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Bounds for the Atom–Bond Sum–Connectivity Index of Graphs

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Abstract

The *atom-bond sum-connectivity* ($ABSC$) index of a graph G is defined as $ABSC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}$, where d_u and d_v represent the degrees of u and v in G, respectively. In this paper, we give some sharp bounds for the $ABSC$ index in terms of the first Zagreb index, the harmonic index, the sum-connectivity index, the minimum and maximum degrees, the clique number, and the chromatic number. We also find a lower bounds for the $ABSC$ index of trees with given number of vertices and maximum degree.

1 Introduction

In chemical graph theory, vertices correspond to atoms, and edges correspond to bonds in a molecule. Using graphs, one can capture essential features of molecular structures and explore how they affect various chemical properties and behaviors [5, 8]. In graph-theoretical terms, a number represents graph's structure is called a topological index. To measure the degree of branching of saturated hydrocarbons, in 1975 Randić introduced the Randić (R) index [28]. For a graph G the Randić index is defined as:

$$R(G) := \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}, \tag{1}$$

where d_u and d_v represent the degrees of u and v in G, respectively.

Estrada et al. [15] modified the Randić index by taking into consideration not only the degrees of the end-vertices of the edges, but also the degrees of the edges by introducing the atom-bond connectivity (ABC) index of graphs. This parameter was studied in the papers [1,6,9,12–14,17,19,22]. The ABC index of a graph G is defined as:

$$ABC(G) := \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}. \tag{2}$$

A probabilistic interpretation of the ABC index is given in [14]. It indicates that the terms defining this index represent the probability of visiting a nearest neighbor edge from one side or the other of a given edge in a graph.

On the other side, Zhou et al. [31] modified the Randić index by replacing $d_u d_v$ with $d_u + d_v$ in the formula (1), and named this new index the sum-connectivity index. For a graph G, the sum-connectivity (SC) index is defined as:

$$SC(G) := \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}. \tag{3}$$

By amalgamating the core idea of the atom-bond connectivity index and sum-connectivity index, Ali et al. [2, 3] proposed the *atom-bond sum-connectivity* ($ABSC$) index of graphs. The $ABSC$ index of a graph G is defined as:

$$ABSC(G) := \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}}. \tag{4}$$

The formula (4) can also be written as:

$$ABSC(G) := \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}}. \tag{5}$$

Several authors have been attracted towards the $ABSC$ index in the last few years for example [4, 10, 11, 21, 23, 25–27, 30]. All these papers are related to extreme bounds for the $ABSC$ index of graphs.

Let $\mathcal{T}_n$ be the set of trees with n vertices and $\mathcal{T}_{n,\Delta}$ be the set of trees with n vertices and maximum degree Δ. Denote by P_n and S_n the path and star with n vertices, respectively. A *pendent vertex* in a graph is a vertex with degree one. A pendent vertex in a tree is also called a *leave.* A *stem* is a vertex adjacent to a pendent vertex. A *strong stem* is a stem adjacent to at least two pendent vertices. An *end-stem* is a stem whose all neighbors, except at most one, are pendent vertices. A vertex's *progenitor* (parent vertex) in a rooted tree is the vertex connected to it along the path to the root. The term *clique number* of a graph G refers to the number of vertices in a largest clique (the set of vertices with all pairs adjacent) and it is denoted as $\alpha(G)$. The *chromatic number* of a graph G is the least number of colors required to color all of its vertices while ensuring that no two neighboring vertices receive the same color. This number is represented by the symbol $\chi(G)$. For convenience, we will use $I(uv) = \sqrt{1 - \frac{2}{d_u + d_v}}$. Some significant results that will assist us in proving our primary findings are provided below.

Lemma 1. ([2]) *Let $n \geq 4$ and $T \in \mathcal{T}_n$. Then $ABSC(T) \geq ABSC(P_n)$, the equality holds if and only if $T \cong P_n$.*

A well-known inequality, the Diaz-Metcalf inequality, was first published in [24].

Lemma 2. ([24]) *Let a_i and b_i, where $i = 1, 2, \ldots, n$, be real numbers such that $Aa_i \leq b_i \leq Ba_i$ for each $i = 1, 2, \ldots, n$ with $0 < A \leq B$. Then*

$$(A + B) \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^2 + AB \sum_{i=1}^{n} a_i^2,$$

the equality holds if and only if either $b_i = Aa_i$ *or* $b_i = Ba_i$ *for each* $i = 1, 2, \ldots, n$.

