_1x_1}g^2_{x_1x_1}],
\end{aligned}
$$

where

$$
f^2_{x_1x_1} = -6\alpha(1-\alpha)^{-\frac{2}{3}} - 6P_0(1-\alpha)^{\frac{2}{3}},\ f^2_{x_1y_1} = -3Q_0(1-\alpha)^{\frac{2}{3}},
$$

$$
\begin{aligned}
g^2_{x_1y_1} =& 3(1-\alpha)\beta_0 + 2P_0(1-\alpha)^3\beta_0 + 3P_0(1-\alpha)^{\frac{2}{3}},\\
f^2_{x_1x_1x_1} =& -6\left[\frac{\alpha}{1-\alpha} + 3P_0(1-\alpha)^{\frac{1}{3}}\right], \quad g^2_{y_1y_1y_1} = -6Q_0^2(1-\alpha)^4\beta_0,
\end{aligned}
$$

$$
\begin{aligned}
g^2_{x_1x_1} =& \frac{2}{Q_0}[3\alpha(1-\alpha)^{-\frac{2}{3}}\beta_0 + 3P_0(1-\alpha)\beta_0 + 3P_0(1-\alpha)^{-\frac{2}{3}}\\
& + P_0^2(1-\alpha)^3\beta_0 + 3P_0^2(1-\alpha)^{\frac{2}{3}}],\ g^2_{y_1y_1} = 2Q_0(1-\alpha)^3\beta_0,\\
g^2_{x_1x_1y_1} =& 6\left[(1-\alpha)^{\frac{1}{3}}\beta_0 + P_0(1-\alpha)^{\frac{1}{3}} - P_0^2(1-\alpha)^4\beta_0\right].
\end{aligned}
$$

According to [32], the Hopf bifurcation is supercritical (resp. subcritical) and the periodic solution is stable (resp. unstable) if $\mathrm{Re}(c(\beta_0)) < 0$ (resp. $\mathrm{Re}(c(\beta_0)) > 0$). This ends the proof.

3 Hopf bifurcation and Turing instability of system (3)

3.1 Hopf bifurcation and its directions

Define $S := \{(u, v) \in H^2([0, l\pi]) \times H^2([0, l\pi]) : u_x = v_x = 0,\ x = 0, l\pi\}$, where $H^2([0, l\pi])$ is the standard Sobolev space. Let $\widetilde{u} = u - u_*$ and $\widetilde{v} = v - v_*$ and remove the tildes, the linearized system of (3) at (u_*, v_*) can be written as follows:

$$
\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = [J_0(\beta) + D] \begin{pmatrix} u \\ v \end{pmatrix} := L(\beta) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{7}
$$

where $D = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}$, $J_0(\beta) = \begin{pmatrix} -\frac{3\alpha}{u_*} & \alpha - 1 \\ \frac{3\alpha\beta}{u_*} & \alpha(1-\alpha)\beta \end{pmatrix}$, and $L(\beta)$ is a linear operator with domain $D_L := S_{\mathbb{C}} = S \oplus iS = \{a + ib : a, b \in S\}$. Consider the general solution of system (3) below:

$$
\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{\lambda_k t} \cos\frac{k}{l}x, \tag{8}
$$

where $\lambda_k \in \mathbb{C}$ is the temporal spectrum, k is the spatial spectrum, a_k, b_k are nonzero coefficients for $k \in \mathbb{N}_0 := \{0, 1, 2\cdots\}$ and $l > 0$. Substituting (8) into (7), we derive

$$\sum_{k=0}^{\infty} \left(\lambda_k I - J_k(\beta)\right) \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos\frac{k}{l}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad k \in \mathbb{N}_0, \tag{9}$$

where

$$J_k(\beta) = \begin{pmatrix} -\frac{3\alpha}{u_*} - d_1\left(\frac{k}{l}\right)^2 & \alpha - 1 \\ \frac{3\alpha\beta}{u_*} & \alpha(1-\alpha)\beta - d_2\left(\frac{k}{l}\right)^2 \end{pmatrix}.$$

Hence, we have the characteristic equation below:

$$\lambda_k{}^2 - T_k(\beta)\lambda_k + D_k(\beta) = 0, \tag{10}$$

where

$$\begin{aligned} T_k(\beta) &= T_0(\beta) - (d_1 + d_2)\left(\frac{k}{l}\right)^2, \\ D_k(\beta) &= d_1 d_2 \left(\frac{k}{l}\right)^4 + \left[\frac{3\alpha}{u_*}d_2 - \alpha(1-\alpha)\beta d_1\right]\left(\frac{k}{l}\right)^2 + D_0(\beta), \end{aligned}$$

where $T_0(\beta)$ and $D_0(\beta)$ are defined before.

Theorem 3. *Suppose that $0 < \alpha < 1$ and spatial scale l is large. Let*

$$\beta_k := \frac{(d_1+d_2)k^2}{\alpha(1-\alpha)l^2} + \frac{3}{(1-\alpha)u_*} = \frac{(d_1+d_2)k^2}{\alpha(1-\alpha)l^2} + \beta_0.$$

System (3) enjoys the Hopf bifurcation when $\beta = \beta_k$ for some $0 \le k \le k_*$, where $k_* = \left\lfloor l\sqrt{\frac{\alpha\beta_0(d_2-d_1)(1-\alpha)}{d_1(d_1+d_2)}}\right\rfloor$ with $d_2 > d_1$ and $\lfloor\cdot\rfloor$ is a floor function.

Proof. If $\beta = \beta_k$, we have $T_k(\beta_k) = 0$ for $0 \le k \le k_*$. Moreover,

$$\begin{aligned} D_k(\beta_k) &> \left[\frac{3\alpha}{u_*}d_2 - \alpha(1-\alpha)\beta_k d_1\right]\left(\frac{k}{l}\right)^2 \\ &\ge \alpha(1-\alpha)\left[(d_2-d_1)\beta_0 - \frac{d_1(d_1+d_2)k_*^2}{\alpha(1-\alpha)l^2}\right]\left(\frac{k}{l}\right)^2 \ge 0. \end{aligned}$$

On the other hand, one yields

$$\frac{d}{d\beta}\mathrm{Re}(\lambda)\Big|_{\beta=\beta_k} = \frac{d}{d\beta}\left(\frac{T_k(\beta)}{2}\right)\Big|_{\beta=\beta_k} = \frac{1}{2}\alpha(1-\alpha) > 0.$$

Thus, system (3) undergoes the Hopf bifurcation. The proof is finished.

Theorem 3 explains that there is a Hopf bifurcation that will occur in PDE system (3) as $\beta = \beta_k$ for $0 \le k \le k_*$. Hence, our following task is determining its direction by virtue of the center manifold reduction and normal form theory, see Refs. [33,34]. Without loss of generality, we shall consider the case of $\beta = \beta_0$. Then, we establish the following result.

Theorem 4. *Assume that* $0 < \alpha < 1$ *is satisfied. If* $Re(\gamma_1(\beta_0)) < 0$ *(resp.* $Re(\gamma_1(\beta_0)) > 0$*), then the Hopf bifurcation of system* (3) *is supercritical (resp. subcritical), where* $\gamma_1(\beta_0)$ *has been defined in* (14)*.*

Proof. Let $\widetilde{u} = u - u_*$, $\widetilde{v} = v - v_*$ and remove the tildes. In this manner, system (3) at $\beta = \beta_0$ can be written as

$$\begin{cases} \frac{\partial H}{\partial t} = LH + F(H, \beta_0), \\ \frac{\partial H}{\partial x}(0,t) = \frac{\partial H}{\partial x}(l\pi, t) = (0,0)^T, \end{cases} \tag{11}$$

where $H = (u,v)^T$, $L = L(\beta_0)$, $F(H,\beta_0) = (f^1(u,v,\beta_0), g^1(u,v,\beta_0))^T$ with f^1 and g^1 are defined as before. As in [34], $F(H,\beta_0)$ has the form:

$$F(H,\beta_0) = \frac{1}{2}M(H,H) + \frac{1}{6}N(H,H,H) + O(|H|^4),$$

where for $U, V, w \in H^2([0,l\pi]) \times H^2([0,l\pi])$, M and N fulfill

$$M(U,V) = \begin{pmatrix} M_1(U,V) \\ M_2(U,V) \end{pmatrix}, \qquad N(U,V,w) = \begin{pmatrix} N_1(U,V,w) \\ N_2(U,V,w) \end{pmatrix},$$

with

$$M_1(U,V) = a_{11}u_1v_1 + a_{12}(u_1v_2 + u_2v_1),$$
$$M_2(U,V) = b_{11}u_1v_1 + b_{12}(u_1v_2 + u_2v_1) + b_{22}u_2v_2,$$

$$N_1(U,V,w) = a_{11}^* u_1 v_1 w_1 + a_{12}^*(u_1 v_1 w_2 + u_1 v_2 w_1 + u_2 v_1 w_1),$$
$$N_2(U,V,w) = b_{11}^* u_1 v_1 w_1 + b_{12}^*(u_1 v_1 w_2 + u_1 v_2 w_1 + u_2 v_1 w_1) + b_{22}^* u_2 v_2 w_2$$

for any $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, $w = (w_1, w_2)^T$, where

$$a_{11} = -6\alpha(1-\alpha)^{-\frac{2}{3}},\ a_{12} = -3(1-\alpha)^{\frac{2}{3}},\ b_{11} = \frac{18\alpha}{(1-\alpha)^2},$$
$$b_{12} = 9(1-\alpha)^{-\frac{1}{3}},\ b_{22} = 6(1-\alpha)^{\frac{5}{3}},\ a_{11}^* = -\frac{6\alpha}{1-\alpha},\ a_{12}^* = -6(1-\alpha)^{\frac{1}{3}},$$
$$b_{11}^* = 18\alpha(1-\alpha)^{-\frac{7}{3}},\ b_{12}^* = -\frac{18}{(1-\alpha)},\ b_{22}^* = -18(1-\alpha)^{\frac{8}{3}}.$$

For $\beta = \beta_0$, let L^* be the corresponding adjoint operator of L. Then we get $L^*H = [J_0^*(\beta_0) + D]H$, where $J_0^*(\beta_0) = \begin{pmatrix} -\frac{3\alpha}{u_*} & \frac{9\alpha}{(1-\alpha)u_*^2} \\ \alpha - 1 & \frac{3\alpha}{u_*} \end{pmatrix}$.

