MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **95** (2026) 467–485
ISSN: 0340–6253
doi: 10.46793/match.95-2.10325

On the Vertex Energy of Small Integral Trees

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(Received September 25, 2025)

Abstract

Introduced by Arizmendi et al. in the year 2018, the energy $\mathcal{E}_G(v)$ of a vertex v of a graph gives the graph energy distribution over all its vertices and is found to be very useful in understanding the influence of individual vertices on the overall graph energy. In this paper, we determine the vertex energies of some particular classes of trees. Consequently, we obtain the vertex energies of all integral trees on at most 30 vertices.

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1 Introduction

The concept of graph energy, introduced by Gutman [6], has been a very significant graph invariant in the field of mathematical chemistry. Owing to its close association with the total π-electron energy of molecules, obtained from simple H$\ddot{u}$ckel orbital calculations, graph energy and its variants have been studied extensively in literature.

Given a simple undirected graph $G = (V, E)$ with $V(G) = \{v_1, \cdots, v_n\}$, its adjacency matrix $A(G) = [a_{ij}]_{n\times n}$ has entries 1 or 0 according as the vertices v_i and v_j are adjacent or non-adjacent. The energy $\mathcal{E}(G)$ of G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i's are the eigenvalues of $A(G)$, which are all real, owing to the fact that $A(G)$ is real symmetric.

In the year 2018, Arizmendi et al. [1] introduced the concept of the energy $\mathcal{E}_G(v_i)$ of a vertex v_i of the graph G as

$$\mathcal{E}_G(v_i) = |A|_{ii}, \quad i = 1, 2, \cdots, n, \quad \text{where} \quad |A| = (AA^*)^{1/2}.$$

The author in [1] have shown that the energy $\mathcal{E}(G)$ is the sum of the individual energies of all its vertices, thereby validating the significance of vertex energy in analyzing the contribution of an individual vertex to the total graph energy. Also, they have deduced various inequalities for the vertex energy and have provided certain examples and counterexamples of natural conjectures for this parameter to explain its role in further studies on graph energy. Ramane et al. [9] calculated the vertex energies of subdivision graphs of some commonly studied graph structures such as complete graph, complete bipartite graph. Later, Gutman and Furtula [7] have shown how to numerically compute the energy of vertices of a graph using the eigenvalue matrix and orthonormal matrix of the eigenvector matrix of $A(G)$. In this article, we highlight the scope for calculating numerical values of vertex energies, along with their analysis and potential applications.

No.	n	*Name of the tree*	*Tree*	*Spectrum*
1	1	K_1	0	0
2	2	K_2	01	1
3	5	$K_{1,4}$	01111	$0^3, 2$
4	6	$K_{1,2} \sim K_{1,2}$	012211	$0^2, 1, 2$
5	7	$SK_{1,3}$	0121212	$0, 1^2, 2$
6	10	$K_{1,9}$	01^9	$0^8, 3$
7	14	$K_{1,6} \sim K_{1,6}$	$012^6 1^6$	$0^{10}, 2, 3$
8	17	$K_{1,16}$	01^{16}	$0^{15}, 4$
9	17	$SK_{1,8}$	$0(12)^8$	$0, 1^7, 3$
10	17	$K_{1,7} \sim SK_{1,4}$	$012^7(12)^4$	$0^7, 1^3, 2, 3$
11	19	$K_{1,5} \sim SK_{1,6}$	$012^5(12)^6$	$0^5, 1^5, 2, 3$
12	25	T_1	$01(23333)^3 22(12)^3$	$0^{11}, 1^3, 2^3, 3$
13	26	$K_{1,25}$	01^{25}	$0^{24}, 5$
14	26	T_2	$0(12222)^5$	$0^{16}, 2^4, 3$
15	26	$K_{1,12} \sim K_{1,12}$	$012^{12}1^{12}$	$0^{22}, 3, 4$

Table 1. Integral trees of order at most 30

A tree is a simple undirected graph that is connected and acyclic. In mathematical chemistry, trees are very often used to represent and analyze chemical and molecular structures and their intrinsic properties. A graph is said to be integral [8] if the spectrum of its adjacency matrix has only integral eigenvalues. In the literature, several studies have focused on determining and characterizing integral graphs. For a concise survey on integral trees and integral graphs, we refer the reader to [2]. As noted in [4] and [5], only 15 out of 14,830,871,802 non-isomorphic trees of order up to 30 are integral. Table 1 presents a complete enumeration of all such integral trees of order at most 30.