Lemma 3. ([20]) *Let* a *and* b *be real numbers such that* $a \geq b \geq 0$. *Then* $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$, *the equality holds if and only if* $a = b$ *or* $b = 0$.

Lemma 4. ([29]) *Let* G *be a connected* K_{q+1}*-free graph of order* n *and having* m *edges. Then*

$$m \leq \left(1 - \frac{1}{q}\right) \cdot \frac{n^2}{2},$$

the equality holds if and only if G *is a complete* q*-partite graph in which all classes are of equal cardinality.*

For a graph G, the harmonic (H) index is defined as [16]:

$$H(G) := \sum_{uv \in E(G)} \frac{2}{d_u + d_v}. \tag{6}$$

Lemma 5. ([7]) *Let* G *be a simple graph with chromatic number* $\chi(G)$ *and harmonic index* $H(G)$. *Then* $\chi(G) \leq 2H(G)$, *the equality holds if and only if* G *is a complete graph possibly with some additional isolated vertices.*

The remainder of this paper is organized as follows: In Section 2, we obtain certain lower bounds for the $ABSC$ index of graphs, specifically the least $ABSC$ index of trees with a given maximum degree. In Section 3, we give some upper bounds for the $ABSC$ index of graphs. In Section 4, we conclude our paper.

2 Lower bounds for the $ABSC$ index of graphs

In this section, we will find lower bounds for the atom-bond sum-connectivity index of graphs.

Theorem 1. *Let G be a connected graph with m edges, maximum degree Δ and minimum degree δ. Then*

$$m\sqrt{\frac{\delta-1}{\Delta}} \le ABSC(G) \le m\sqrt{\frac{\Delta-1}{\delta}},$$

the equality holds if and only if G is a regular graph.

Proof. For $uv \in E(G)$, we have

$$\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} = \sqrt{\frac{1}{d_u+d_v}} \cdot \sqrt{d_u+d_v-2}.$$

It follows from

$$\sqrt{2\left(\delta-1\right)} \le \sqrt{d_u+d_v-2} \le \sqrt{2\left(\Delta-1\right)}$$

and

$$\sqrt{\frac{1}{2\Delta}} \le \sqrt{\frac{1}{d_u+d_v}} \le \sqrt{\frac{1}{2\delta}}$$

Thus,

$$m\sqrt{\frac{\delta-1}{\Delta}} \le ABSC(G) \le m\sqrt{\frac{\Delta-1}{\delta}}.$$

Since $\delta = \Delta$ only holds when G is a regular graph and vice versa, the equalities only hold when G is a regular graph. ■

The first Zagreb (M_1) index [18] of a graph G is defined as:

$$M_1(G) := \sum_{uv\in E(G)} \left(d_u+d_v\right). \tag{7}$$

Theorem 2. *Let G be a connected graph with m edges, maximum degree Δ and minimum degree $\delta \ge 2$. Then*

$$ABSC(G) \ge \frac{M_1(G) + 2\sqrt{\delta\Delta(\delta-1)(\Delta-1)}H(G) - 2m}{2\left(\sqrt{\delta(\delta-1)} + \sqrt{\Delta(\Delta-1)}\right)}, \tag{8}$$

the equality holds if and only if G is a regular graph.

Proof. We know that

$$\sqrt{2(\delta-1)} \leq \sqrt{d_u+d_v-2} \leq \sqrt{2(\Delta-1)},$$

$$\sqrt{\frac{1}{d_u+d_v}}\sqrt{2\delta}\sqrt{2(\delta-1)} \leq \sqrt{d_u+d_v-2} \leq \sqrt{\frac{1}{d_u+d_v}}\sqrt{2\Delta}\sqrt{2(\Delta-1)},$$

and

$$\sqrt{\frac{1}{d_u+d_v}}2\sqrt{\delta(\delta-1)} \leq \sqrt{d_u+d_v-2} \leq \sqrt{\frac{1}{d_u+d_v}}2\sqrt{\Delta(\Delta-1)}.$$