In what follows, it is easily check that $\langle L^*U, V\rangle = \langle U, LV\rangle$ for any $U, V \in H^2([0, l\pi]) \times H^2([0, l\pi])$, where the inner production in $H^2([0, l\pi]) \times H^2([0, l\pi])$ is defined as $\langle U, V\rangle = \frac{1}{l\pi} \times \int_0^{l\pi} \overline{U}^T V dx$. Choosing

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3\alpha}{(\alpha-1)u_*} + \frac{w_0}{\alpha-1} i \end{pmatrix},$$
$$\xi^* = \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = \frac{\alpha-1}{2w_0} \begin{pmatrix} \frac{w_0}{\alpha-1} + \frac{3\alpha}{(1-\alpha)u_*} i \\ i \end{pmatrix}$$

such that $L\xi = iw_0\xi$, $L^*\xi^* = -iw_0\xi^*$, $\langle \xi^*, \xi\rangle = 1$, and $\langle \xi^*, \overline{\xi}\rangle = 0$. Using [34], we have $H = z\xi + \overline{z}\overline{\xi} + w,\ z = \langle \xi^*, H\rangle,\ w = (w_1, w_2)^T$ and

$$\begin{cases} u = z + \overline{z} + w_1, \\ v = \frac{3\alpha}{(\alpha-1)u_*}(z + \overline{z}) + \frac{w_0}{\alpha-1} i(z - \overline{z}) + w_2. \end{cases}$$

Then, system (11) in (z, w) coordinates becomes

$$\begin{cases} \frac{dz}{dt} = iw_0 z + \langle \xi^*, \widehat{f}\rangle, \\ \frac{dw}{dt} = Lw + R(z, \overline{z}, w), \end{cases} \tag{12}$$

where $\widehat{f}=F(z\xi+\overline{z}\overline{\xi}+w,\beta_0)$, $R(z,\overline{z},w)=\widehat{f}-\langle\xi^*,\widehat{f}\rangle\xi-\langle\overline{\xi}^*,\widehat{f}\rangle\overline{\xi}$. We get

$$\begin{aligned}\widehat{f}=&\frac{1}{2}M(\xi,\xi)z^2+M(\xi,\overline{\xi})z\overline{z}+\frac{1}{2}M(\overline{\xi},\overline{\xi})\overline{z}^2+O(|z|^3,|z|\cdot|w|,|w|^2),\\ \langle\xi^*,\widehat{f}\rangle=&\frac{1}{2}\langle\xi^*,M(\xi,\xi)\rangle z^2+\frac{1}{2}\langle\xi^*,M(\xi,\overline{\xi})\rangle z\overline{z}+\frac{1}{2}\langle\xi^*,M(\overline{\xi},\overline{\xi})\rangle\overline{z}^2\\ &+O(|z|^3,|z|\cdot|w|,|w|^2),\\ \langle\overline{\xi}^*,\widehat{f}\rangle=&\frac{1}{2}\langle\overline{\xi}^*,M(\xi,\xi)\rangle z^2+\frac{1}{2}\langle\overline{\xi}^*,M(\xi,\overline{\xi})\rangle z\overline{z}+\frac{1}{2}\langle\overline{\xi}^*,M(\overline{\xi},\overline{\xi})\rangle\overline{z}^2\\ &+O(|z|^3,|z|\cdot|w|,|w|^2),\\ R(z,\overline{z},w)=&\frac{1}{2}z^2R_{20}+z\overline{z}R_{11}+\frac{1}{2}\overline{z}^2R_{02}+O(|z|^3,|z|\cdot|w|,|w|^2),\end{aligned}$$

where

$$\begin{aligned}R_{20}&=M(\xi,\xi)-\langle\xi^*,M(\xi,\xi)\rangle\xi-\langle\overline{\xi}^*,M(\xi,\xi)\rangle\overline{\xi},\\ R_{11}&=M(\xi,\overline{\xi})-\langle\xi^*,M(\xi,\overline{\xi})\rangle\xi-\langle\overline{\xi}^*,M(\xi,\overline{\xi})\rangle\overline{\xi},\\ R_{02}&=M(\overline{\xi},\overline{\xi})-\langle\xi^*,M(\overline{\xi},\overline{\xi})\rangle\xi-\langle\overline{\xi}^*,M(\overline{\xi},\overline{\xi})\rangle\overline{\xi}.\end{aligned}$$

Some calculations show that $R_{20}=R_{11}=R_{02}=(0,0)^T$. This shows $R(z,\overline{z},w)=O(|z|^3,|z|\cdot|w|,|w|^2)$. It follows from [34] that system (12) has a center manifold and we can express w as follows:

$$w=\frac{1}{2}z^2w_{20}+z\overline{z}w_{11}+\frac{1}{2}\overline{z}^2w_{02}+O(|z|^3).$$

We can see that $\frac{dw}{dt}=\frac{\partial w}{\partial\overline{z}}\frac{d\overline{z}}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=Lw+R(z,\overline{z},w)$. And we have $w_{20}=[2iw_0-L]^{-1}R_{20}=(0,0)^T, w_{11}=-L^{-1}R_{11}=(0,0)^T$ and $w_{02}=[-2iw_0-L]^{-1}R_{02}=(0,0)^T$. As a consequence, we get $w=O(|z|^3)$. Hence, system (3) has a center manifold takes the form:

$$\frac{dz}{dt}=iw_0z+\langle\xi^*,\widehat{f}\rangle=iw_0z+\sum_{2\le i+j\le 3}\frac{c_{ij}}{i!j!}z^i\overline{z}^j+O(|z|^4),\tag{13}$$

where $c_{20}=\langle\xi^*,M(\xi,\xi)\rangle, c_{11}=\langle\xi^*,M(\xi,\overline{\xi})\rangle, c_{02}=\langle\xi^*,M(\overline{\xi},\overline{\xi})\rangle$ and $c_{21}=\langle\xi^*,N(\xi,\xi,\overline{\xi})\rangle$. From [34], (13) has the Poincaré normal form:

$$\frac{dz}{dt}=iw_0z+z\sum_{j=1}^{k}\gamma_j(\beta_0)(z\overline{z})^j,$$

where z is a complex variable, $k \geq 1$ and $\gamma_j(\beta_0)$ are complex-valued coefficients. Then, we have

$$\gamma_1(\beta_0) = \frac{c_{20}c_{11}i}{2w_0} + \frac{|c_{02}|^2}{6iw_0} + \frac{|c_{11}|^2}{iw_0} + \frac{c_{21}}{2}. \tag{14}$$

Therefore, we can get

$$\mathrm{Re}(\gamma_1(\beta_0)) = -\frac{1}{2w_0}[\mathrm{Re}(c_{20})\mathrm{Im}(c_{11}) + \mathrm{Im}(c_{20})\mathrm{Re}(c_{11})] + \frac{1}{2}\mathrm{Re}(c_{21}),$$

where

$$\begin{aligned}
c_{20} &= A_1 + \frac{3\alpha}{u_*}A_2 + \left(\frac{w_0}{\alpha-1}A_3 + \frac{1}{w_0}A_4\right)i,\\
c_{11} &= B_1 + \frac{3\alpha}{u_*}B_2 + \left(\frac{w_0}{\alpha-1}B_3 + \frac{1}{w_0}B_4\right)i,\\
c_{21} &= \frac{1}{2}C_1 + \frac{1}{\alpha-1}C_2 + \left(\frac{1}{2(\alpha-1)}C_3 + \frac{1}{2w_0}C_4\right)i
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= \frac{1}{2}a_{11} + b_{12}, \quad A_2 = \frac{b_{22}}{\alpha-1} - \frac{b_{12}}{w_0}, \quad A_3 = a_{12} - \frac{1}{2}b_{22},\\
A_4 &= \frac{9\alpha^2(b_{22}-2a_{12})}{2(1-\alpha)u_*^2} + \frac{3\alpha a_{11}}{2u_*} + \frac{(1-\alpha)b_{11}}{2},\\
B_1 &= \frac{1}{2}a_{11}, \quad B_2 = \frac{a_{12}}{\alpha-1}, \quad B_3 = -\frac{1}{2}b_{22},\\
B_4 &= \frac{9\alpha^2(2a_{12}-b_{22})}{2(\alpha-1)u_*^2} + \frac{3\alpha(a_{11}-2b_{12})}{2u_*} + \frac{(1-\alpha)b_{12}}{2},\\
C_1 &= a_{11}^* + b_{12}^*, \quad C_2 = \frac{9\alpha^2 b_{22}^*}{2(\alpha-1)u_*^2} + \frac{w_0^2 b_{22}^*}{2(\alpha-1)} + \frac{3\alpha a_{12}^*}{u_*},
\end{aligned}$$

$$\begin{aligned}
C_3 &= \frac{27\alpha^2 a_{12}^*}{w_0 u_*^2} - \frac{27\alpha^3 b_{22}^*}{(\alpha-1)w_0 u_*^3} - \frac{3\alpha w_0 b_{22}^*}{(\alpha-1)u_*} + w_0 a_{12}^*,\\
C_4 &= \frac{3\alpha(a_{11}^* - 3b_{12}^*)}{u_*} + (1-\alpha)b_{11}^*.
\end{aligned}$$

Using [35, 36], we can claim that the Hopf bifurcation is supercritical as

$\mathrm{Re}(\gamma_1(\beta_0)) < 0$ and subcritical as $\mathrm{Re}(\gamma_1(\beta_0)) > 0$. This ends the proof.

