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The Steiner Gutman Index of Trees

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Abstract

The concept of *Gutman index* $Gut(G)$ of a connected graph G was introduced in 1994. The Steiner distance in a graph, introduced by Chartrand et al. in 1989, is a natural generalization of the concept of classical graph distance. The *Steiner Gutman k-index* $SGut_k(G)$ introduced by Mao et al. in 2018, is defined by $SGut_k(G) = \sum_{S\subseteq V(G),|S|=k}\left[\prod_{v\in S} d_G(v)\right] d_G(S)$, where $d_G(S)$ is the Steiner distance of S and $d_G(v)$ is the degree of v in G. In this paper, we obtained the relations between Steiner Gutman k-index and Gutman index, Wiener index and Steiner Wiener k-index of trees with $k = 3, 4$.

1 Introduction

All graphs in this paper are undirected, finite, connected and simple. We refer to [2] for graph theoretical notation and terminology not described here. Let G be a graph with n vertices and m edges. Its vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$ and its edge set $E(G)$. For a vertex $v_i \in V(G)$, the *degree* of v_i in graph G, denoted by $d_G(v_i)$, is the number of edges of G incident with v_i. The minimum vertex degree is denoted by δ and the maximum by Δ. If the vertices v_i and v_j are adjacent, then the edge connecting them is labeled by e_{ij}. If G is a connected graph and $u, v \in V(G)$, then the *distance* between u and v, denoted $d_G(u,v)$, is the length of a shortest path connecting u and v.

In graph theory applied to chemical problems, a large number of molecular structure descriptors, so-called "topological indices", has been studied [25]. Many of these descriptors are defined in terms of vertex degrees; see [4, 14, 25]. Equally many of these descriptors are in terms of distance between vertices; see [25, 26]. These are also several degree-and-distance-based topological indices; see [9–12, 15].

In [15], the *Gutman index* of a graph G is defined as

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} [d_G(u)d_G(v)]\, d_G(u,v),$$

where $d_G(u)$ is the degree of vertex $u \in V(G)$, and $d_G(u,v)$ is the distance between the vertices $u, v \in V(G)$. For more details on Gutman index, we refer to [6, 10, 13, 25].

The Steiner distance of a graph, introduced by Chartrand et al. in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G = (V, E)$ and a set $S \subseteq V$, an *S-Steiner tree* or a *Steiner tree connecting* S (or simply, an *S-tree*) is a subgraph $H = (V', E')$ of G that is a tree with $S \subseteq V'$. The *Steiner distance* $d_G(S)$ among the vertices of S (or simply the *distance* of S) is the minimum size of a connected subgraph of G such that $S \subseteq V(H)$. It is clear that H must be a tree, and if $|S| = k$, then $d(S) \geq k - 1$. For more details on Steiner distance, we refer to [1, 3, 5, 7, 8, 24].

The *Wiener index* $W(G)$ of a connected graph G, introduced by Wiener in 1947, is defined as $W(G) = \sum_{\{u,v\} \in V(G)} d_G(u,v)$, where $d_G(u,v)$ is the distance between the vertices u and v in G [17]. As a generalization of Wiener index, the *the Steiner Wiener k-index* or *k-center Steiner Wiener index* $SW_k(G)$ of a connected graph G was introduced by Li et al. in [17]:

$$SW_k(G) = \sum_{S \subseteq V(G), |S| = k} d_G(S).$$

It is clear that for a connected graph G of order n,

$$SW_2(G) = W(G), \ SW_1(G) = 0, \ SW_n(G) = n - 1.$$

For more details on the Steiner Wiener index, we refer to [17–21].

Mao et al. [23] generalize the concept of Gutman index by Steiner distance. The *Steiner Gutman k-index* $SGut_k(G)$ of G is defined by

$$SGut_k(G) = \sum_{S \subseteq V(G), |S| = k} \left[\prod_{v \in S} d_G(v) \right] d_G(S).$$

The *degree distance* of a graph G is defined as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} [d_G(u) + d_G(v)] \, d_G(u, v).$$

And Gutman [23] generalized the concept of degree distance by Steiner degree distance. The *k-center Steiner degree distance* $SDD_k(G)$ of G is defined by

$$SDD_k(G) = \sum_{S \subseteq V(G), |S| = k} \left[\sum_{v \in S} d_G(v) \right] d_G(S).$$

For more details on the Steiner degree distance, we refer to [23].