Since the spectrum of each tree is symmetric around 0 owing to the fact that it is bipartite, only the non-negative half of the spectrum is enlisted in the table. Also, each tree is listed as a sequence, with the zero representing the starting vertex, called the root, and each increasing sub-sequence representing a new walk, using the depth-first approach, starting from the vertex represented by the initial number of the sub-sequence.

Further, the exponent k to a sub-sequence represents k walks starting from the preceding vertex of the sub-sequence. For example, the tree T_1 represented by the sequence $01(23333)^3 22(12)^3$ is shown in Figure. 1. Also, some of the enlisted trees are named as follows.

(i) The star $K_{1,n}$ has $n+1$ vertices, with the root as the central vertex.

(ii) The graph SG is the subdivision graph of the the graph G, with the root being the same for both the graphs.

(iii) If G_1 and G_2 are rooted trees, then $G_1 \sim G_2$ is the tree obtained by adding an edge between their roots.

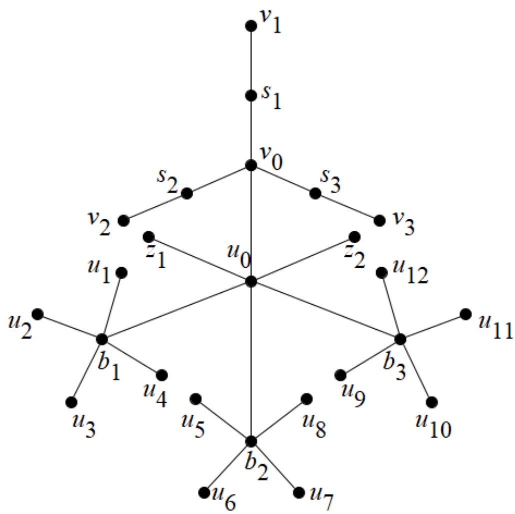

Figure 1. The tree T_1 represented by the sequence $01(23333)^3 22(12)^3$

In the present study, we work on establishing closed form expressions to determine the vertex energies of some particular classes of trees. Subsequently, we obtain the vertex energies of all integral trees of order up to 30.

The following results are helpful for discussion in the later sections of the paper.

Lemma 1. *[1] Let G be a graph of order n. Then,*

$$\mathcal{E}_G(v_i) = \sum_{j=1}^{n} p_{ij} \left|\lambda_j\right|, \quad i = 1, \cdots, n,$$

where λ_j denotes the j^{th} eigenvalue of $A(G)$ and

$$\sum_{i=1}^{n} p_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^{n} p_{ij} = 1.$$

Further, $p_{ij} = u_{ij}^2$, where $U = (u_{ij})$ is the matrix of orthonormal vectors of the eigenvectors of $A(G)$.

Lemma 2. *[1] Let G be a graph of order n. For $k \in \mathbb{N}$, let $\phi_i\left(A^k\right)$ be the k^{th} moment of A w. r. t. the linear functional ϕ_i. Then,*

$$\phi_i\left(A^k\right) = \sum_{j=1}^{n} p_{ij}\left(\lambda_j^k\right), \qquad i = 1, \cdots, n,$$

with $\phi_i\left(A^k\right)$ being the number of $v_i - v_i$ walks of G of length k.

Lemma 3. *[10] Let G_1 and G_2 be graphs with roots at u and v respectively. Then, the characteristic polynomial of $G_1 \sim G_2$ is*

$$P\left(G_1 \sim G_2; \lambda\right) = P(G_1; \lambda)P(G_2; \lambda) - P(G_1 - u; \lambda)P(G_2 - v; \lambda).$$

Theorem 1. *[1] For the star graph $G = K_{1,n}$, with v_0 being the central vertex and $v_1, \cdots, v_n$ the leaves,*

$$\mathcal{E}_G(v_i) = \begin{cases} \sqrt{n} & \text{if } i = 0, \\ \frac{1}{\sqrt{n}} & \text{otherwise.} \end{cases}$$

2 Vertex energies of some trees

In this section, we determine the vertex energies of some special trees, namely the subdivision graph of the star graph $K_{1,n}$ and the tree $K_{1,n} \sim K_{1,n}$.