3.2 Turing instability of the positive equilibrium

In this part, our purpose is to explore the Turing instability for E_* of system (3). To this end, benefiting from (i) and (ii) of Theorem 1, we always require $0 < \beta < \beta_0$ to hold in the next analysis. Let

$$\tau_h := \min_{1 \le \frac{k}{l} \le h} \frac{\frac{3\alpha}{u_*} d_2 k^2 + D_0(\beta) l^2}{[\alpha(1-\alpha)\beta - d_2 \left(\frac{k}{l}\right)^2] k^2} \quad \text{for} \quad k \in \mathbb{N}_0 \backslash \{0\}.$$

Now, one builds the following.

Theorem 5. *Suppose that* $0 < \alpha < 1$ *and* $0 < \beta < \beta_0$. *Then, system* (3) *yields the following conclusions.*

(i) Equilibrium E_* *is locally asymptotically stable if one of the below conditions is fulfilled:*

$$(a) \quad \frac{\alpha(1-\alpha)\beta}{d_2} \le 1,$$

$$(b) \quad h^2 < \frac{\alpha(1-\alpha)\beta}{d_2} \le (h+1)^2, \ d_1 < \tau_h, \ h \in \mathbb{N}_0 \backslash \{0\};$$

(ii) Equilibrium E_* *is unstable as*

$$(c) \quad h^2 < \frac{\alpha(1-\alpha)\beta}{d_2} \le (h+1)^2, \ d_1 > \tau_h, \ h \in \mathbb{N}_0 \backslash \{0\}.$$

Proof. Obviously, we have $T_k(\beta) < 0$ for any $k \in \mathbb{N}_0$. Therefore, the Turing instability for E_* of system (3) depends on the sign of $D_k(\beta)$. To explore the sign of $D_k(\beta)$, we rewrite $D_k(\beta)$ as

$$D_k(\beta) = \left(\left(\frac{k}{l}\right)^2 - \frac{\alpha(1-\alpha)\beta}{d_2} \right) d_1 d_2 \left(\frac{k}{l}\right)^2 + \frac{3\alpha}{u_*} d_2 \left(\frac{k}{l}\right)^2 + D_0(\beta).$$

If the condition (a) is satisfied, then $\left(\frac{k}{l}\right)^2 - \frac{\alpha(1-\alpha)\beta}{d_2} \ge \left(\frac{k}{l}\right)^2 - 1 \ge 0$. As such, $D_k(\beta) \ge 3\alpha(1-\alpha)^3 d_2 \left(\frac{k}{l}\right)^2 + D_0(\beta) > 0$. Therefore, E_* is stable.

Now, if (b) holds, one yields

$$d_2\left(h^2-\left(\frac{k}{l}\right)^2\right)<\alpha(1-\alpha)\beta-d_2\left(\frac{k}{l}\right)^2\leq d_2\left[(h+1)^2-\left(\frac{k}{l}\right)^2\right].$$

Since $1\leq\frac{k}{l}\leq h$, which means that $\alpha(1-\alpha)\beta-d_2\left(\frac{k}{l}\right)^2>d_2(h^2-\left(\frac{k}{l}\right)^2)\geq 0$. And if $d_1<\tau_h$, we have $d_1<\tau_h\leq\frac{\frac{3\alpha}{u_*}d_2k^2+D_0(\beta)l^2}{[\alpha(1-\alpha)\beta-d_2\left(\frac{k}{l}\right)^2]k^2}$. This gives

$$\frac{3\alpha}{u_*}d_2\left(\frac{k}{l}\right)^2+D_0(\beta)>\left(\frac{\alpha(1-\alpha)\beta}{d_2}-\left(\frac{k}{l}\right)^2\right)d_1d_2\left(\frac{k}{l}\right)^2.$$

Then, we can get

$$\begin{aligned}D_k(\beta)&=\left(\left(\frac{k}{l}\right)^2-\frac{\alpha(1-\alpha)\beta}{d_2}\right)d_1d_2\left(\frac{k}{l}\right)^2+\frac{3\alpha}{u_*}d_2\left(\frac{k}{l}\right)^2+D_0(\beta)\\&>d_1d_2\left(\frac{k}{l}\right)^2\left[\left(\left(\frac{k}{l}\right)^2-\frac{\alpha(1-\alpha)\beta}{d_2}\right)+\left(\frac{\alpha(1-\alpha)\beta}{d_2}-\left(\frac{k}{l}\right)^2\right)\right]\\&=0.\end{aligned}$$

As a consequence, E_* is stable. In the same way, if the condition (c) is valid, there exists at least one of $D_1(\beta)$, $D_2(\beta)$, $\cdots$, $D_h(\beta)$ to be negative. This implies that E_* is unstable. The proof is completed.

Remark 1. Theorem 5 shows that neither (a) nor (b) leads to Turing instability. Nevertheless, when (c) is true, E_* becomes unstable for system (3) at a critical wavenumber $k_c^2=\sqrt{\frac{3\alpha(1-\alpha)^2\beta}{u_*d_1d_2}}$. So, this is the condition that leads to Turing instability of E_*.

3.3 Turing instability of the periodic solution

Theorem 2 suggests that system (2) enjoys a stable periodic solution owing to a supercritical Hopf bifurcation. Therefore, in this part, we will introduce a perturbation term on the basis of system (2) and then study the stability of such a periodic solution. A perturbed system has the form:

$$\left(I+\sigma\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\right)\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} \alpha-u^3v \\ \beta(u^3v-\frac{v}{v+1}) \end{pmatrix}, \tag{15}$$

where I is a second-order identity matrix, σ is a disturbance parameter and $|\sigma|$ is sufficiently small. System (15) can be read as follows:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\frac{1}{D(\sigma)}\begin{pmatrix} 1+d_2\sigma & 0 \\ 0 & 1+d_1\sigma \end{pmatrix}\begin{pmatrix} \alpha-u^3v \\ \beta(u^3v-\frac{v}{v+1}) \end{pmatrix}, \tag{16}$$

where $D(\sigma)>0$. The Jacobian matrix of system (16) at E_* is

$$J(\beta,\sigma)=\begin{pmatrix} J_{11}(\beta,\sigma) & J_{12}(\beta,\sigma) \\ J_{21}(\beta,\sigma) & J_{22}(\beta,\sigma) \end{pmatrix},$$

where

$$J_{11}(\beta,\sigma)=-\frac{(1+d_2\sigma)\frac{3\alpha}{u_*}}{D(\sigma)},\ J_{12}(\beta,\sigma)=\frac{(1+d_2\sigma)(\alpha-1)}{D(\sigma)},$$
$$J_{21}(\beta,\sigma)=\frac{(1+d_1\sigma)\frac{3\alpha}{u_*}\beta}{D(\sigma)},\ J_{22}(\beta,\sigma)=\frac{(1+d_1\sigma)\alpha(1-\alpha)\beta}{D(\sigma)}.$$

The characteristic equation of $J(\beta,\sigma)$ can be written as

$$\lambda^2-T(\beta,\sigma)\lambda+D(\beta,\sigma)=0, \tag{17}$$

where

$$T(\beta,\sigma)=\frac{\left(T_0(\beta)+\sigma\left(\alpha(1-\alpha)\beta d_1-\frac{3\alpha}{u_*}d_2\right)\right)}{D(\sigma)},\ D(\beta,\sigma)=\frac{D_0(\beta)}{D(\sigma)},$$

where $T_0(\beta)$ and $D_0(\beta)$ have been defined before. Obviously, we have $T(\beta,0)=T_0(\beta)$ and $D(\beta,0)=D_0(\beta)$ since $D(0)=1$. Note that $T(\beta_0,0)=T_0(\beta_0)$ and $\partial_\beta T(\beta_0,0)=T_0'(\beta_0)=\alpha(1-\alpha)\neq 0$ where $\partial_\beta T=\frac{\partial T}{\partial\beta}$. Then, according to the implicit function theorem, there exists a $\sigma_1>0$ and a continuously differentiable function

$$\beta_\sigma = \frac{3(1+d_2\sigma)}{(1-\alpha)(1+d_1\sigma)u_*},$$

with $\sigma \in (-\sigma_1, \sigma_1)$ such that $T(\beta_\sigma, \sigma) = 0$. So, equation (17) has a pair of imaginary eigenvalues $\widehat{\lambda}_{1,2} = \pm i\sqrt{D(\beta_\sigma, \sigma)}$ at $\beta = \beta_\sigma$, where

$$D(\beta_\sigma, \sigma) = \frac{D_0(\beta_\sigma)}{D(\sigma)} > 0.$$

Let $\widehat{\lambda} := \widehat{l}(\beta) \pm i\widehat{w}(\beta)$ be the complex characteristic roots of (17), then

$$\widehat{l}(\beta) = \frac{T(\beta,\sigma)}{2}, \quad \widehat{w}(\beta) = \frac{1}{2}\sqrt{4D(\beta,\sigma) - T^2(\beta,\sigma)}.$$

When $\sigma = 0$, we know $\beta_\sigma = \beta_0$ and

$$\frac{d}{d\beta}\mathrm{Re}(\widehat{\lambda})\Big|_{\beta=\beta_\sigma} = \frac{d}{d\beta}\mathrm{Re}(\widehat{\lambda})\Big|_{\beta=\beta_0} = \frac{1}{2}\alpha(1-\alpha) > 0.$$

This means that β_σ is the Hopf bifurcation point of system (15). According to [35, 37], we can obtain the following.