Mao et al. obtained the exact values of the Steiner Gutman k-index of the complete graphs, complete bipartite graph, path and star. When G is a connected graph, especially when it is a tree, Mao et al. also get the expression of $SGut_k(G)$ for $k = n, n-1$. And get sharp lower and upper bounds for $SGut_k(G)$ in terms of degree, or both order and size, or order. And comparison between $SDD_k(G)$ and $SGut_k(G)$ of graphs is given. And some results as following.

Proposition 1.1. *[23] Let K_n be the complete graph of order n, and let k be an integer such that $2 \le k \le n$. Then*

$$SGut_k(K_n) = \binom{n}{k} (n-1)^k (k-1).$$

Proposition 1.2. *[23] Let $K_{a,b}$ be the complete bipartite graph of order $a+b\,(1 \le a \le b)$, and let k be an integer such that $2 \le k \le a+b$. Then*

$$SGut_k\left(K_{a,b}\right)=\begin{cases} ka^k\binom{b}{k}+kb^k\binom{a}{k}+(k-1)\sum_{x=1}^{k-1}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & if 1\le k\le a, \\ ka^k\binom{b}{k}+(k-1)\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & if a< k\le b, \\ (k-1)\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & if b< k\le a+b\ . \end{cases}$$

Corollary 1.1. *[23] Let S_n be the star of order $n\,(n \ge 3)$, and let k be an integer such that $2 \le k \le n$. Then*

$$SGut_k\left(S_n\right) = (kn-2k+1)\binom{n-1}{k-1}(n-1)\,.$$

Proposition 1.3. *[23] Let P_n be the path of order n, and let k be an integer such that $2 \le k \le n-2$. Then*

$$SGut_k\left(P_n\right) = 2^k\,(k-1)\binom{n}{k+1} + 2^{k-2}\,(n-1)\binom{n-2}{k-2}.$$

Proposition 1.4. *[23] Let G be a connected graph of order n, and let k be an integer such that $2 \le k \le n$. Then*

$$\delta^k\binom{n}{k}(k-1) \le SGut_k\left(G\right) \le \Delta^k\,(k-1)\binom{n+1}{k+1}.$$

Moreover, the bounds are sharp.

Theorem 1.1. *[23] Let G be a connected graph n vertices and m edges, and let k be an integer such that $2 \le k \le n$. Then*

$$(n-1)\left(\frac{2m}{k}\right)^k\binom{n-1}{k-1}^k \ge SGut_k\left(G\right) \ge \begin{cases} 2m\,(k-1)\binom{n-1}{k-1} & if\ \delta \ge 2, \\ (k-1)\binom{n}{k} & if\ \delta = 1. \end{cases}$$

Moreover, the upper and lower bounds are sharp.

In this paper, we obtained the relations between Steiner Gutman k-index and Gutman index , Wiener index and Steiner Wiener k-index of trees with $k = 3, 4$.

2 Main results

We defined the *k-Steiner Gutman Transmission* $sg_k(G,v)$ of a vertex $v \in V(G)$,

$$sg_k(G,v) = \sum_{S\subseteq V(G),|S|=k,v\in S} \left[\prod_{u\in S} d_G(u)\right] d_G(S).$$

Let T be a tree of order n and label the vertices of T by $v_1, v_2, \ldots, v_n$. Then when $k=3$, we have

$$SGut_k(T) = \frac{1}{3}\sum_{i=1}^{n} sg_k(T, v_i).$$

And we defined the *k-Steiner Gutman generalized Transmission*

$$sg_k(G, v_1, v_2, \ldots, v_t)$$

of vertices set $\{v_1, v_2, \ldots, v_t\} \subseteq V(G)$,

$$sg_k(G, v_1, v_2, \ldots, v_t) = \sum_{S\subseteq V(G),|S|=k,\{v_1,v_2,\ldots,v_t\}\subseteq S} \left[\prod_{u\in S} d_G(u)\right] d_G(S).$$