Setting $2\sqrt{\delta(\delta-1)}$ as A, $2\sqrt{\Delta(\Delta-1)}$ as B , $\sqrt{\frac{1}{d_u+d_v}}$ as a_i and $\sqrt{d_u+d_v-2}$ as b_i, by Lemma 2 we have

$$\begin{aligned}
&ABSC(G) = \\
&\sum_{uv\in E(G)}\sqrt{\frac{1}{d_u+d_v}}\cdot\sqrt{d_u+d_v-2} \\
= \quad & \frac{\left[2\sqrt{\delta(\delta-1)}+2\sqrt{\Delta(\Delta-1)}\right]}{2\sqrt{\delta(\delta-1)}+2\sqrt{\Delta(\Delta-1)}}\sum_{uv\in E(G)}\left(\sqrt{\frac{1}{d_u+d_v}}\cdot\sqrt{d_u+d_v-2}\right) \\
\geq \quad & \frac{\left[\sum_{uv\in E(G)}(d_u+d_v-2)+\left(4\sqrt{\delta(\delta-1)}\sqrt{\Delta(\Delta-1)}\right)\left(\sum_{uv\in E(G)}\frac{1}{d_u+d_v}\right)\right]}{2\sqrt{\delta(\delta-1)}+2\sqrt{\Delta(\Delta-1)}} \\
= \quad & \frac{M_1(G)+2\sqrt{\delta\Delta(\delta-1)(\Delta-1)}H(G)-2m}{2\left(\sqrt{\delta(\delta-1)}+\sqrt{\Delta(\Delta-1)}\right)}.
\end{aligned}$$

This completes the proof. ■

Theorem 3. *Let G be a graph with n vertices and m edges. Then*

$$ABSC(G) \geq m - \sqrt{2}\ SC(G), \tag{9}$$

the equality holds if and only if $G \cong \frac{n}{2}K_2$ (n is even).

Proof. By Lemma 3, we have

$$ABSC(G) \quad = \quad \sum_{uv\in E(G)}\sqrt{\frac{1}{d_u+d_v}}\cdot\sqrt{d_u+d_v-2}$$

$$\begin{aligned}
&\geq \sum_{uv\in E(G)} \sqrt{\frac{1}{d_u+d_v}} \cdot \left(\sqrt{d_u+d_v}-\sqrt{2}\right)\\
&= \sum_{uv\in E(G)} - \sum_{uv\in E(G)} \sqrt{\frac{2}{d_u+d_v}}\\
&= m-\sqrt{2}SC(G),
\end{aligned}$$

where the equality holds if and only if $G \cong \frac{n}{2}K_2$ (n is even). This completes the proof. ■

Corollary. *Let G be a graph with n vertices, m edges and minimum degree δ. Then*

$$ABSC(G) \geq m\left(1-\frac{1}{\sqrt{\delta}}\right),$$

the equality holds if and only if $G \cong \frac{n}{2}K_2$(n is even).

Proof. Since

$$SC(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u+d_v}} \leq \frac{m}{\sqrt{2\delta}}, \tag{10}$$

from Theorem 3, we get the required result. Moreover the equality holds in (10) if and only if $G \cong \frac{n}{2}K_2$(n is even). ■

Now we will find the minimum value of the $ABSC$ index of $T \in \mathcal{T}_{n,\Delta}$. We apply some transformations to streamline the problem of minimizing the $ABSC$ index of $T \in \mathcal{T}_{n,\Delta}$. From this point on, we assume that T is a rooted tree with root r and that r is a vertex with maximum degree Δ and $N(r) = \{r_1, r_2, \ldots, r_\Delta\}$.

Theorem 4. *Let $n \geq 5$ and $T \in \mathcal{T}_{n,\Delta}$. Then we have*

$$ABSC(T) \geq \begin{cases} \Delta\left(\sqrt{\frac{\Delta}{\Delta+2}}+\sqrt{\frac{1}{3}}\right)+\frac{\sqrt{2}}{2}\left(n-2\Delta-1\right), & \Delta \leq \frac{n-1}{2};\\[2ex] (n-\Delta-1)\left(\sqrt{\frac{\Delta}{\Delta+1}}+\sqrt{\frac{1}{3}}\right)+(2\Delta-n+1)\sqrt{\frac{\Delta-1}{\Delta+1}}, & \Delta > \frac{n-1}{2}. \end{cases}$$

Before we prove Theorem 4, we need to prove a few supporting results.

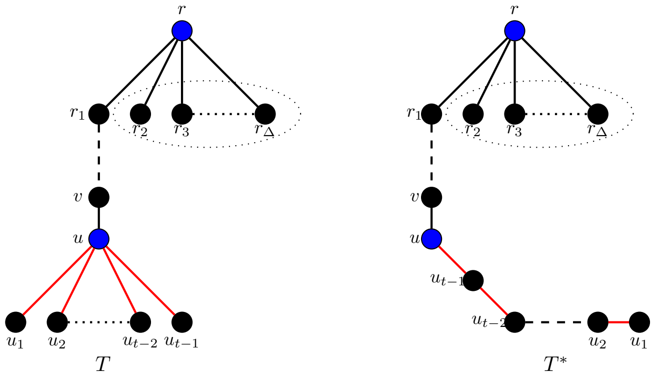

Figure 1. The trees T and T^* in the proof of Lemma 6.