Lemma 1. *Assume that* $0 < \alpha < 1$. *System* (2) *possesses a stable periodic solution* $(u^p(t), v^p(t))$ *bifurcating from* (u_*, v_*) *at* $\beta = \beta_0$*, where* $p > 0$ *is the minimum positive period. Then, there exists a* $\sigma_2 > 0$ *such that for any* $\sigma \in (-\sigma_2, \sigma_2)$*, system* (15) *has a periodic solution* $(u^p(t,\sigma), v^p(t,\sigma))$ *depending on* σ*,* $p(\sigma)$ *is its minimum positive period. Moreover, we get*

(i) $p(\sigma) \to p$ *and* $(u^p(t,\sigma), v^p(t,\sigma)) \to (u^p(t), v^p(t)),\ \ \sigma \to 0$*;*

(ii) $p(\sigma) = \frac{2\pi}{\widehat{w}(\beta_\sigma)}\left(1 + \left(\frac{\widehat{l}'(\beta_\sigma) Im(c(\beta_\sigma))}{\widehat{w}(\beta_\sigma) Re(c(\beta_\sigma))} - \frac{\widehat{w}'(\beta_\sigma)}{\widehat{w}(\beta_\sigma)}\right)(\beta - \beta_\sigma)\right) + O((\beta-\beta_\sigma)^2)$, $\beta \to \beta_\sigma$*, where*

$$c(\beta_\sigma) = \frac{i}{2\widehat{w}(\beta_\sigma)}(r_{20}(\sigma)r_{11}(\sigma) - 2|r_{11}(\sigma)|^2 - \frac{1}{3}|r_{02}(\sigma)|^2) + \frac{r_{21}(\sigma)}{2},$$
$$\widehat{l}(\beta) = \frac{T(\beta,\sigma)}{2}, \quad \widehat{w}(\beta) = \frac{1}{2}\sqrt{4D(\beta,\sigma) - T^2(\beta,\sigma)}.$$

Proof. Since (i) is a direct subsequence of Theorem 1 in Chapter 14 of [38], we will not elaborate on it further here. Now, we only show the validity (ii) of Lemma 1. Firstly, system (16) can expressed as follows:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J(\beta,\sigma)\begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{D(\sigma)}\begin{pmatrix} 1+d_2\sigma & 0 \\ 0 & 1+d_1\sigma \end{pmatrix}\begin{pmatrix} f^1(u,v,\beta) \\ g^1(u,v,\beta) \end{pmatrix}, \tag{18}$$

where $J(\beta,\sigma)$, $f^1(u,v,\beta)$ and $g^1(u,v,\beta)$ have been defined before. Define

$$T(\sigma) := \begin{pmatrix} 1 & 0 \\ P(\sigma) & Q(\sigma) \end{pmatrix},$$

where

$$P(\sigma) = \frac{D(\sigma)\widehat{l}(\beta) - J_{11}(\beta,\sigma)}{J_{12}(\beta,\sigma)}, \quad Q(\sigma) = -\frac{D(\sigma)\widehat{w}(\beta)}{J_{12}(\beta,\sigma)},$$
$$\widehat{l}(\beta) = \frac{T(\beta,\sigma)}{2}, \quad \widehat{w}(\beta) = \frac{1}{2}\sqrt{4D(\beta,\sigma) - T^2(\beta,\sigma)}.$$

Then, by using $(u,v)^T = T(\sigma)(x_2,y_2)^T$, rewriting system (18) as follows

$$\begin{pmatrix} x_{2t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \widehat{l}(\beta) & -\widehat{w}(\beta) \\ \widehat{w}(\beta) & \widehat{l}(\beta) \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} \widetilde{F}(x_2,y_2,\beta) \\ \widetilde{G}(x_2,y_2,\beta) \end{pmatrix}, \tag{19}$$

where

$$\widetilde{F}(x_2,y_2,\beta) = \frac{1+d_2\sigma}{D(\sigma)} f^1\left(x_2, \frac{D(\sigma)\widehat{l}(\beta) - J_{11}(\beta,\sigma)}{J_{12}(\beta,\sigma)}x_2 - \frac{D(\sigma)\widehat{w}(\beta)}{J_{12}(\beta,\sigma)}y_2, \beta\right),$$

$$\begin{aligned}\widetilde{G}(x_2,y_2,\beta) = {} & \frac{D(\sigma)\widehat{l}(\beta) - J_{11}(\beta,\sigma)}{D(\sigma)\widehat{w}(\beta)}\widetilde{F}(x_2,y_2,\beta) - \frac{J_{12}(\beta,\sigma)(1+d_1\sigma)}{D^2(\sigma)\widehat{w}(\beta)} \\ & \times g^1\left(x_2, \frac{D(\sigma)\widehat{l}(\beta) - J_{11}(\beta,\sigma)}{J_{12}(\beta,\sigma)}x_2 - \frac{D(\sigma)\widehat{w}(\beta)}{J_{12}(\beta,\sigma)}y_2, \beta\right).\end{aligned}$$

Note that when $\beta = \beta_\sigma$, system (19) can be written as

$$\begin{pmatrix} x_{2t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 0 & -\widehat{w}(\beta) \\ \widehat{w}(\beta) & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \widetilde{F}(x_2,y_2,\beta_\sigma) \\ \widetilde{G}(x_2,y_2,\beta_\sigma) \end{pmatrix}.$$

Then, according to [34], we have

$$c(\beta_\sigma) = \frac{i}{2\widehat{w}(\beta_\sigma)}\left(r_{20}(\sigma)r_{11}(\sigma) - 2|r_{11}(\sigma)|^2 - \frac{1}{3}|r_{02}(\sigma)|^2\right) + \frac{r_{21}(\sigma)}{2},$$

with

$$\begin{aligned}
r_{11}(\sigma) =& \frac{1}{4}(\widetilde{F}_{x_2x_2} + \widetilde{F}_{y_2y_2} + i(\widetilde{G}_{x_2x_2} + \widetilde{G}_{y_2y_2})),\\
r_{02}(\sigma) =& \frac{1}{4}(\widetilde{F}_{x_2x_2} - \widetilde{F}_{y_2y_2} - 2\widetilde{G}_{x_2y_2} + i(\widetilde{G}_{x_2x_2} - \widetilde{G}_{y_2y_2} + 2\widetilde{F}_{x_2y_2})),\\
r_{20}(\sigma) =& \frac{1}{4}(\widetilde{F}_{x_2x_2} - \widetilde{F}_{y_2y_2} + 2\widetilde{G}_{x_2y_2} + i(\widetilde{G}_{x_2x_2} - \widetilde{G}_{y_2y_2} - 2\widetilde{F}_{x_2y_2})),\\
r_{21}(\sigma) =& \frac{1}{8}(\widetilde{F}_{x_2x_2x_2} + \widetilde{F}_{x_2y_2y_2} + \widetilde{G}_{x_2x_2y_2} + \widetilde{G}_{y_2y_2y_2} + i(\widetilde{G}_{x_2x_2x_2}\\
&+ \widetilde{G}_{x_2y_2y_2} - \widetilde{F}_{x_2x_2y_2} - \widetilde{F}_{y_2y_2y_2})),
\end{aligned}$$

where all variables are evaluated at $(\beta, x_2, y_2) = (\beta_\sigma, 0, 0)$. Owing to $p(\sigma)$ is the minimum period of periodic solution $(u^p(t,\sigma), v^p(t,\sigma))$. Then, following page 90 of [34], we get

$$p(\sigma) = \frac{2\pi}{\widehat{w}(\beta_\sigma)}\left(1 + \left(\frac{\widehat{l}'(\beta_\sigma)\mathrm{Im}(c(\beta_\sigma))}{\widehat{w}(\beta_\sigma)\mathrm{Re}(c(\beta_\sigma))} - \frac{\widehat{w}'(\beta_\sigma)}{\widehat{w}(\beta_\sigma)}\right)(\beta - \beta_\sigma)\right) + O(2).$$

This completes the proof.

When $\beta \to \beta_0$, we have the following result.

Theorem 6. *$p(\sigma)$ is the minimum positive period defined in Lemma 1. When β is sufficiently close to β_0, we have*

$$p'(0) = \frac{\pi}{D_0(\beta_0)}(\zeta_1(\beta_0)d_1 + \zeta_2(\beta_0)d_2) + O(\beta - \beta_0),$$

where

$$\begin{aligned}
\zeta_1(\beta_0) &= \sqrt{D_0(\beta_0)} + \alpha(1-\alpha)\beta_0\frac{Im(c(\beta_0))}{Re(c(\beta_0))},\\
\zeta_2(\beta_0) &= \sqrt{D_0(\beta_0)} - \frac{3\alpha}{u_*}\cdot\frac{Im(c(\beta_0))}{Re(c(\beta_0))},
\end{aligned}$$

where $c(\beta_0)$ can be found in (6).