We can get

$$SGut_k(T) = \frac{1}{\binom{k}{t}} \sum_{1\leq i_1<i_2<\ldots<i_t\leq n} sg_k(T, v_{i_1}, v_{i_2}, \ldots, v_{i_t}).$$

Theorem 2.1. *If T be a tree of order $n\,(n\geq 3)$, then*

$$\frac{n-2}{2}Gut(T) \leq SGut_3(T) \leq \frac{\Delta(n-2)}{2}Gut(T).$$

Proof. Let T be a tree of order $n\,(n\geq 3)$. Then for any 3 vertices $\{u,v,w\} \subseteq V(T)$, we have

$$d_T(u,v,w) = \frac{1}{2}\left(d_T(u,v) + d_T(u,w) + d_T(v,w)\right).$$

So we can get

$$\begin{aligned}
&\sum_{i=3}^{n} d_T(v_1)\, d_T(v_2)\, d_T(v_i)\, d_T(v_1, v_2, v_i) \\
&\leq \frac{\Delta}{2} \sum_{i=3}^{n} [d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) + d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i) \\
&+ d_T(v_2)\, d_T(v_i)\, d_T(v_2, v_i)] \\
&= \frac{\Delta}{2} \left[(n-3)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) + \sum_{i=2}^{n} d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i) \right. \\
&\left. + \sum_{i=3}^{n} d_T(v_2)\, d_T(v_i)\, d_T(v_2, v_i) \right] \\
&= \frac{\Delta}{2} \left[(n-4)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) + sg_2(T, v_1) + sg_2(T, v_2) \right].
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=4}^{n} d_T(v_1)\, d_T(v_3)\, d_T(v_i)\, d_T(v_1, v_3, v_i) \\
&\leq \frac{\Delta}{2} \sum_{i=4}^{n} [d_T(v_1)\, d_T(v_3)\, d_T(v_1, v_3) + d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i) \\
&+ d_T(v_3)\, d_T(v_i)\, d_T(v_3, v_i)] \\
&= \frac{\Delta}{2} \left[(n-3)\, d_T(v_1)\, d_T(v_3)\, d_T(v_1, v_3) + \sum_{i=4}^{n} d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i) \right. \\
&\left. + \sum_{i=4}^{n} d_T(v_3)\, d_T(v_i)\, d_T(v_3, v_i) \right] \\
&= \frac{\Delta}{2} [(n-5)\, d_T(v_1)\, d_T(v_3)\, d_T(v_1, v_3) + sg_2(T, v_1) + sg_2(T, v_3) \\
&- d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) - d_T(v_2)\, d_T(v_3)\, d_T(v_2, v_3)] \\
&\leq \frac{\Delta}{2} [(n-4)\, d_T(v_1)\, d_T(v_3)\, d_T(v_1, v_3) + sg_2(T, v_1) + sg_2(T, v_3)] \\
&- d_T(v_1)\, d_T(v_2)\, d_T(v_3)\, d_T(v_1, v_2, v_3).
\end{aligned}$$