Theorem 2. *Let $G = SK_{1,n}$ be the subdivision graph of the star graph $K_{1,n}$ with v_0 being the central vertex, $v_1, \cdots, v_n$ the leaves and $s_1, \cdots, s_n$ the vertices obtained by subdivision, then*

$$\mathcal{E}_G(v) = \begin{cases} \dfrac{n}{\sqrt{n+1}} & \text{if } v = v_0, \\ \dfrac{n^2 - 1 + \sqrt{n+1}}{n(n+1)} & \text{if } v = v_i, i = 1, \cdots, n, \\ \dfrac{n - 1 + \sqrt{n+1}}{n} & \text{otherwise} \end{cases}$$

Proof. As discussed in [3], the characteristic polynomial of $G = SK_{1,n}$ is given by

$$P(G; \lambda) = \lambda (\lambda + 1)^{n-1} (\lambda - 1)^{n-1} \left(\lambda + \sqrt{n+1}\right) \left(\lambda - \sqrt{n+1}\right).$$

Accordingly, the spectrum of G is given by $0, \pm 1^{n-1}, \pm\sqrt{n+1}$ so that the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = -\sqrt{n+1}$ and $\lambda_5 = \sqrt{n+1}$.

Observing the different vertex symmetries in the graph, we have that all the vertices $v_i, i = 1, \cdots, n$ have the same energy, and $s_i, i = 1, \cdots, n$ have the same energy. Further, since there are only five distinct eigenvalues, by Lemma 2, the energy of the vertices of G can be obtained by solving three 5×5 systems of equations, one each for the central vertex v_0, the leaves v_i and the vertices s_i, from calculating the first moments on one side and directly counting walks on the other, considering the following three cases.

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights of the vertex v_0 in G. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= n \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 0 \\ p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= n(n+1). \end{aligned}$$

Solving the system of equations, we get

$$p_{11} = \frac{1}{n+1}, p_{12} = 0, p_{13} = 0, p_{14} = \frac{n}{2\left(n+1\right)}, p_{15} = \frac{n}{2\left(n+1\right)}.$$

Thus, the energy of the vertex v_0 is given by

$$\mathcal{E}_G(v_0) = \sum_{j=1}^{5} p_{1j}\left|\lambda_j\right| = \frac{n}{\sqrt{n+1}}.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $v_i, i = 1, \cdots, n$ in G. Then, we have

$$\begin{aligned}
p_{21} + p_{22} + p_{23} + p_{24} + p_{25} &= 1\\
p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 + p_{25}\lambda_5 &= 0\\
p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 + p_{25}\lambda_5^2 &= 1\\
p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 + p_{25}\lambda_5^3 &= 0\\
p_{21}\lambda_1^4 + p_{22}\lambda_2^4 + p_{23}\lambda_3^4 + p_{24}\lambda_4^4 + p_{25}\lambda_5^4 &= 2.
\end{aligned}$$

Solving the system of equations, we get

$$\begin{aligned}
p_{21} &= \frac{1}{n+1}, p_{22} = \frac{n-1}{2n}, p_{23} = \frac{n-1}{2n},\\
p_{24} &= \frac{1}{2n\left(n+1\right)}, p_{25} = \frac{1}{2n\left(n+1\right)}.
\end{aligned}$$

Thus, the energy of each of the vertices $v_i, i = 1, \cdots, n$, is given by

$$\mathcal{E}_G(v_i) = \sum_{j=1}^{5} p_{2j}\left|\lambda_j\right| = \frac{n^2 - 1 + \sqrt{n+1}}{n\left(n+1\right)}.$$

Case 3: Let $p_{31}, p_{32}, p_{33}, p_{34}, p_{35}$ be the weights of any vertex $s_i, i = 1, \cdots, n$ in G. Then, we have

$$\begin{aligned}
p_{31} + p_{32} + p_{33} + p_{34} + p_{35} &= 1\\
p_{31}\lambda_1 + p_{32}\lambda_2 + p_{33}\lambda_3 + p_{34}\lambda_4 + p_{35}\lambda_5 &= 0
\end{aligned}$$

$$p_{31}\lambda_1^2 + p_{32}\lambda_2^2 + p_{33}\lambda_3^2 + p_{34}\lambda_4^2 + p_{35}\lambda_5^2 = 2$$
$$p_{31}\lambda_1^3 + p_{32}\lambda_2^3 + p_{33}\lambda_3^3 + p_{34}\lambda_4^3 + p_{35}\lambda_5^3 = 0$$
$$p_{31}\lambda_1^4 + p_{32}\lambda_2^4 + p_{33}\lambda_3^4 + p_{34}\lambda_4^4 + p_{35}\lambda_5^4 = n + 3.$$