Proof. Differentiating $p(\sigma)$ with respect to σ, we can get

$$p'(\sigma) = -\frac{2\pi}{\widehat{w}^2(\beta_\sigma)}\frac{d(\widehat{w}(\beta_\sigma))}{d\sigma} - \frac{2\pi}{\widehat{w}(\beta_\sigma)}\left(\frac{\widehat{l}'(\beta_\sigma)\mathrm{Im}(c(\beta_\sigma))}{\widehat{w}(\beta_\sigma)\mathrm{Re}(c(\beta_\sigma))} - \frac{\widehat{w}'(\beta_\sigma)}{\widehat{w}(\beta_\sigma)}\right)\frac{d(\beta_\sigma)}{d\sigma}$$
$$+ O\big((\beta - \beta_\sigma)\big).$$

In the sequel, our main goal is to obtain $p'(0)$. We notice that $\beta_\sigma = \beta_0$, $\widehat{w}(\beta_\sigma) = w(\beta_0) = \sqrt{D_0(\beta_0)}$, $\mathrm{Re}(c(\beta_\sigma)) = \mathrm{Re}(c(\beta_0))$, $\mathrm{Im}(c(\beta_\sigma)) = \mathrm{Im}(c(\beta_0))$ and $\widehat{l}'(\beta_\sigma) = \frac{T_0'(\beta_0)}{2}$ when $\sigma = 0$, where $c(\beta_0)$ is defined in (6).

Firstly, we compute $\frac{d\beta_\sigma}{d\sigma}\Big|_{\sigma=0}$. At $\beta = \beta_\sigma$, from $T(\beta, \sigma) = 0$, we get

$$T(\beta_\sigma) + \sigma\left(\alpha(1-\alpha)\beta_\sigma d_1 - \frac{3\alpha}{u_*}d_2\right) = 0. \tag{20}$$

Differentiating (20) with σ and setting $\sigma = 0$, we can obtain

$$\frac{d\beta_\sigma}{d\sigma}\Big|_{\sigma=0} = \frac{1}{T_0'(\beta_0)}\left(\frac{3\alpha}{u_*}d_2 - \alpha(1-\alpha)\beta_0 d_1\right) := \frac{H(\beta_0)}{T_0'(\beta_0)}, \tag{21}$$

where $H(\beta_0) = \frac{3\alpha}{u_*}d_2 - \alpha(1-\alpha)\beta_0 d_1$.

Secondly, we need to calculate $\widehat{w}'(\beta_0)$. As β tends to β_σ, we have

$$\widehat{w}(\beta) = \frac{1}{2}\sqrt{4D(\beta,\sigma) - T^2(\beta,\sigma)}.$$

Differentiating $\widehat{w}(\beta)$ with β, we will get

$$\widehat{w}'(\beta) = \frac{2\partial_\beta D(\beta,\sigma) - T(\beta,\sigma)\partial_\beta T(\beta,\sigma)}{2\sqrt{4D(\beta,\sigma) - T^2(\beta,\sigma)}}.$$

Owing to we have $T(\beta_\sigma, \sigma) = 0$ and $\partial_\beta D(\beta_\sigma,\sigma) = \frac{D'(\beta_\sigma)}{D(\sigma)}$, then

$$\widehat{w}'(\beta_0) = \widehat{w}'(\beta_\sigma)\Big|_{\sigma=0} = \frac{\partial_\beta D(\beta_\sigma,\sigma)}{\sqrt{4D(\beta_\sigma,\sigma)}}\Big|_{\sigma=0} = \frac{D_0'(\beta_0)}{2\sqrt{D_0(\beta_0)}}. \tag{22}$$

Finally, completing $\frac{d}{d\sigma}(\widehat{w}(\beta_\sigma))\Big|_{\sigma=0}$. Since $\widehat{w}(\beta_\sigma) = \sqrt{D(\beta_\sigma,\sigma)}$, we have

$$\frac{d}{d\sigma}\widehat{w}(\beta_\sigma) = \frac{1}{2\sqrt{D(\beta_\sigma,\sigma)}}\frac{dD(\beta_\sigma,\sigma)}{d\sigma}. \tag{23}$$

Since $D(\beta_\sigma,\sigma) = \frac{D(\beta_\sigma)}{D(\sigma)}$, we obtain

$$\frac{dD(\beta_\sigma,\sigma)}{d\sigma} = -\frac{D'(\sigma)}{D^2(\sigma)}D(\beta_\sigma) + \frac{dD(\beta_\sigma)}{d\sigma}\frac{1}{D(\sigma)}. \tag{24}$$

When $\sigma = 0$, we have $D(0) = 1$ and

$$-\frac{D'(0)}{D^2(0)}D_0(\beta_0) = -(d_1+d_2)D_0(\beta_0), \ \frac{dD(\beta_\sigma)}{d\sigma}\Big|_{\sigma=0} = D_0'(\beta_0)\frac{H(\beta_0)}{T_0'(\beta_0)}. \tag{25}$$

So, by (24) and (25), we can get

$$\frac{dD(\beta_\sigma,\sigma)}{d\sigma}\Big|_{\sigma=0} = -(d_1+d_2)D_0(\beta_0) + D_0'(\beta_0)\frac{H(\beta_0)}{T_0'(\beta_0)}. \tag{26}$$

From (23) and (26), we obtain

$$\begin{aligned}\frac{d\widehat{w}(\beta_\sigma)}{d\sigma}\Big|_{\sigma=0} &= \frac{1}{2\sqrt{D_0(\beta_0)}}\left(D_0'(\beta_0)\frac{H(\beta_0)}{T_0'(\beta_0)} - (d_1+d_2)D_0(\beta_0)\right)\\ &= -\frac{1}{2}\sqrt{D_0(\beta_0)}\left(d_1+d_2 - \frac{H(\beta_0)D_0'(\beta_0)}{D_0(\beta_0)T_0'(\beta_0)}\right).\end{aligned} \tag{27}$$

In light of (21), (22) and (27), we can derive

$$\begin{aligned}p'(0) = &-\frac{2\pi}{\widehat{w}^2(\beta_0)}\frac{d\widehat{w}(\beta_\sigma)}{d\sigma}\Big|_{\sigma=0} - \frac{2\pi}{\widehat{w}(\beta_0)}\left(\frac{\widehat{l}'(\beta_0)\mathrm{Im}(c(\beta_0))}{\widehat{w}(\beta_0)\mathrm{Re}(c(\beta_0))} - \frac{\widehat{w}'(\beta_0)}{\widehat{w}(\beta_0)}\right)\frac{d\beta_\sigma}{d\sigma}\Big|_{\sigma=0}\\ &+ O(\beta-\beta_0)\\ = &\frac{\pi}{D_0(\beta_0)}\left(\sqrt{D_0(\beta_0)}d_1 + \sqrt{D_0(\beta_0)}d_2 - \frac{\mathrm{Im}(c(\beta_0))H(\beta_0)}{\mathrm{Re}(c(\beta_0))}\right) + O(\beta-\beta_0),\end{aligned} \tag{28}$$

where

$$\mathrm{Im}(c(\beta_0)) =$$

$$\frac{1}{16}(g^2_{x_1x_1x_1} + g^2_{x_1y_1y_1} - f^2_{x_1x_1y_1}) - \frac{1}{16w(\beta_0)}((g^2_{x_1x_1} + g^2_{y_1y_1})^2 + (f^2_{x_1x_1})^2)$$

$$+\frac{1}{32w(\beta_0)}\big(f^2_{x_1x_1}(f^2_{x_1x_1}+2g^2_{x_1y_1})-(g^2_{x_1x_1}+g^2_{y_1y_1})(g^2_{x_1x_1}-g^2_{y_1y_1}-2f^2_{x_1y_1})\big)$$
$$-\frac{1}{96w(\beta_0)}\big((f^2_{x_1x_1}-2g^2_{x_1y_1})^2+(g^2_{x_1x_1}-g^2_{y_1y_1}+f^2_{x_1y_1})^2\big)$$

with

$$f^2_{x_1x_1y_1}=-6Q_0(1-\alpha)^{\frac{1}{3}},\quad g^2_{x_1y_1y_1}=-6P_0Q_0(1-\alpha)^4\beta_0,$$
$$g^2_{x_1x_1x_1}=\frac{6}{Q_0}\left[\frac{\alpha(\beta_0+P_0)}{1-\alpha}+3P_0(1-\alpha)^{\frac{1}{3}}(\beta_0+P_0)-P_0^3(1-\alpha)^4\beta_0\right].$$

As a result, through direct calculations, we can get

$$\frac{\mathrm{Im}(c(\beta_0))}{\mathrm{Re}(c(\beta_0))}=\frac{(1-\alpha)^2\digamma_1+(1-\alpha)\digamma_2+(1-u_*)\digamma_3}{12\sqrt{\alpha(1-\alpha)}\digamma_4},\tag{29}$$

where

$$\digamma_1=12\alpha(27\alpha-5)-u_*\mu^2-36u_*\delta^2,$$
$$\digamma_2=24\alpha(6\alpha^2-11\alpha+3u_*),\quad \digamma_3=108\alpha(2\alpha^2-3\alpha+3u_*-2),$$
$$\digamma_4=9\big((1-\alpha)^{\frac{2}{3}}-(1-\alpha)^{\frac{4}{3}}\big)-6\alpha(1-\alpha)^{\frac{1}{3}}(4-3\alpha)-15\alpha^3+13\alpha^2-4\alpha+3,$$

with

$$\mu=10\alpha(1-\alpha)^{-\frac{2}{3}}-6(1-\alpha)^{-\frac{1}{3}}+12\alpha(1-\alpha)^{\frac{1}{3}},$$
$$\delta=\alpha^{\frac{3}{2}}(1-\alpha)^{-\frac{7}{6}}(2-\alpha)-\alpha^{\frac{1}{2}}(1-\alpha)^{-\frac{1}{6}}(2-\alpha)-\alpha^{\frac{1}{2}}(1-\alpha)^{-\frac{5}{6}}.$$

Substituting (29) into (28), we can get the following expression

$$p'(0)=\frac{\pi}{D_0(\beta_0)}\left(\zeta_1(\beta_0)d_1+\zeta_2(\beta_0)d_2\right)+O\big(\beta-\beta_0\big),$$

where

$$\zeta_1(\beta_0)=\sqrt{D_0(\beta_0)}+\alpha(1-\alpha)\beta_0\frac{\mathrm{Im}(c(\beta_0))}{\mathrm{Re}(c(\beta_0))},$$
$$\zeta_2(\beta_0)=\sqrt{D_0(\beta_0)}-\frac{3\alpha}{u_*}\cdot\frac{\mathrm{Im}(c(\beta_0))}{\mathrm{Re}(c(\beta_0))}.$$

The proof is concluded.