$$\vdots$$

$$d_T(v_1)\, d_T(v_{n-1})\, d_T(v_n)\, d_T(v_1, v_{n-1}, v_n)$$
$$\leq \frac{\Delta}{2}[d_T(v_1)\, d_T(v_{n-1})\, d_T(v_1, v_{n-1}) + d_T(v_1)\, d_T(v_n)\, d_T(v_1, v_n)$$
$$+ d_T(v_{n-1})\, d_T(v_n)\, d_T(v_{n-1}, v_n)]$$
$$= \frac{\Delta}{2}[(n-4)\, d_T(v_1)\, d_T(v_{n-1})\, d_T(v_1, v_{n-1}) - (n-5)\, d_T(v_1)\, d_T(v_{n-1})$$
$$d_T(v_1, v_{n-1}) + sg_3(T, v_1) + sg_3(T, v_{n-1}) - \sum_{i=2}^{n-1} d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i)$$
$$- \sum_{i=1}^{n-2} d_T(v_i)\, d_T(v_{n-1})\, d_T(v_i, v_{n-1})]$$
$$\leq [(n-4)\, d_T(v_1)\, d_T(v_{n-1})\, d_T(v_1, v_{n-1}) + sg_2(T, v_1) + sg_2(T, v_{n-1})]$$
$$- d_T(v_1)\, d_T(v_2)\, d_T(v_{n-1})\, d_T(v_1, v_2, v_{n-1}) - d_T(v_1)\, d_T(v_3)\, d_T(v_{n-1})$$
$$d_T(v_1, v_3, v_{n-1}) - \ldots - d_T(v_1)\, d_T(v_{n-2})\, d_T(v_{n-1})\, d_T(v_1, v_{n-2}, v_{n-1}).$$

Summing the above we can get

$$sg_3(T, v_1) = \sum_{j=2}^{n-1} \sum_{i=j+1}^{n} d_T(v_1)\, d_T(v_j)\, d_T(v_i)\, d_T(v_1, v_j, v_i)$$
$$= \sum_{i=3}^{n} d_T(v_1)\, d_T(v_2)\, d_T(v_i)\, d_T(v_1, v_2, v_i)$$
$$+ \sum_{i=4}^{n} d_T(v_1)\, d_T(v_3)\, d_T(v_i)\, d_T(v_1, v_3, v_i)$$
$$+ \ldots + d_T(v_1)\, d_T(v_{n-1})\, d_T(v_n)\, d_T(v_1, v_{n-1}, v_n)$$
$$\leq \sum_{j=2}^{n-1} \frac{\Delta}{2}[(n-4)\, d_T(v_1)\, d_T(v_j)\, d_T(v_1, v_j) + sg_2(v_1) + sg_2(v_j)]$$
$$- \sum_{j=2}^{n-1} \sum_{k=2}^{j-1} d_T(v_1)\, d_T(v_k)\, d_T(v_j)\, d_T(v_1, v_k, v_j)$$
$$\leq (n-3)\, \Delta sg_2(T, v_1) + \Delta Gut(T) - sg_3(T, v_1).$$

So

$$sg_3(T, v_1) \leq \frac{\Delta(n-3)}{2} sg_2(T, v_1) + \frac{\Delta}{2} Gut(T).$$

And we have

$$\begin{aligned} SGut_3(T) &= \frac{1}{3} \sum_{i=1}^{n} sg_3(T, v_i) \\ &\leq \frac{1}{3} \left[\Delta(n-3) Gut(T) + \frac{\Delta n}{2} Gut(T) \right] \\ &= \frac{\Delta(n-2)}{2} Gut(T). \end{aligned}$$

By the same way, we can get:

$$SGut_3(T) \geq \frac{n-2}{2} Gut(T).$$

So

$$\frac{n-2}{2} Gut(T) \leq SGut_3(T) \leq \frac{\Delta(n-2)}{2} Gut(T).$$

The proof is completed. ■

Corollary 2.1. *[15, 27] For a tree T of order n, we have*

$$Gut(T) = 4W(T) - (2n-1)(n-1).$$

Corollary 2.2. *For a tree T of order n, we have*
$2(n-2)W(T) - \frac{(2n-1)(n-1)(n-2)}{2} \leq SGut_3(T) \leq 2\Delta(n-2)W(T) - \frac{\Delta(2n-1)(n-1)(n-2)}{2}$.

Proof. It is immediate from the theorem and corollary above. ■

Theorem 2.2. *[17] For a tree T of order n, we have*

$$SW_3(T) = \frac{(n-2)}{2} W(T).$$

Corollary 2.3. *For a tree T of order n, we have*
$4SW_3(T) - \frac{(2n-1)(n-1)(n-2)}{2} \leq SGut_3(T) \leq 4\Delta SW_3(T) - \frac{\Delta(2n-1)(n-1)(n-2)}{2}$.