Solving the system of equations, we get

$$p_{31} = 0, p_{32} = \frac{n-1}{2n}, p_{33} = \frac{n-1}{2n}, p_{34} = \frac{1}{2n}, p_{35} = \frac{1}{2n}.$$

Thus, the energy of each of the vertices $s_i, i = 1, \cdots, n$, is given by

$$\mathcal{E}_G(v_i) = \sum_{j=1}^{5} p_{3j} |\lambda_j| = \frac{n - 1 + \sqrt{n+1}}{n}. \qquad \blacksquare$$

Lemma 4. *The characteristic polynomial of $K_{1,n} \sim K_{1,n}$ is given by*

$$P(K_{1,n} \sim K_{1,n}; \lambda) = \lambda^{2n-2}\left(\lambda^2 + \lambda - n\right)\left(\lambda^2 - \lambda - n\right).$$

Proof. Taking $G_1 = G_2 = K_{1,n}$ in Lemma 3, we get

$$\begin{aligned} P(K_{1,n} \sim K_{1,n}; \lambda) &= P(K_{1,n}; \lambda)^2 - P\left(\overline{K_n}; \lambda\right)^2 \\ &= \left(\lambda^{n-1}\left(\lambda + \sqrt{n}\right)\left(\lambda - \sqrt{n}\right)\right)^2 - (\lambda^n)^2 \\ &= \lambda^{2n-2}\left(\left(\lambda^2 - n\right)^2 - \lambda^2\right) \\ &= \lambda^{2n-2}\left(\lambda^2 + \lambda - n\right)\left(\lambda^2 - \lambda - n\right). \end{aligned} \qquad \blacksquare$$

Theorem 3. *For the graph $G = K_{1,n} \sim K_{1,n}$, with u_0 and v_0 being the central vertices of each star and $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ their leaves respectively,*

$$\mathcal{E}_G(v) = \begin{cases} \dfrac{2n+1}{\sqrt{4n+1}} & \text{if } v = u_0, v_0, \\ \dfrac{2}{\sqrt{4n+1}} & \text{otherwise.} \end{cases}$$

Proof. As seen from Lemma 4, the characteristic equation of G is given by

$$P(G; \lambda) = \lambda^{2n-2}\left(\lambda^2 + \lambda - n\right)\left(\lambda^2 - \lambda - n\right).$$

Thus, the spectrum of G is given by $0^{2n-2}, \frac{-1\pm\sqrt{4n+1}}{2}, \frac{1\pm\sqrt{4n+1}}{2}$. Accordingly, the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = \frac{-1-\sqrt{4n+1}}{2}, \lambda_3 = \frac{-1+\sqrt{4n+1}}{2}, \lambda_4 = \frac{1-\sqrt{4n+1}}{2}$ and $\lambda_5 = \frac{1+\sqrt{4n+1}}{2}$.

As there exist two vertex symmetries in the graph, the vertices u_0 and v_0 have the same energy and the vertices $u_i, v_i, i = 1, \cdots, n$, have the same energy. Further, since there are only five distinct eigenvalues, by Lemma 2, the energy of the vertices of G can be obtained by solving three 5×5 systems of equations, one for the central vertices u_0 and v_0, and the other for the leaves $u_i, v_i, i = 1, \cdots, n$, from calculating the first moments on one side and directly counting walks on the other, considering the following two cases.

Case 1: For the vertices u_0 and v_0, let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights in G. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= n+1 \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 0 \\ p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= n^2 + 3n + 1. \end{aligned}$$

Solving the system of equations, we get

$$\begin{aligned} p_{11} &= 0, p_{12} = \frac{1+\sqrt{4n+1}}{4\sqrt{4n+1}}, p_{13} = \frac{-1+\sqrt{4n+1}}{4\sqrt{4n+1}}, \\ p_{14} &= \frac{-1+\sqrt{4n+1}}{4\sqrt{4n+1}}, p_{15} = \frac{1+\sqrt{4n+1}}{4\sqrt{4n+1}}. \end{aligned}$$