Theorem 7. *If $p'(0) < 0$, that is $\zeta_1(\beta_0)d_1 + \zeta_2(\beta_0)d_2 < 0$. Then the periodic solution $(u^p(t), v^p(t))$ of system (2) will undergoes Turing instability.*

Proof. At $(u^p(t), v^p(t))$, local linearized system of (3) has the form:

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = D \begin{pmatrix} \Delta u & 0 \\ 0 & \Delta v \end{pmatrix} + J^p(t) \begin{pmatrix} u \\ v \end{pmatrix},$$

where $D = \mathrm{diag}(d_1, d_2)$ and

$$J^p(t) = \begin{pmatrix} -3(u^p(t))^2 v^p(t) & -(u^p(t))^3 \\ 3(u^p(t))^2 v^p(t)\beta & ((u^p(t))^3 - \frac{1}{(v^p(t)+1)^2})\beta \end{pmatrix}.$$

Let ε_n and $\eta_n(x)$ be the eigenvalue and characteristic function of $-\Delta$ in domain Ω respectively, where $n \in \mathbb{N}_0 \backslash \{0\}$. Setting $(u, v)^T = (m(t), n(t))^T \sum_{n=0}^{\infty} \varepsilon_n \eta_n(x)$, we have

$$\begin{pmatrix} m'(t) \\ n'(t) \end{pmatrix} = -\varrho D \begin{pmatrix} m(t) \\ n(t) \end{pmatrix} + J^p(t) \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}, \tag{30}$$

where $\varrho = \varepsilon_n \geq 0$ for $n \in \mathbb{N}_0 \backslash \{0\}$. When diffusion is absent, the system (30) can be written as

$$\begin{pmatrix} m'(t) \\ n'(t) \end{pmatrix} = J^p(t) \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}. \tag{31}$$

Let $\Phi(t)$ as the fundamental solution matrix of (31) and $\Phi(0) = I$, where I is 2×2 identity matrix. Assume κ_i are the eigenvalues of $\Phi(p)$ and let $(p_i, q_i)^T$ be the corresponding characteristic function. We have

$$\Phi(p) \begin{pmatrix} p_i \\ q_i \end{pmatrix} = \kappa_i \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad i = 1, 2, \tag{32}$$

where $\kappa_i (i = 1, 2)$ is the Floquet multiplier of (31). Define

$$\begin{pmatrix} \overline{\Lambda}_i(t) \\ \overline{\Upsilon}_i(t) \end{pmatrix} := \Phi(t) \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad i = 1, 2. \tag{33}$$

Clearly, one has

$$\begin{pmatrix} \overline{\Lambda}_i(0) \\ \overline{\Upsilon}_i(0) \end{pmatrix} = \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad i = 1, 2. \tag{34}$$

So, system (32) can be rewritten as

$$\Phi(p) \begin{pmatrix} \overline{\Lambda}_i(0) \\ \overline{\Upsilon}_i(0) \end{pmatrix} = \kappa_i \begin{pmatrix} \overline{\Lambda}_i(0) \\ \overline{\Upsilon}_i(0) \end{pmatrix}, \qquad i = 1, 2. \tag{35}$$

Without loss of generality, we may suppose $\kappa_1 = 1$. Since $(u^p(t), v^p(t))$ is stable for system (2), so the Floquet multiplier κ_2 satisfy $|\kappa_2| < 1$, see [33,37]. Let $\Phi(t, \varrho)$ is the fundamental solution matrix of (30), namely

$$\partial_t \Phi(t, \varrho) = -\varrho D \Phi(t, \varrho) + J^p(t) \Phi(t, \varrho).$$

Clearly, we find $\Phi(t, 0) = \Phi(t)$ and $\Phi(0, \varrho) = I$. Let $\eta_i(\varrho)$ and $(P_i(\varrho), Q_i(\varrho))^T$ are the eigenvalues and corresponding eigenfunctions of $\Phi(p, \varrho)$ such that

$$\Phi(p, \varrho) \begin{pmatrix} P_i(\varrho) \\ Q_i(\varrho) \end{pmatrix} = \eta_i(\varrho) \begin{pmatrix} P_i(\varrho) \\ Q_i(\varrho) \end{pmatrix}, \qquad i = 1, 2, \tag{36}$$

where $\eta_i(\varrho)(i = 1, 2)$ represents the Floquet multipliers of (30). We assume $\eta_i(0) = \kappa_1$ and $(P_i(\varrho), Q_i(\varrho))^T = (p_i, q_i)^T$. Define

$$\begin{pmatrix} \overline{\Lambda}_i(t, \varrho) \\ \overline{\Upsilon}_i(t, \varrho) \end{pmatrix} := \Phi(t, \varrho) \begin{pmatrix} P_i(\varrho) \\ Q_i(\varrho) \end{pmatrix}, \qquad i = 1, 2. \tag{37}$$

By employing $\Phi(0, \varrho) = I$ and (36), we have

$$\begin{pmatrix} \overline{\Lambda}_i(0, \varrho) \\ \overline{\Upsilon}_i(0, \varrho) \end{pmatrix} = \begin{pmatrix} P_i(\varrho) \\ Q_i(\varrho) \end{pmatrix}, \qquad i = 1, 2. \tag{38}$$

In light of (36) and (38), we obtain

$$\Phi(p, \varrho) \begin{pmatrix} \overline{\Lambda}_i(0, \varrho) \\ \overline{\Upsilon}_i(0, \varrho) \end{pmatrix} = \eta_i(\varrho) \begin{pmatrix} \overline{\Lambda}_i(0, \varrho) \\ \overline{\Upsilon}_i(0, \varrho) \end{pmatrix}, \qquad i = 1, 2.$$

Particulary, by (33), (34), (35) and (37), we get

$$\begin{aligned}\begin{pmatrix} \overline{\Lambda}_i(t,0) \\ \overline{\Upsilon}_i(t,0) \end{pmatrix} &= \Phi(t,0)\begin{pmatrix} P_i(0) \\ Q_i(0) \end{pmatrix} = \Phi(t)\begin{pmatrix} p_i \\ q_i \end{pmatrix} \\ &= \Phi(t)\begin{pmatrix} \overline{\Lambda}_i(0) \\ \overline{\Upsilon}_i(0) \end{pmatrix} = \begin{pmatrix} \overline{\Lambda}_i(t) \\ \overline{\Upsilon}_i(t) \end{pmatrix}, \qquad i=1,2.\end{aligned} \tag{39}$$

From the definition of $(\overline{\Lambda}_1(t,\varrho), \overline{\Upsilon}_1(t,\varrho))^T$ in (37), we find

$$\begin{pmatrix} \partial_t\overline{\Lambda}_1(t,\varrho) \\ \partial_t\overline{\Upsilon}_1(t,\varrho) \end{pmatrix} = -\varrho D\begin{pmatrix} \overline{\Lambda}_1(t,\varrho) \\ \overline{\Upsilon}_1(t,\varrho) \end{pmatrix} + J^p(t)\begin{pmatrix} \overline{\Lambda}_1(t,\varrho) \\ \overline{\Upsilon}_1(t,\varrho) \end{pmatrix}. \tag{40}$$

Differentiating (40) with ϱ at $\varrho = 0$, we have

$$\begin{pmatrix} \partial_t\overline{\Lambda}_{1\varrho}(t,0) \\ \partial_t\overline{\Upsilon}_{1\varrho}(t,0) \end{pmatrix} = -D\begin{pmatrix} \overline{\Lambda}_1(t) \\ \overline{\Upsilon}_1(t) \end{pmatrix} + J^p(t)\begin{pmatrix} \overline{\Lambda}_{1\varrho}(t,0) \\ \overline{\Upsilon}_{1\varrho}(t,0) \end{pmatrix}, \tag{41}$$

where $\partial_\varrho\overline{\Lambda}_1(t,\varrho) = \overline{\Lambda}_{1\varrho}(t,0)$, $\partial_\varrho\overline{\Upsilon}_1(t,0) = \overline{\Upsilon}_{1\varrho}(t,0)$. Besides, from (36) to (38), we can deduce

$$\begin{pmatrix} \overline{\Lambda}_1(p,\varrho) \\ \overline{\Upsilon}_1(p,\varrho) \end{pmatrix} = \Phi(p,\varrho)\begin{pmatrix} P_1(\varrho) \\ Q_1(\varrho) \end{pmatrix} = \eta_1(\varrho)\begin{pmatrix} \overline{\Lambda}_1(0,\varrho) \\ \overline{\Upsilon}_1(0,\varrho) \end{pmatrix}. \tag{42}$$

Differentiating both sides of (42) with respect to ϱ, we can get

$$\begin{pmatrix} \overline{\Lambda}_{1\varrho}(p,\varrho) \\ \overline{\Upsilon}_{1\varrho}(p,\varrho) \end{pmatrix} = \eta_1'(\varrho)\begin{pmatrix} \overline{\Lambda}_1(0,\varrho) \\ \overline{\Upsilon}_1(0,\varrho) \end{pmatrix} + \eta_1(\varrho)\begin{pmatrix} \overline{\Lambda}_{1\varrho}(0,\varrho) \\ \overline{\Upsilon}_{1\varrho}(0,\varrho) \end{pmatrix}.$$