Lemma 6. *Let $T \in \mathcal{T}_{n,\Delta}$ be a rooted tree and r be the root vertex of T with $d_T(r) = \Delta$. If there is an end-stem with degree at least three other than the root vertex r in T, then there exists a tree T^* with $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$, such that $ABSC(T) > ABSC(T^*)$.*

Proof. Let $u \neq r$ be an end-stem of T with $d_T(u) = t \geq 3$. Suppose that v is the parent vertex of u and $N(u) = \{v, u_1, u_2, \ldots, u_{t-1}\}$. Let $\Omega = \{uv, uu_1, uu_2, \ldots, uu_{t-1}\}$ and $T^* = T - \{uu_1, uu_2, \ldots, uu_{t-2}\} + \{u_1u_2, u_2u_3, \ldots, u_{t-2}u_{t-1}\}$. Then $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$, thus we have

$$\begin{aligned} ABSC(T) &= \sum_{uv\notin\Omega} I(uv) + \sum_{uv\in\Omega} I(uv) \\ &= \sum_{uv\notin\Omega} I(uv) + \sqrt{1 - \frac{2}{t + d_T(v)}} + (t-2)\sqrt{1 - \frac{2}{t+1}} \\ &+ \sqrt{1 - \frac{2}{t+1}} \end{aligned}$$

and

$$ABSC(T^*) = \sum_{uv\notin\Omega} I(uv) + \sqrt{1 - \frac{2}{2 + d_T(v)}} + (t-2)\sqrt{1 - \frac{2}{2+2}}$$

$$+ \quad \sqrt{1 - \frac{2}{2+1}}.$$

Now using $d_T(u) = t \geq 3$, we get

$$\begin{aligned} ABSC(T) - ABSC(T^*) \quad &\geq \quad \sqrt{1 - \frac{2}{t + d_T(v)}} - \sqrt{1 - \frac{2}{2 + d_T(v)}} \\ &+ \quad \sqrt{1 - \frac{2}{t+1}} - \sqrt{1 - \frac{2}{2+1}} > 0. \end{aligned}$$

■

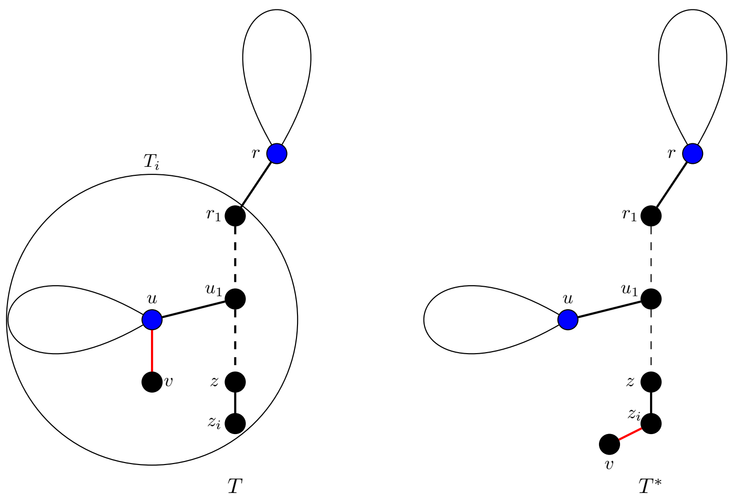

Figure 2. The trees T and T^* in the proof of Lemma 7.

Lemma 7. *Let $T \in \mathcal{T}_{n,\Delta}$ be a rooted tree and r is the root vertex of T with $d_T(r) = \Delta$. If there is a stem with degree at least three other than the root vertex r in T, then there exists a tree T^* with $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$, such that $ABSC(T) > ABSC(T^*)$.*

Proof. Let $u \neq r$ be a stem of T with $d_T(u) = t \geq 3$ and $N(u) =$

$\{v, u_1, u_2, \ldots, u_{t-1}\}$ where vertex u_1 is the parent of u and v is a pendent vertex in T. If u is an end-stem, then by Lemma 6, the conclusion holds. Thus we can assume that u is a stem, but not an end-stem. Let $rr_i \in E(T)$ and T_i be the component of $T - rr_i$ containing r_i. Let $z_i \in V(T_i)$ be one pendent vertex with maximum distance from r_i in T_i and vertex z is a parent of z_i. If $d_T(z) \geq 3$, then by Lemma 6, we can obtain a contradiction. Thus $d_T(z) = 2$. Let $\Omega = \{zz_i, uv, uu_1, uu_2, \ldots, uu_{t-1}\}$ and $T^* = T - uv + vz_i$. Then $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$, thus we have