Thus, we have

$$\mathcal{E}_G(u_0) = \mathcal{E}_G(v_0) = \sum_{j=1}^{5} p_{1j} \left|\lambda_j\right| = \frac{2n+1}{\sqrt{4n+1}}.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $u_i, v_i, i =$

$1, \cdots, n$ in G. Then, we have

$$\begin{aligned} p_{21}+p_{22}+p_{23}+p_{24}+p_{25} &= 1 \\ p_{21}\lambda_1+p_{22}\lambda_2+p_{23}\lambda_3+p_{24}\lambda_4+p_{25}\lambda_5 &= 0 \\ p_{21}\lambda_1^2+p_{22}\lambda_2^2+p_{23}\lambda_3^2+p_{24}\lambda_4^2+p_{25}\lambda_5^2 &= 1 \\ p_{21}\lambda_1^3+p_{22}\lambda_2^3+p_{23}\lambda_3^3+p_{24}\lambda_4^3+p_{25}\lambda_5^3 &= 0 \\ p_{21}\lambda_1^4+p_{22}\lambda_2^4+p_{23}\lambda_3^4+p_{24}\lambda_4^4+p_{25}\lambda_5^4 &= n+1. \end{aligned}$$

Solving the system of equations, we get

$$p_{21} = \frac{n-1}{n}, p_{22} = \frac{-1+\sqrt{4n+1}}{4n\sqrt{4n+1}}, p_{23} = \frac{1+\sqrt{4n+1}}{4n\sqrt{4n+1}},$$
$$p_{24} = \frac{1+\sqrt{4n+1}}{4n\sqrt{4n+1}}, p_{25} = \frac{-1+\sqrt{4n+1}}{4n\sqrt{4n+1}}.$$

Thus, the energy of each of the vertices $u_i, v_i, i = 1, \cdots, n$, is given by

$$\mathcal{E}_G(u_i) = \mathcal{E}_G(v_i) = \sum_{j=1}^{5} p_{2j}\,|\lambda_j| = \frac{2}{\sqrt{4n+1}}. \qquad \blacksquare$$

3 Vertex energy of small integral trees

In this section, we determine the vertex energy of all integral trees of order at most 30 using the results in the previous sections. To begin with, we introduce some notations used in the process.

Let G be a graph with k distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ and r vertex symmetries. Then, it is easy to observe that there are r sets of vertices, each having the same vertex energy. Further, using Lemma 1, they can be determined by solving r different linear systems of k equations in k unknowns of the form

$$\begin{aligned} p_{i1}+p_{i2}+\cdots+p_{ik} &= \phi_i(1) \\ p_{i1}\lambda_1+p_{i2}\lambda_2+\cdots+p_{ik}\lambda_k &= \phi_i(A) \\ p_{i1}\lambda_1^2+p_{i2}\lambda_2^2+\cdots+p_{ik}\lambda_k^2 &= \phi_i(A^2) \end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$p_{i1}\lambda_1^{k-1} + p_{i2}\lambda_2^{k-1} + \cdots + p_{ik}\lambda_k^{k-1} = \phi_i(A^{k-1}),$$

where $\phi_i(A^j)$ represents the number of $v_i - v_i$ walks of length $j = 1, \cdots, k-1$ for each $i = 1, \cdots, r$.

Thus, if

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1}, \end{pmatrix}$$

$$P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{r1} \\ p_{12} & p_{22} & \cdots & p_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ p_{1k} & p_{2k} & \cdots & p_{rk}, \end{pmatrix}$$

and

$$Y = \begin{pmatrix} \phi_1(1) & \phi_2(1) & \cdots & \phi_r(1) \\ \phi_1(A) & \phi_2(A) & \cdots & \phi_r(A) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(A^{k-1}) & \phi_2(A^{k-1}) & \cdots & \phi_r(A^{k-1}) \end{pmatrix},$$

then the complete system can be written as

$$JP = Y$$

Further, since J is non-singular, the complete solution to this system is given by

$$P = J^{-1}Y$$

using which the energies of each vertex of a particular symmetry can be computed.