Proof. It is immediate from the Theorem 2.2 and corollary 2.2. ■

By proving Theorem 2.1, we can obtain the next theorem:

Theorem 2.3. *If T be a tree of order $n\,(n \geq 4)$, then*

$$SGut_4\,(T) \leq \frac{\Delta^2}{6}\,(n-2)\,(n-3)\,Gut\,(T)\,.$$

Proof. Let T be a tree of order $n\,(n \geq 4)$. Then for any 4 vertices $\{u, v, x, y\} \subseteq V\,(T)$, we have $d_T\,(S) \leq \frac{1}{3}(d_T(u,v) + d_T(u,x) + d_T(u,y) + d_T(v,x) + d_T(v,y) + d_T(x,y))$. So we can get

$$\begin{aligned}
&\sum_{i=4}^{n} d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_3)\,d_T\,(v_i)\,d_T\,(v_1, v_2, v_3, v_i)\\
&\leq \frac{\Delta^2}{3}\sum_{i=4}^{n}[d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_1, v_2) + d_T\,(v_1)\,d_T\,(v_3)\,d_T\,(v_1, v_3)\\
&+ d_T\,(v_2)\,d_T\,(v_3)\,d_T\,(v_2, v_3) + d_T\,(v_1)\,d_T\,(v_i)\,d_T\,(v_1, v_i)\\
&+ d_T\,(v_2)\,d_T\,(v_i)\,d_T\,(v_2, v_i) + d_T\,(v_3)\,d_T\,(v_i)\,d_T\,(v_3, v_i)]\\
&= \frac{\Delta^2}{3}[(n-5)\,d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_1, v_2) + (n-5)\,d_T\,(v_1)\,d_T\,(v_3)\,d_T\,(v_1, v_3)\\
&+ (n-5)\,d_T\,(v_2)\,d_T\,(v_3)\,d_T\,(v_2, v_3) + sg_2\,(T, v_1) + sg_2\,(T, v_2) + sg_2\,(T, v_3)].
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=5}^{n} d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_4)\,d_T\,(v_i)\,d_T\,(v_1, v_2, v_4, v_i)\\
&\leq \frac{\Delta^2}{3}\sum_{i=5}^{n}[d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_1, v_2) + d_T\,(v_1)\,d_T\,(v_4)\,d_T\,(v_1, v_4)\\
&+ d_T\,(v_2)\,d_T\,(v_4)\,d_T\,(v_2, v_4) + d_T\,(v_1)\,d_T\,(v_i)\,d_T\,(v_1, v_i)\\
&+ d_T\,(v_2)\,d_T\,(v_i)\,d_T\,(v_2, v_i) + d_T\,(v_4)\,d_T\,(v_i)\,d_T\,(v_4, v_i)]\\
&= \frac{\Delta^2}{3}[(n-5)\,d_T\,(v_1)\,d_T\,(v_2)\,d_T\,(v_1 v_2) + (n-5)\,d_T\,(v_1)\,d_T\,(v_4)\,d_T\,(v_1, v_4)\\
&+ (n-5)\,d_T\,(v_2)\,d_T\,(v_4)\,d_T\,(v_2 v_4) + sg_2\,(T, v_1) + sg_2\,(T, v_2) + sg_2\,(T, v_4)
\end{aligned}$$

$$- d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) - d_T(v_1)\, d_T(v_3)\, d_T(v_1, v_3)$$
$$- d_T(v_1)\, d_T(v_4)\, d_T(v_1, v_4) - d_T(v_2)\, d_T(v_3)\, d_T(v_2, v_3)$$
$$- d_T(v_2)\, d_T(v_4)\, d_T(v_2, v_4) - d_T(v_3)\, d_T(v_4)\, d_T(v_1, v_4)]$$
$$\le \frac{\Delta^2}{3}[(n-5)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1 v_2) + (n-5)\, d_T(v_1)\, d_T(v_4)\, d_T(v_1, v_4)$$
$$+ (n-5)\, d_T(v_2)\, d_T(v_4)\, d_T(v_2 v_4) + sg_2(T, v_1) + sg_2(T, v_2) + sg_2(T, v_4)]$$
$$- d_T(v_1)\, d_T(v_2)\, d_T(v_4)\, d_T(v_i)\, d_T(v_1, v_2, v_4, v_i)\,.$$