$$\begin{aligned} ABSC(T) &= \sum_{uv \notin \Omega} I(uv) + \sum_{uv \in \Omega} I(uv) \\ &= \sum_{uv \notin \Omega} I(uv) + \sqrt{1 - \frac{2}{1+t}} + \sqrt{1 - \frac{2}{1+2}} \\ &+ \sum_{i=1}^{t-1} \sqrt{1 - \frac{2}{d_T(u_i) + t}} \end{aligned}$$

and

$$\begin{aligned} ABSC(T^*) &= \sum_{uv \notin \Omega} I(uv) + \sqrt{1 - \frac{2}{2+2}} + \sqrt{1 - \frac{2}{1+2}} \\ &+ \sum_{i=1}^{t-1} \sqrt{1 - \frac{2}{d_T(u_i) + t - 1}}. \end{aligned}$$

Now using $d_T(u) = t \geq 3$, we get

$$\begin{aligned} ABSC(T) - ABSC(T^*) &= \sqrt{1 - \frac{2}{1+t}} - \sqrt{1 - \frac{2}{2+2}} \\ &+ \sum_{i=1}^{t-1} \sqrt{1 - \frac{2}{d_T(u_i) + t}} \\ &- \sum_{i=1}^{t-1} \sqrt{1 - \frac{2}{d_T(u_i) + t - 1}} > 0. \end{aligned}$$

■

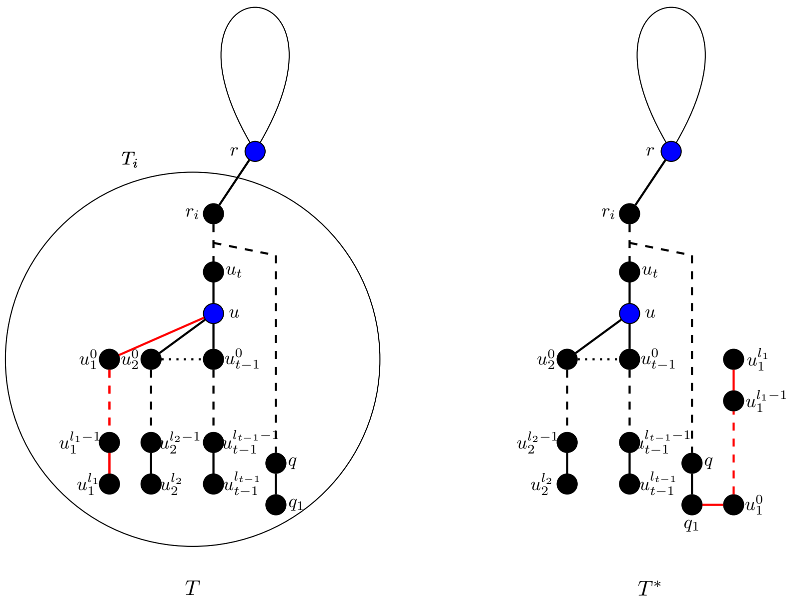

Figure 3. The trees T and T^* in the proof of Lemma 8.

Lemma 8. *Let $T \in \mathcal{T}_{n,\Delta}$ be a rooted tree and r be the root vertex of T with $d_T(r) = \Delta$. If there is a vertex with degree at least three other than the root vertex r in T, then there exists a tree T^* with $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$, such that $ABSC(T) > ABSC(T^*)$.*

Proof. Let $u \neq r$ be a vertex with $d_T(u) = t \geq 3$ such that $d_T(u, r)$ is as large as possible. Assume $N(u) = \left\{u_1^0, u_2^0, \ldots, u_{t-1}^0, u_t\right\}$, and u_t is the parent of vertex u. Let $rr_i \in E(T)$ and T_i be the component of $T - rr_i$ containing r_i. Suppose that the path $u_i^0 u_i^1 \ldots u_i^{l_i}$ is the longest path in T_i starting from u_i^0 where $i = 1, 2, l \ldots, t-1$. Further suppose that $q_i \in V(T_i)$ is a pendent vertex such that $d_{T_i}(r_i, q_i)$ is maximum. Let q be the parent of vertex q_i. Now we can suppose that $q_i \notin \left\{u_1^{l_1}, \ldots, u_{t-2}^{l_{t-2}}\right\}$. By Lemmas 6 and 7 and that $d_T(u, r)$ is as large as possible, $d_T(q) = 2$ and all descendants of u except pendent vertices, have degree two. We distinguish three cases:

Case 1. $t = 3$.