Theorem 4. *For the graph $G = K_{1,7} \sim SK_{1,4}$, with u_0 and v_0 being the roots, $u_1, \cdots, u_7$ and $v_1, \cdots, v_4$ being the leaves of $K_{1,7}$ and $SK_{1,4}$*

respectively and $s_1, \cdots, s_4$ *the vertices obtained by subdivision in* $SK_{1,4}$,

$$\mathcal{E}_G(v) = \begin{cases} 1.96667 & \textit{if } v = v_0, \\ 2.8 & \textit{if } v = u_0, \\ 0.36667 & \textit{if } v = u_1, \cdots, u_7, \\ 1.3 & \textit{if } v = s_1, \cdots, s_4, \\ 0.86667 & \textit{otherwise.} \end{cases}$$

Proof. As seen in Table 1, the characteristic polynomial of $G = K_{1,7} \sim SK_{1,4}$ is given by

$$P(G; \lambda) = \lambda^7 (\lambda + 1)^3 (\lambda - 1)^3 (\lambda + 2)(\lambda - 2)(\lambda + 3)(\lambda - 3).$$

The spectrum of G is given by $0^7, \pm 1^3, \pm 2, \pm 3$ so that the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = 1, \lambda_6 = 2$ and $\lambda_7 = 3$. The vertex set of G can be partitioned into 5 sets, based on 5 different vertex symmetries, namely $\{u_0\}$, $\{v_0\}$, $\{u_1, \cdots, u_7\}$, $\{s_1, \cdots, s_4\}$ and $\{v_1, \cdots, v_4\}$, so that all the vertices belonging to a set have the same energy.

Therefore, as discussed in the earlier part of the section, we have the matrices J of order 7×7, P of order 7×5 and Y of order 7×5 given by

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_7 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_7^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^6 & \lambda_2^6 & \cdots & \lambda_7^6 \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{51} \\ p_{12} & p_{22} & \cdots & p_{52} \\ \vdots & \vdots & \vdots & \vdots \\ p_{17} & p_{27} & \cdots & p_{57} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 36 & 68 & 8 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 288 & 596 & 68 & 50 & 8 \end{pmatrix}$$

where Y is computed by counting the number of $v_i - v_i$ walks of length $j = 1, \cdots, 6$ for each $i = 1, \cdots, 5$. Solving the system $JP = Y$, we get

$$P = \begin{pmatrix} 0.177778 & 0.4 & 0.044444 & 0.025 & 0.002778 \\ 0.225 & 0.1 & 0.025 & 0.1 & 0.025 \\ 0 & 0 & 0 & 0.375 & 0.375 \\ 0.194444 & 0 & 0.861111 & 0 & 0.194444 \\ 0 & 0 & 0 & 0.375 & 0.375 \\ 0.225 & 0.1 & 0.025 & 0.1 & 0.025 \\ 0.177778 & 0.4 & 0.04444 & 0.025 & 0.0027778 \end{pmatrix}$$

Using the values of $p'_{ij}s$ in Lemma 1, we arrive at the required result. ■

Theorem 5. *For the graph $G = K_{1,5} \sim SK_{1,6}$, with u_0 and v_0 being the roots, $u_1, \cdots, u_5$ and $v_1, \cdots, v_6$ being the leaves of $K_{1,5}$ and $SK_{1,6}$ respectively and $s_1, \cdots, s_6$ the vertices obtained by subdivision in $SK_{1,6}$,*

$$\mathcal{E}_G(v) = \begin{cases} 2.433333 & \text{if } v = v_0, \\ 2.4 & \text{if } v = u_0, \\ 0.433333 & \text{if } v = u_1, \cdots, u_5, \\ 1.266667 & \text{if } v = s_1, \cdots, s_6, \\ 0.9 & \text{otherwise.} \end{cases}$$

Proof. As seen in Table 1, the characteristic polynomial of $G = K_{1,5} \sim SK_{1,6}$ is given by

$$P(G; \lambda) = \lambda^5 (\lambda + 1)^5 (\lambda - 1)^5 (\lambda + 2)(\lambda - 2)(\lambda + 3)(\lambda - 3).$$

The spectrum of G is given by $0^5, \pm 1^5, \pm 2, \pm 3$ so that the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = 1, \lambda_6 = 2$ and $\lambda_7 = 3$. Here, the vertex set of G can be partitioned into 5 sets, based on 5 different vertex symmetries, namely $\{u_0\}$, $\{v_0\}$, $\{u_1, \cdots, u_5\}$, $\{s_1, \cdots, s_6\}$ and $\{v_1, \cdots, v_6\}$, so that all the vertices belonging to a set have the same energy.