$$\vdots$$

$$d_T(v_1)\, d_T(v_2)\, d_T(v_{n-1})\, d_T(v_n)\, d_T(v_1, v_2, v_{n-1}, v_n)$$
$$\le \frac{\Delta^2}{3}[(n-5)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1 v_2) + (n-5)\, d_T(v_1)\, d_T(v_{n-1})$$
$$d_T(v_1, v_{n-1}) + (n-5)\, d_T(v_2)\, d_T(v_{n-1})\, d_T(v_2 v_{n-1}) + sg_2(T, v_1)$$
$$+ sg_2(T, v_2) + sg_2(T, v_{n-1}) - (n-6)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1 v_2)$$
$$- (n-6)\, d_T(v_1)\, d_T(v_{n-1})\, d_T(v_1, v_{n-1}) - (n-6)\, d_T(v_2)\, d_T(v_{n-1})$$
$$d_T(v_2 v_{n-1}) - \sum_{i=2}^{n-1} d_T(v_1)\, d_T(v_i)\, d_T(v_1, v_i) - \sum_{i=3}^{n-1} d_T(v_2)\, d_T(v_i)\, d_T(v_2, v_i)$$
$$- \sum_{i=1}^{n-2} d_T(v_i)\, d_T(v_{n-1})\, d_T(v_i, v_{n-1})]$$
$$\le \frac{\Delta^2}{3}[(n-5)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1 v_2) + (n-5)\, d_T(v_1)\, d_T(v_{n-1})$$
$$d_T(v_1, v_{n-1}) + (n-5)\, d_T(v_2)\, d_T(v_{n-1})\, d_T(v_2 v_{n-1}) + sg_2(T, v_1) + sg_2(T, v_2)$$
$$+ sg_2(T, v_{n-1})] - \sum_{i=3}^{n-2} d_T(v_1)\, d_T(v_2)\, d_T(v_i)\, d_T(v_{n-1})\, d_T(v_1, v_2, v_i, v_{n-1})\,.$$

Summing the above we can get

$$sg_4(T,v_1,v_2)=\sum_{j=3}^{n-1}\sum_{i=j+1}^{n} d_T(v_1)\,d_T(v_2)\,d_T(v_j)\,d_T(v_i)\,d_T(v_1,v_2,v_j,v_i)$$

$$\leq \frac{\Delta^2}{3}[(n-5)(n-3)\,d_T(v_1)\,d_T(v_2)\,d_T(v_1,v_2)$$

$$+(n-5)\sum_{j=3}^{n-1} d_T(v_1)\,d_T(v_i)\,d_T(v_1,v_i)+(n-4)\,sg_2(T,v_1)$$

$$+(n-5)\sum_{j=3}^{n-1} d_T(v_2)\,d_T(v_i)\,d_T(v_2,v_i)+(n-4)\,sg_2(T,v_2)$$

$$+\sum_{j=3}^{n} sg_2(T,v_i)-sg_2(T,v_n)]$$

$$-\sum_{j=3}^{n-2}\sum_{i=j+1}^{n-1} d_T(v_1)\,d_T(v_2)\,d_T(v_j)\,d_T(v_i)\,d_T(v_1,v_2,v_j,v_i)$$

$$=\frac{\Delta^2}{3}[(n-5)(n-3)\,d_T(v_1)\,d_T(v_2)\,d_T(v_1v_2)$$

$$+(n-5)\sum_{j=3}^{n-1} d_T(v_1)\,d_T(v_i)\,d_T(v_1,v_i)+(n-5)\,sg_2(T,v_1)$$

$$+(n-5)\sum_{j=3}^{n-1} d_T(v_2)\,d_T(v_i)\,d_T(v_2,v_i)+(n-5)\,sg_2(T,v_2)$$

$$+2Gut(T)-sg_2(T,v_n)]$$

$$-\sum_{j=3}^{n-1}\sum_{i=j+1}^{n} d_T(v_1)\,d_T(v_2)\,d_T(v_j)\,d_T(v_i)\,d_T(v_1,v_2,v_j,v_i)$$