Let $T^* = T - uu_1^0 + q_i u_1^0$. Then we have $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$. Let $\Omega = \{q_i q, uu_1^0, uu_2^0, uu_3\}$. Then we have

$$\begin{aligned} ABSC(T) &= \sum_{uv\notin\Omega} I(uv) + \sum_{uv\in\Omega} I(uv) \\ &= \sum_{uv\notin\Omega} I(uv) + \sqrt{1-\frac{2}{3+2}} + \sqrt{1-\frac{2}{3+2}} \\ &+ \sqrt{1-\frac{2}{3+d_T(u_3)}} + \sqrt{1-\frac{2}{2+1}} \end{aligned}$$

and

$$\begin{aligned} ABSC(T^*) &= \sum_{uv\notin\Omega} I(uv) + \sqrt{1-\frac{2}{2+2}} + \sqrt{1-\frac{2}{2+2}} \\ &+ \sqrt{1-\frac{2}{2+d_T(u_3)}} + \sqrt{1-\frac{2}{2+2}}. \end{aligned}$$

Thus,

$$ABSC(T) - ABSC(T^*) > 2\sqrt{1-\frac{2}{3+2}} + \sqrt{1-\frac{2}{2+1}} - 3\sqrt{1-\frac{2}{2+2}} > 0.$$

Case 2. $t = 4$.

Let $T^* = T - uu_1^0 + q_i u_1^0$. Then we have $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$. Let $\Omega = \{q_i q, uu_1^0, uu_2^0, uu_3^0, uu_4\}$. Then we have

$$\begin{aligned} ABSC(T) &= \sum_{uv\notin\Omega} I(uv) + \sum_{uv\in\Omega} I(uv) \\ &= \sum_{uv\notin\Omega} I(uv) + 3\sqrt{1-\frac{2}{4+2}} + \sqrt{1-\frac{2}{4+d_T(u_4)}} \\ &+ \sqrt{1-\frac{2}{2+1}} \end{aligned}$$

and

$$ABSC(T^*) = \sum_{uv \notin \Omega} I(uv) + 2\sqrt{1 - \frac{2}{3+2}} + 2\sqrt{1 - \frac{2}{3+2}} + \sqrt{1 - \frac{2}{3 + d_T(u_4)}}.$$

Thus,

$$\begin{aligned} ABSC(T) - ABSC(T^*) &> 3\sqrt{1 - \frac{2}{4+2}} - 2\sqrt{1 - \frac{2}{3+2}} + \sqrt{1 - \frac{2}{2+1}} \\ &- 2\sqrt{1 - \frac{2}{2+2}} > 0. \end{aligned}$$

Case 3. $t \geq 5$.

For $t \geq 5$, we consider further two cases, $q_i \neq u_{t-1}^{l_{t-1}}$ and $q_i = u_{t-1}^{l_{t-1}}$. First we consider $q_i \neq u_{t-1}^{t-1}$. Let $\Omega = \left\{uu_i^0, u_i^{l_i} u_i^{l_i - 1} \mid 1 \leq i \leq t-1\right\} \cup \{q_i q, uu_t\}$. Let $T^* = T - \left\{uu_1^0, uu_2^0, l \ldots, uu_{t-1}^0\right\} + \left\{u_1^0 q_i, u_2^0 u_1^{l_1}, u_3^0 u_2^{l_2}, l \ldots, u_{t-1}^0 u_{t-2}^{l_{t-2}}\right\}$. Then we have $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$. Thus we have

$$\begin{aligned} ABSC(T) &= \sum_{uv \notin \Omega} I(uv) + \sum_{uv \in \Omega} I(uv) \\ &= \sum_{uv \notin \Omega} I(uv) + (t-1)\sqrt{1 - \frac{2}{2+1}} + \sqrt{1 - \frac{2}{2+1}} \\ &+ (t-1)\sqrt{1 - \frac{2}{t+2}} + \sqrt{1 - \frac{2}{t + d_T(u_t)}} \end{aligned}$$

and

$$\begin{aligned} ABSC(T^*) &= \sum_{uv \notin \Omega} I(uv) + (t-1)\sqrt{1 - \frac{2}{2+2}} + \sqrt{1 - \frac{2}{2+1}} \\ &+ (t-1)\sqrt{1 - \frac{2}{2+2}} + \sqrt{1 - \frac{2}{1 + d_T(u_t)}}. \end{aligned}$$