The matrices J of order 7×7, P of order 7×5 and Y of order 7×5, pertaining to the graph are therefore given by

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_7 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_7^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^6 & \lambda_2^6 & \cdots & \lambda_7^6 \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{51} \\ p_{12} & p_{22} & \cdots & p_{52} \\ \vdots & \vdots & \vdots & \vdots \\ p_{17} & p_{27} & \cdots & p_{57} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 60 & 42 & 6 & 10 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 528 & 330 & 42 & 78 & 10 \end{pmatrix}$$

where Y is computed by counting the number of $v_i - v_i$ walks of length $j = 1, \cdots, 6$ for each $i = 1, \cdots, 5$. Solving the system $JP = Y$, we get

$$P = \begin{pmatrix} 0.355556 & 0.2 & 0.022222 & 0.05 & 0.005556 \\ 0.075 & 0.3 & 0.075 & 0.033333 & 0.083333 \\ 0 & 0 & 0 & 0.416667 & 0.416667 \\ 0.138889 & 0 & 0.805556 & 0 & 0.138889 \\ 0 & 0 & 0 & 0.416667 & 0.416667 \\ 0.075 & 0.3 & 0.075 & 0.033333 & 0.083333 \\ 0.355556 & 0.2 & 0.022222 & 0.05 & 0.005556 \end{pmatrix}$$

Using the values of $p'_{ij}s$ in Lemma 1 gives the required result. ■

Theorem 6. *For the graph $G = T_1$, with u_0 being the root, v_0 being the subdivided rooted vertex, b_1, b_2 & b_3 being the branching vertices, $u_1, \cdots,$ u_{12} being the leaves associated with branching vertices, z_1 & z_2 being leaves associated with u_0,s_1, s_2 & s_3 being the vertices obtained by subdivision and*

v_1, v_2 & v_3 *the leaves associated with these vertices,*

$$\mathcal{E}_G(v) = \begin{cases} 1.733333 & \text{if } v = v_0, \\ 2.25 & \text{if } v = u_0, \\ 2.183333 & \text{if } v = b_1, b_2, b_3, \\ 0.483333 & \text{if } v = u_1, \cdots, u_{12}, \\ 0.583333 & \text{if } v = z_1, z_2, \\ 1.316667 & \text{if } v = s_1, s_2, s_3, \\ 0.85 & \text{if } v = v_1, v_2, v_3. \end{cases}$$

Proof. As seen in Table 1, the characteristic polynomial of $G = T_1$ is given by

$$P(G; \lambda) = \lambda^{11} (\lambda + 1)^3 (\lambda - 1)^3 (\lambda + 2)^3 (\lambda - 2)^3 (\lambda + 3)(\lambda - 3).$$

The spectrum of G is given by $0^{11}, \pm 1^3, \pm 2^3, \pm 3$ so that the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = 1, \lambda_6 = 2$ and $\lambda_7 = 3$. Figure 1 shows the graph T_1. As seen from the Figure 1, the vertex set of G can be partitioned into 7 sets, based on 7 different vertex symmetries, namely $\{u_0\}$, $\{v_0\}$, $\{b_1, b_2, b_3\}$, $\{u_1, \cdots, u_{12}\}$, $\{z_1, z_2\}$, $\{s_1, s_2, s_3\}$ and $\{v_1, v_2, v_3\}$, so that all the vertices belonging to a set have the same energy.

Therefore, we have the matrices J of order 7×7,P of order 7×7 and Y of order 7×7 given by

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_7 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_7^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^6 & \lambda_2^6 & \cdots & \lambda_7^6 \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{71} \\ p_{12} & p_{22} & \cdots & p_{72} \\ \vdots & \vdots & \vdots & \vdots \\ p_{17} & p_{27} & \cdots & p_{77} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 5 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 51 & 30 & 5 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 168 & 456 & 211 & 30 & 51 & 36 & 7 \end{pmatrix}$$

where Y is computed by counting the number of $v_i - v_i$ walks of length $j = 1, \cdots, 6$ for each $i = 1, \cdots, 7$. Solving the system $JP = Y$, we get

$$P = \begin{pmatrix} 0.088889 & 0.3125 & 0.1125 & 0.0125 & 0.034722 & 0.0125 & 0.001389 \\ 0.3 & 0 & 0.366667 & 0.091667 & 0 & 0.133333 & 0.033333 \\ 0 & 0.187500 & 0.020833 & 0.020833 & 0.187500 & 0.354167 & 0.354167 \\ 0.222222 & 0 & 0 & 0.75 & 0.555556 & 0 & 0.222222 \\ 0 & 0.1875 & 0.020833 & 0.020833 & 0.1875 & 0.354167 & 0.354167 \\ 0.3 & 0 & 0.366667 & 0.091667 & 0 & 0.133333 & 0.033333 \\ 0.088889 & 0.3125 & 0.1125 & 0.0125 & 0.034722 & 0.0125 & 0.001389 \end{pmatrix}$$