$$+\sum_{i=3}^{n-1} d_T(v_1)\,d_T(v_2)\,d_T(v_i)\,d_T(v_n)\,d_T(v_1,v_2,v_i,v_n)$$

$$\leq \frac{\Delta^2}{3}[(n-3)(n-4)\,d_T(v_1)\,d_T(v_2)\,d_T(v_1,v_2)+2Gut(T)$$

$$+(n-4)\sum_{i=3}^{n-1} d_T(v_1)\,d_T(v_i)\,d_T(v_1,v_i)+\sum_{i=3}^{n-1} d_T(v_2)\,d_T(v_i)\,d_T(v_2,v_i)$$

$$+(n-4)\,(sg_2(T,v_1)+sg_2(T,v_2))+(n-v_4)\,d_T(v_1)\,d_T(v_n)\,d_T(v_1,v_n)$$

$$+ (n-4)\, d_T(v_2)\, d_T(v_n)\, d_T(v_2, v_n)] - sg_4(T, v_1, v_2)$$
$$= \frac{\Delta^2}{3}[(n-4)(n-5)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2) + 2Gut(T)$$
$$+ 2(n-4)(sg_2(T, v_1) + sg_2(T, v_2))] - sg_4(T, v_1, v_2).$$

So

$$sg_4(T, v_1, v_2)$$
$$\le \frac{\Delta^2}{6}[(n-4)(n-5)\, d_T(v_1)\, d_T(v_2)\, d_T(v_1, v_2)$$
$$+ 2(n-4)(sg_2(T, v_1) + sg_2(T, v_2)) + 2Gut(T)].$$

And we have

$$SGut_4(T)$$
$$= \frac{1}{6}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} sg_4(T, v_i, v_j)$$
$$\le \frac{1}{6}[\frac{\Delta^2}{6}(n-4)(n-5)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} d_T(v_i)\, d_T(v_j)\, d_T(v_i, v_j)$$
$$+ \frac{\Delta^2}{3}(n-4)(n-1)\sum_{i=1}^{n} sg_2(T, v_i) + \frac{n(n-1)\Delta^2}{6}Gut(T)]$$
$$= \frac{1}{6}[\frac{\Delta^2}{6}(n-4)(n-5)\, Gut(T) + \frac{2\Delta^2}{3}(n-4)(n-1)\, Gut(T)$$
$$+ \frac{n(n-1)\Delta^2}{6}Gut(T)]$$
$$= \frac{\Delta^2}{6}(n-2)(n-3)\, Gut(T).$$

The proof is completed. ■

Corollary 2.4. *For a tree T of order $n\,(n \geq 4)$, we have*

$$SGut_4\,(T) \leq \frac{2\Delta^2}{3}(n-2)(n-3)W\,(T) - \frac{\Delta^2}{6}(n-1)(n-2)(n-3)(2n-1)\,.$$

Proof. It is immediate from the Theorem 2.3 and corollary 2.1.

■

Corollary 2.5. *For a tree T of order $n\,(n \geq 4)$, we have*

$$SGut_4\,(T) \leq \frac{4\Delta^2}{3}\,(n-3)\,SW_3\,(T) - \frac{\Delta^2}{6}\,(n-1)\,(n-2)\,(n-3)\,(2n-1)\,.$$

Proof. It is immediate from the Theorem 2.2 and corollary 2.5. ■

We get a conjecture at last.

Conjecture 1. *If T be a tree of order $n\,(n \geq k \geq 4)$, then*

$$SGut_k\,(T) \leq \frac{2\Delta^{k-2}}{k\,(k-1)}\,(n-k+1)\,(n-k+2)\,Gut\,(T)\,.$$

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