By using (39) and note that $\eta_1(0) = \kappa_1 = 1$, when $\varrho = 0$, we have

$$\begin{pmatrix} \overline{\Lambda}_{1\varrho}(p,0) \\ \overline{\Upsilon}_{1\varrho}(p,0) \end{pmatrix} = \eta_1'(0)\begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix} + \begin{pmatrix} \overline{\Lambda}_{1\varrho}(0,0) \\ \overline{\Upsilon}_{1\varrho}(0,0) \end{pmatrix}. \tag{43}$$

Using Lemma 1, $(u^p(t,\sigma), v^p(t,\sigma))$ is the periodic solution of (15), namely

$$(I+\sigma D)\begin{pmatrix} \partial_t u^p(t,\sigma) \\ \partial_t v^p(t,\sigma) \end{pmatrix} = \begin{pmatrix} \alpha - (u^p(t,\sigma))^3 v^p(t,\sigma) \\ \beta(u^p(t,\sigma))^3 v^p(t,\sigma) - \frac{v^p(t,\sigma)}{v^p(t,\sigma)+1} \end{pmatrix}. \tag{44}$$

Differentiating (44) with σ and setting $\sigma = 0$, we obtain

$$\begin{pmatrix} \partial_t u^p_\sigma(t,0) \\ \partial_t v^p_\sigma(t,0) \end{pmatrix} = -D \begin{pmatrix} \overline{\Lambda}_1(t) \\ \overline{\Upsilon}_1(t) \end{pmatrix} + J^p(t) \begin{pmatrix} u^p_\sigma(t,0) \\ v^p_\sigma(t,0) \end{pmatrix}, \tag{45}$$

where $\partial_\sigma u^p(t,0) = u^p_\sigma(t,0)$, $\partial_\sigma v^p(t,0) = v^p_\sigma(t,0)$, $\partial_t u^p(t,0) = \overline{\Lambda}_1(t)$ and $\partial_t v^p(t,0) = \overline{\Upsilon}_1(t)$. Moreover, one has

$$\begin{pmatrix} u^p(t,\sigma) \\ v^p(t,\sigma) \end{pmatrix} = \begin{pmatrix} u^p(t+p(\sigma),\sigma) \\ v^p(t+p(\sigma),\sigma) \end{pmatrix}. \tag{46}$$

Differentiating (46) about σ and evaluating at $\sigma = 0$ and $t = 0$, we get

$$\begin{pmatrix} u^p_\sigma(p,0) \\ v^p_\sigma(p,0) \end{pmatrix} = -p'(0) \begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix} + \begin{pmatrix} u^p_\sigma(0,0) \\ v^p_\sigma(0,0) \end{pmatrix}, \tag{47}$$

where $u^p(t,0) = u^p(t)$, $v^p(t,0) = v^p(t)$ and $p(0) = p$.

By utilizing (41), (43), (45) and (47), we can derive

$$\begin{pmatrix} \partial_t \overline{\Lambda}_{1\varrho}(t,0) \\ \partial_t \overline{\Upsilon}_{1\varrho}(t,0) \end{pmatrix} - \begin{pmatrix} \partial_t u^p_\sigma(t,0) \\ \partial_t v^p_\sigma(t,0) \end{pmatrix} = J^p(t) \left(\begin{pmatrix} \overline{\Lambda}_{1\varrho}(t,0) \\ \overline{\Upsilon}_{1\varrho}(t,0) \end{pmatrix} - \begin{pmatrix} u^p_\sigma(t,0) \\ v^p_\sigma(t,0) \end{pmatrix} \right) \tag{48}$$

and

$$\begin{aligned} \begin{pmatrix} \overline{\Lambda}_{1\varrho}(p,0) \\ \overline{\Upsilon}_{1\varrho}(p,0) \end{pmatrix} - \begin{pmatrix} u^p_\sigma(p,0) \\ v^p_\sigma(p,0) \end{pmatrix} &= (\eta'(0) + p'(0)) \begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix} \\ &\quad + \begin{pmatrix} \overline{\Lambda}_{1\varrho}(0,0) \\ \overline{\Upsilon}_{1\varrho}(0,0) \end{pmatrix} - \begin{pmatrix} u^p_\sigma(0,0) \\ v^p_\sigma(0,0) \end{pmatrix}. \end{aligned} \tag{49}$$

Define

$$\Psi(t) := \begin{pmatrix} \overline{\Lambda}_{1\varrho}(t,0) \\ \overline{\Upsilon}_{1\varrho}(t,0) \end{pmatrix} - \begin{pmatrix} u^p_\sigma(t,0) \\ v^p_\sigma(t,0) \end{pmatrix}.$$

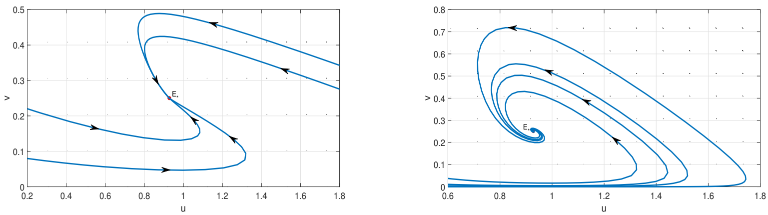
Consequently, (48) and (49) equivalent to the following form

$$\frac{d}{dt}\Psi(t) = J^p(t)\Psi(t),\ \ \Psi(p) - \Psi(0) = (\eta'(0) + p'(0)) \begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix}. \qquad (50)$$

Due to $(\overline{\Lambda}_1(0), \overline{\Upsilon}_1(0))^T$ and $(\overline{\Lambda}_2(0), \overline{\Upsilon}_2(0))^T$ are linearly independent, there exist constants k_1 and k_2 such that for any $(\varphi_1, \varphi_2)^T \in R^2$, there is

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = k_1 \begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix} + k_2 \begin{pmatrix} \overline{\Lambda}_1(0) \\ \overline{\Upsilon}_1(0) \end{pmatrix}. \qquad (51)$$

Let $\Psi(t) = \Phi(t)(\varphi_1, \varphi_2)^T$ be the general solution of $\Psi'(t) = J^p(t)\Psi(t)$. Then, substituting (51) into (50), we can find $\eta'(0) + p'(0) = 0$. Therefore, if $p'(0) < 0$, we have $\eta'(0) > 0$. By using [37], we know $(u^p(t), v^p(t))$ become unstable, that is the periodic solution $(u^p(t), v^p(t))$ of system (2) undergoes the Turing instability. The proof is finished.



(a) $\alpha = 0.2,\ \beta = 0.2$ (b) $\alpha = 0.2,\ \beta = 0.9$

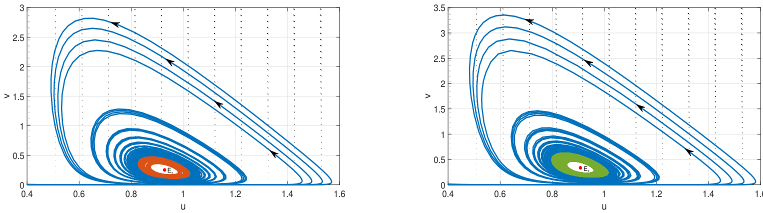
Figure 2. Phase portraits of enzyme-catalyzed system (2). (a) $E_* = (0.9283, 0.2500)$ is a stable node for $\alpha = 0.2$, $\beta = 0.2$; (b) $E_* = (0.9283, 0.2500)$ is a stable focus for $\alpha = 0.2$, $\beta = 0.9$.

4 Numerical simulations

In this section, we are committed to checking our previous theoretical analysis by utilizing the numerical approach.

When the parameters are selected as $\alpha = 0.2$ and $\beta = 0.2$, we will have $\beta_0 = 4.0396$, $\beta_1 = 0.2251$ and $\beta_2 = 72.4871$. Thereby, condition $0 < \beta \leq \beta_1 < \beta_0 < \beta_2$ is always satisfied. According to Theorem 1,

$E_* = (0.9283, 0.2500)$ of system (2) is a stable node, as shown in Fig. 2(a). When we fix the parameters $\alpha = 0.2$ and $\beta = 0.9$, we can conclude that $0 < \beta_1 < \beta < \beta_0 < \beta_2$. So, $E_* = (0.9283, 0.2500)$ of system (2) is a stable focus, see Fig. 2(b). In short, Fig. 2 illustrates the theoretical validity of Theorem 1.



(a) $\alpha = 0.2,\ \beta = 4.05$ (b) $\alpha = 0.25,\ \beta = 4.43$

Figure 3. Supercritical Hopf bifurcation happens to system (2). (a) $\alpha = 0.2$ and $\beta = 4.05$; (b) $\alpha = 0.25$ and $\beta = 4.43$.

In what follows, let us construct the stable periodic solution, which is induced by a supercritical type Hopf bifurcation based on Theorem 2. Choosing $\alpha = 0.2$ and $\beta = 4.05$, we can calculate that $u_* = 0.9238$, $v_* = 0.2500$, $\beta_0 = 4.0396 < \beta$ and $\mathrm{Re}(c(\beta_0)) = -0.0677 < 0$. This implies that the system (2) admits a supercritical Hopf bifurcation and generates stable periodic solutions. This judgment can be confirmed by Fig. 3(a). On the other hand, taking $\alpha = 0.25$ and $\beta = 4.43$, we can get $u_* = 0.9086$, $v_* = 0.3333$, $\beta_0 = 4.4026 < \beta$ and $\mathrm{Re}(c(\beta_0)) = -0.3697 < 0$. Benefiting from Theorem 2, the direction of Hopf bifurcation is supercritical, and the bifurcating periodic solutions are stable, as illustrated in Fig. 3(b).