Thus,

$$ABSC(T) - ABSC(T^*) > (t-1)\left[\sqrt{\frac{1}{3}} + \sqrt{1 - \frac{2}{t+2}} - \sqrt{2}\right] > 0, \text{ for } t \geq 5.$$

Now suppose that $q_i = u_{t-1}^{l_{t-1}}$. Let $T^* = T - \{uu_1^0, uu_2^0, l \ldots, uu_{t-2}^0\} +$

$\{u_1^0 q_i, u_1^{t_1} u_2^0, u_2^{l_2} u_3^0, l \ldots, u_{t-3}^{l_{t-3}} u_{t-2}^0\}$. Then $|V(T^*)| = n$ and $\Delta(T^*) = \Delta$. Let $\Omega = \left\{ uu_i^0, u_i^{l_i} u_i^{l_i - 1} \mid 1 \leq i \leq t-1 \right\} \cup \{uu_t\}$. Then we have

$$\begin{aligned}
ABSC(T) &= \sum_{uv \notin \Omega} I(uv) + \sum_{uv \in \Omega} I(uv) \\
&= \sum_{uv \notin \Omega} I(uv) + (t-1)\sqrt{1 - \frac{2}{1+2}} + (t-1)\sqrt{1 - \frac{2}{t+2}} \\
&+ \sqrt{1 - \frac{2}{t + d_T(u_t)}}
\end{aligned}$$

and

$$\begin{aligned}
ABSC(T^*) &= \sum_{uv \notin \Omega} I(uv) + (t-2)\sqrt{1 - \frac{2}{2+2}} + \sqrt{1 - \frac{2}{1+2}} \\
&+ (t-1)\sqrt{1 - \frac{2}{2+2}} + \sqrt{1 - \frac{2}{2 + d_T(u_t)}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
ABSC(T) - ABSC(T^*) &> (t-2)\left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}}\right) \\
&+ (t-1)\left(\sqrt{1 - \frac{2}{t+2}} - \sqrt{\frac{1}{2}}\right) > 0, \text{ for } t \geq 5.
\end{aligned}$$

This completes the proof. ∎

A *spider* is a tree with only one vertex whose degree is greater than two and that vertex is called the central vertex of the spider. A leg in a spider is a path from the vertex with maximum degree to one pendent vertex.

Lemma 9. *Let T be a spider with n vertices and $k \geq 3$ legs. If there is a leg of length larger than two and a leg of length one. Then there exists a spider T^* with k legs and n vertices such that $ABSC(T) > ABSC(T^*)$.*

Proof. Let T be a spider and r be the central vertex and root of T. Also, we let $N_T(r) = \{r_1, r_2, \ldots, r_k\}$. Without loosing the generality, we can suppose that $r_k y_1 y_2 \ldots y_l$ is the most longest leg of spider T. Consider a set $\Omega = \{r_1 r, y_l y_{l-1}, y_{l-2} y_{l-1}\}$. Let $T^* = T - y_l y_{l-1} + r_1 y_l$. Then we have

$$\begin{aligned} ABSC(T) &= \sum_{uv \notin \Omega} I(uv) + \sum_{uv \in \Omega} I(uv) \\ &= \sum_{uv \notin \Omega} I(uv) + \sqrt{1 - \frac{2}{k+1}} + \sqrt{1 - \frac{2}{1+2}} + \sqrt{1 - \frac{2}{2+2}} \end{aligned}$$

and

$$ABSC(T^*) = \sum_{uv \notin \Omega} I(uv) + \sqrt{1 - \frac{2}{k+2}} + \sqrt{1 - \frac{2}{1+2}} + \sqrt{1 - \frac{2}{1+2}}.$$

Thus,

$$\begin{aligned} ABSC(T) - ABSC(T^*) &= \sqrt{1 - \frac{2}{k+1}} - \sqrt{1 - \frac{2}{k+2}} + \sqrt{1 - \frac{2}{2+2}} \\ &- \sqrt{1 - \frac{2}{1+2}} > 0, \text{ for } k \geq 3. \end{aligned}$$

This completes the proof. ■

Now we can prove Theorem 4 by using the above lemmas.

Proof of Theorem 4. Let $T^* \in \mathcal{T}_{n,\Delta}$ $(n \geq 5)$ and $ABSC(T^*) = \min\{ABSC(T) \mid T \in \mathcal{T}_{n,\Delta}, n \geq 5\}$. Let r be the root of T^* with $\Delta T^* = \Delta$. If $\Delta = 2$, then $T \cong P_n$. By Lemma 1, the conclusion holds. We may suppose $\Delta \geq 3$. By our choice of T^* as minimum of $ABSC(T)$, we conclude from Lemmas 6, 7 and 8 that T^* is a spider having central vertex r. Further we have two cases now:

Case 1. All legs of T^* have length at least two.