Using the values of $p_{ij}'s$ in Lemma 1 gives the required result. ■

Theorem 7. *For the graph $G = T_2$, with u_0 being the root, $u_1, \cdots, u_5$ being the branching vertices and $v_1, \cdots, v_{20}$ being the leaves,*

$$\mathcal{E}_G(v) = \begin{cases} 1.666667 & \text{if } v = u_0, \\ 2.2 & \text{if } v = u_1, \cdots, u_5, \\ 0.466667 & \text{otherwise.} \end{cases}$$

Proof. As seen in Table 1, the characteristic polynomial of $G = T_2$ is given by

$$P(G; \lambda) = \lambda^{16} (\lambda + 2)^4 (\lambda - 2)^4 (\lambda + 3) (\lambda - 3).$$

The spectrum of G is given by $0^{16}, \pm 2^4, \pm 3$ so that the distinct eigenvalues of G are $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -3, \lambda_4 = 2$ and $\lambda_5 = 3$. The vertex set of G can be partitioned into 3 sets, based on 3 different vertex symmetries, namely $\{u_0\}$, $\{u_1, \cdots, u_5\}$, and $\{v_1, \cdots, v_{20}\}$, so that all the vertices belonging to a set have the same energy. Thus, we have the matrices J of order 5×5, P of order 5×3 and Y of order 5×3 given by

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_5 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_5^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^4 & \lambda_2^4 & \cdots & \lambda_5^4 \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ \vdots & \vdots & \vdots \\ p_{15} & p_{25} & p_{35} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 5 & 5 & 1 \\ 0 & 0 & 0 \\ 45 & 29 & 5 \end{pmatrix}$$

where Y is computed by counting the number of $v_i - v_i$ walks of length $j = 1, \cdots, 4$ for each $i = 1, 2, 3$. Solving the system $JP = Y$, we get

$$P = \begin{pmatrix} 0.277778 & 0.1 & 0.011111 \\ 0 & 0.4 & 0.1 \\ 0.444444 & 0 & 0.777778 \\ 0 & 0.4 & 0.1 \\ 0.277778 & 0.1 & 0.011111 \end{pmatrix}$$

Using the values of $p'_{ij}s$ in Lemma 1 gives the required result. ■

As seen in Table 1, there are 15 integral trees of order at most 30. Out of these, the vertex energies of the trees namely $K_{1,7} \sim SK_{1,4}$, $K_{1,5} \sim SK_{1,6}$, T_1 and T_2 are determined in this section. For the other trees, vertex energies are calculated using suitable values of n as prescribed in Theorems 1, 2 and 3, with the results summarized in Table 2.

Table 2. Vertex energies of the remaining integral trees

No.	*n*	*Name of the tree*	*Number of vertex symmetries*	*Vertex energies*
1	1	K_1	1	0
2	2	K_2	1	1
3	5	$K_{1,4}$	2	*Central vertex:* 2 *Leaves:* 0.5
4	6	$K_{1,2} \sim K_{1,2}$	2	*Central vertices:* 1.667 *Leaves:* 0.667
5	7	$SK_{1,3}$	3	*Central vertex:* 1.5 *Subdivided vertices:* 1.333 *Leaves:* 0.833
6	10	$K_{1,9}$	2	*Central vertex:* 3 *Leaves:* 0.333
7	14	$K_{1,6} \sim K_{1,6}$	2	*Central vertices:* 2.6 *Leaves:* 0.4
8	17	$K_{1,16}$	2	*Central vertex:* 4 *Leaves:* 0.25
9	17	$SK_{1,8}$	3	*Central vertex:* 2.667 *Subdivided vertices:* 1.25 *Leaves:* 0.917
13	26	$K_{1,25}$	2	*Central vertex:* 5 *Leaves:* 0.2
15	26	$K_{1,12} \sim K_{1,12}$	2	*Central vertices:* 3.571 *Leaves:* 0.286

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