For the enzyme-catalyzed system (3), we choose the spatial length control parameter $l = 3$. Now, when $\alpha = 0.28$, $\beta = 5.1$, $d_1 = 0.5$ and $d_2 = 3.0$, we can obtain $E_* = (0.8963, 0.3889), \beta_0 = 4.6488$ and $\mathrm{Re}(\gamma_1(\beta_0) = -9.0642 < 0$. Therefore, according to Theorem 4, we can claim that the Hopf bifurcation of system (3) is supercritical. It has formed a stable and uniform oscillation in time, see Figs. 4(a) and 4(b), respectively. Namely, when the diffusion term is added, although the concentrations of reactants S and P fluctuate unstably over a short period of time, these two reactants can oscillate stably as time progresses, refer to Fig.

4(c). However, these two substances always uniformly distribute in space, see Fig. 4(d), where the initial conditions are $u_0(x) = 0.8963 - 0.01\cos 2x$ and $v_0(x) = 0.3889 - 0.01\cos 2x$.

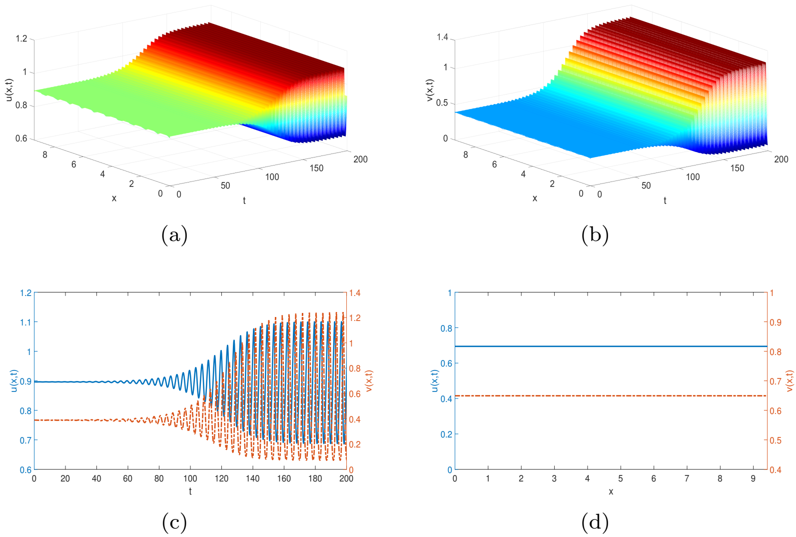


Figure 4. When $l = 3$, $\alpha = 0.28$, $\beta = 5.1$, $d_1 = 0.5$ and $d_2 = 3.0$, the enzyme-catalyzed system (3) admits the stable periodic solutions due to the supercritical Hopf bifurcation, where $u_0(x) = 0.8963 - 0.01\cos 2x$ and $v_0(x) = 0.3889 - 0.01\cos 2x$.

If we take $\alpha = 0.63$, $\beta = 10.8$, $d_1 = 1.25$ and $d_2 = 0.25$, it is easy to calculate $E_* = (0.7179, 1.7027)$, $\beta_0 = 11.2941$ and $\tau_h = -0.1440$. So, the conditions $\beta < \beta_0$ and $d_1 > \tau_h$ of Theorem 5 are satisfied. This shows the stable E_* of system (2) becomes unstable for system (3). Accordingly, Turing instability happens, as illustrated in Figs. 5(a) and 5(b). Thereby, if the diffusion term is presented and the Turing instability emerges, two reactants change from the previous uniform spatial distribution to an uneven one, while the temporal distributions of these two substances are uniform, see Figs. 5(c) and 5(d), where initial conditions are set to be $u_0(x) = 0.7179 - 0.03\cos x$ and $v_0(x) = 1.7027 - 0.03\cos x$.

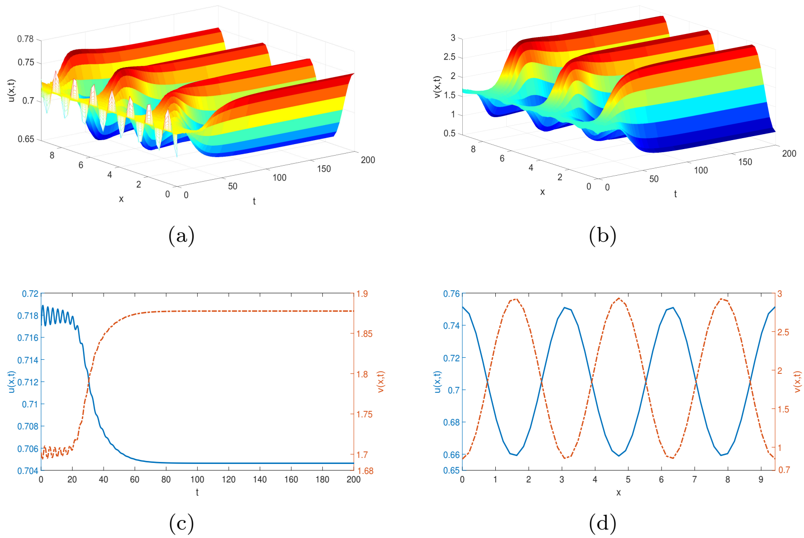


Figure 5. When $l = 3$, $\alpha = 0.63$, $\beta = 10.8$, $d_1 = 1.25$ and $d_2 = 0.25$, system (3) admits Turing instability with $u_0(x) = 0.7179 - 0.03\cos x$ and $v_0(x) = 1.7027 - 0.03\cos x$.

Now, we fix $\alpha = 0.15$, $\beta = 3.72$, $d_1 = 3.5$ and $d_2 = 0.1$, we can readily obtain $\beta_0 = 3.7259$. Under these parameters, we can further yield $E_* = (0.9473, 0.1765), \zeta_1(\beta_0)d_1 + \zeta_2(\beta_0)d_2 = -21.3109 < 0$ and $\beta < \beta_0$. Therefore, according to Theorem 7, it can be concluded that the periodic solution $(u^p(t), v^p(t))$ of system (3) will undergo Turing instability so that there are spatially nonhomogeneous periodic solution that occurs in the enzyme-catalyzed system (3), as shown in Figs. 6(a) and 6(b). This represents the coupling of spatiotemporal non-uniform periodic oscillations, see Figs. 6(c) and 6(d) for the time series and space series diagrams, where $u_0(x) = 0.9473 - 0.01\cos 2x$ and $v_0(x) = 0.1765 - 0.01\cos 2x$.

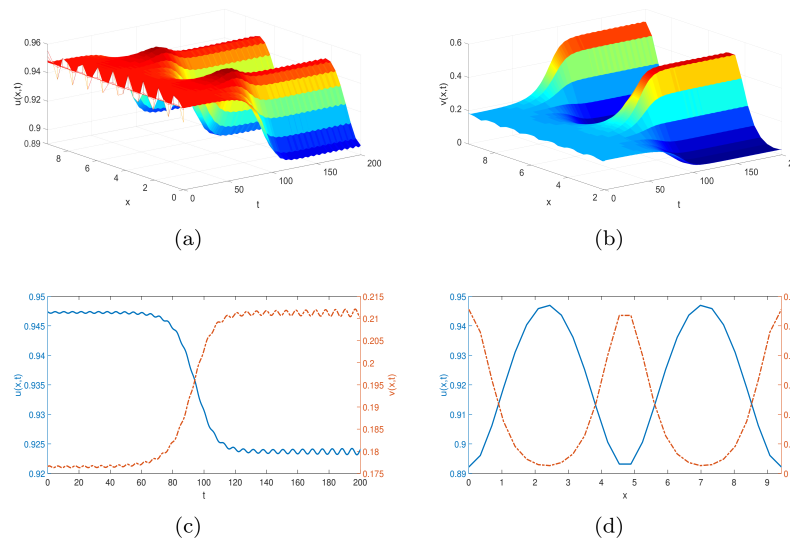


Figure 6. When $l = 3$, $\alpha = 0.15$, $\beta = 3.72$, $d_1 = 3.5$, and $d_2 = 0.1$, the periodic solution suffers from Turing instability with $u_0(x) = 0.9473 - 0.01\cos 2x$ and $v_0(x) = 0.1765 - 0.01\cos 2x$.

5 Conclusions

We investigate the Hopf bifurcation and Turing instability of an enzyme-catalyzed system. Theorem 1 determines the stability of E_* and fully characterizes its dynamical types within specific parameter ranges. Our results illustrate that this equilibrium is a stable/unstable focus or node and center. In Theorem 2, we determine the directions of the Hopf bifurcation via the first Lyapunov coefficient approach. For enzyme-catalyzed system (3), we first explore the spatially inhomogeneous Hopf bifurcation and its natural property by the aid of normal form theory, as described in Theorems 3 and 4, respectively. After that, we are able to research the existence of the Turing instability for the positive equilibrium and the periodic solution, see Theorem 5 and Theorem 7, respectively. Clearly, the most difficult problem is how to determine the Turing instability of the periodic solution. In this work, we can completely address this problem by employing Hopf bifurcation theory and the regular perturbation method. Our theoretical result shows that if $p'(0) < 0$ is satisfied, the periodic solution will become unstable in the Turing sense. Of course, we employ numerical experiments to check the validity of these theoretical conclusions. In short, we successfully explored the existence and detailed properties of the Hopf bifurcation and Turing instability. This paper may provide a methodology to investigate the complicated dynamical profiles of such an enzyme-catalyzed system.

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