It is obvious $\Delta \leq \frac{n-1}{2}$, thus

$$\begin{aligned} ABSC(T^*) &= \Delta\sqrt{1 - \frac{2}{\Delta+2}} + \Delta\sqrt{1 - \frac{2}{1+2}} \\ &+ (n - 2\Delta - 1)\sqrt{1 - \frac{2}{2+2}} \end{aligned}$$

$$= \Delta\left(\sqrt{\frac{\Delta}{\Delta+2}}+\sqrt{\frac{1}{3}}\right)+\frac{\sqrt{2}}{2}(n-2\Delta-1).$$

Case 2. All legs of T^* have length at most 2.

From Case 1, we can suppose that T^* has at least one leg of length 1. If T^* is a star graph then the result will be obvious, so suppose that T^* is not a star graph. Then the number of leaves in $N_{T^*}(r)$ are $2\Delta+1-n$ and we have:

$$\begin{aligned} ABSC(T) &= (2\Delta-n+1)\sqrt{1-\frac{2}{\Delta+1}}+(n-\Delta-1)\sqrt{1-\frac{2}{\Delta+2}} \\ &+ (n-\Delta-1)\sqrt{1-\frac{2}{1+2}} \\ &= (n-\Delta-1)\left(\sqrt{\frac{\Delta}{\Delta+1}}+\sqrt{\frac{1}{3}}\right)+(2\Delta-n+1)\sqrt{\frac{\Delta-1}{\Delta+1}}, \end{aligned}$$

which completes our arguments. ■

3 Upper bounds for the $ABSC$ index of graphs

In this section, we will consider some upper bounds for the atom-bond sum-connectivity index of graphs.

Theorem 5. *Let G be a connected graph having n vertices, m edges, maximum degree Δ, minimum degree δ, and clique number α. Then*

$$ABSC(G)\le\frac{n^2(\alpha-1)}{2\alpha}\sqrt{\frac{\Delta-1}{\delta}}, \qquad (11)$$

the equality holds if and only if G is a complete α-partite graph in which all classes are of equal cardinality.

Proof. We know that

$$ABSC(G)=\sum_{uv\in E(G)}\sqrt{1-\frac{1}{d_u+d_v}}\le m\sqrt{\frac{\Delta-1}{\delta}},$$

since G has a clique number α, so G is $K_{\alpha+1}$-free graph. By Lemma 4, we get

$$m \leq \frac{n^2 (\alpha - 1)}{2\alpha},$$

so from the above arguments we get our required result. Suppose that the equality holds in (11). We can conclude that G is a complete α-partite graph in which all the classes are of equal cardinality.

Conversely, if G is a complete α-partite graph in which all the classes are of equal cardinality, then

$$ABSC(G) = \frac{n^2 (\alpha - 1)}{2\alpha} \sqrt{\frac{\Delta - 1}{\delta}},$$

which completes our arguments. ■

Lemma 10. ([2]) *Let G be a graph with m egdes. Then*

$$ABSC(G) \leq \sqrt{m (m - H(G))}, \tag{12}$$

the equality holds if and only if either $m = 0$ or there is a fixed number k' such that $d_u + d_v = k'$ for every edge $uv \in E(G)$.

Theorem 6. *Let G be a simple graph with m edges and chromatic number $\chi(G)$. Then*

$$ABSC(G) \leq m\sqrt{1 - \frac{\chi(G)}{2m}},$$

the equality holds if and only if G is a complete graph possibly with the some additional isolated vertices.

Proof. By Lemmas 5 and 10, we have

$$\begin{aligned} \frac{\chi(G)}{2} &\leq \frac{m^2 - (ABSC(G))^2}{m}, \\ (ABSC(G))^2 &\leq m^2 - \frac{m\chi(G)}{2}, \\ ABSC(G) &\leq m\sqrt{1 - \frac{\chi(G)}{2m}}. \end{aligned}$$

This completes the proof. ■

4 Conclusions

In this paper, we first presented some lower bounds for the $ABSC$ index of graphs especially for the trees with given maximum degree. Then we gave some upper bounds for the $ABSC$ index of graphs. We found sharp bounds in terms of different graph parameters such as the first Zagreb index, the harmonic index, the sum-connectivity index, the minimum and maximum degrees, the clique number, and the chromatic number. The problem of finding the maximum $ABSC$ index of trees with given maximum degree and characterizing the extremal trees remain to be resolved in the future.

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