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Maximum Atom-Bond Sum-Connectivity Index in Unicyclic Graphs of Fixed Girth

Palaniyappan Nithya[a], Suresh Elumalai[a,*], Selvaraj Balachandran[b], Hechao Liu[c]

[a] *Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu-603 203, India*

[b] *Department of Mathematics, School of Arts, Sciences and Humanities, SASTRA Deemed University, Thanjavur, India*

[c] *Huangshi Key Laboratory of Metaverse and Virtual Simulation, School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China*

nithyapalaniyappan88@gmail.com, sureshkako@gmail.com, balamaths1977@gmail.com, hechaoliu@yeah.net

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Abstract

The ABS (atom-bond sum-connectivity) index of a graph G is given by the formula:

$$ABS(G) = \sum_{xy \in E(G)} \sqrt{\frac{d_x + d_y - 2}{d_x + d_y}},$$

where d_x denotes the degree of vertex x in the graph G. The primary objective of this research paper is to identify the maximum, and second-maximum ABS index among all unicyclic graphs with a fixed girth. Additionally, we provide a characterization of the specific graphs that attain these extreme ABS values.

*Corresponding author.

1 Introduction

Let G be a connected, undirected, simple graph with precisely n vertices. The neighborhood set and degree of a vertex x in G are $N(x)$ and $d(x)$, respectively. A pendent vertex is a vertex of degree 1. An edge incident with a pendent vertex is called a pendent edge. Let $G - x$ represent the resulting graph from G after removing a vertex x and all of its incident edges. Similarly, $G - xy$ represents the resulting graph from G after removing the edge $xy \in E(G)$. On the other hand, $G + xy$ represents the resulting graph from G by adding an edge xy between two non-adjacent vertices x and y of G. Additionally, the renaming of entity B as A is indicated by the notation $A := B$.

Chemical graph theory [27], an essential field within Mathematical Chemistry, is dedicated to representing and examining chemical structures through mathematical models. Within this field, topological indices play a vital role in establishing links between molecular structures and the properties crucial for advancing computer-aided drug design [11, 17]. Among these indices, the atom-bond sum-connectivity (ABS) index stands out, expressing connectivity patterns in molecular graphs as real-number values. The ABS index has garnered considerable attention in literature owing to its capability to provide valuable insights into the structure-property relationship of chemical compounds. Through the quantification of molecular connectivity features, the ABS index serves as a mathematical tool for comprehending and predicting the chemical and physical properties of molecules.

Various topological indices based on degrees have been introduced, as extensively discussed in [13, 16]. Among these, the atom-bond sum-connectivity (ABS) index represents a novel approach, formulated by incorporating fundamental concepts from the atom-bond connectivity (ABC) index and sum-connectivity (SC) index. This modified index first introduced by Furtula et al. [13] in 2013, represents a novel perspective on quantifying molecular structure. Ali et al. [5] have also defined the ABS index for a simple connected graph G, expressed as follows:

$$ABS(G) = \sum_{xy \in E(G)} \sqrt{\frac{d_x + d_y - 2}{d_x + d_y}}.$$

They explored the properties of trees with the extremal ABS index values compared to all other chemical trees of a specified order $n \geq 11$. Expanding on this research, Ali et al. [6], extended the application of ABS to unicyclic graphs, determining both the maximum and minimum ABS index values for unicyclic graphs of the same order and size. In a related study, Gowtham and Gutman [15] established several inequalities that describe the relationship between the SC and ABS indices. Additionally, Alraqad et al. [9, 10] characterized extremal trees and certain classes of graphs with a fixed order, number of pendent vertices, chromatic number, and independence number. Furthermore, [18, 25], provided a comprehensive solution for the largest ABS index observed in trees with a fixed number of leaves and matching number and diameter.

In our earlier investigation [24], we identified the first four minimum ABS indices for unicyclic graphs with a specified girth, highlighting the graphs that exhibit these extremal values. For a more comprehensive exploration of the ABS index and its implications in the realm of chemistry, interested readers can refer to the additional references [1–3, 7, 8, 22, 24].

In [14], Ge et al. derived a relationship between the $ABS(L(G))$ and $ABS(G)$. They also identified the minimum ABS index among all line graphs of unicyclic graphs with a fixed number of vertices. Hussain et al. [19] established the sharp bounds for ABS index in terms of graph invariants. The study also derived a lower bound for the ABS index specifically for trees with the maximum degree. Additionally, in a recent publications [20, 30], authors determined both upper and lower bounds for the ABS index of graphs by using the fixed parameters, which included the chromatic number, clique number, connectivity, matching number and domination number. Very recently, in [23], we determined the minimum values of ABS index among all unicyclic graphs of a given order with maximum degree, also we characterized the extremal graphs attaining these minimum values.

Motivated by [24], in this study, we obtain the maximum, and second-maximum ABS index of unicyclic graphs that consist of at least 5 vertices and girth $\mathcal{R}$, where $(3 \leq \mathcal{R} \leq n)$. Additionally, we provide a characterization of the extremal graphs that correspond to these values.

Consider $\mathbb{U}_n$ to be collection of all unicyclic graphs with $n \geq 5$ vertices. In a similar manner, the collection of all unicyclic graphs that have at least 5 vertices and girth $\mathcal{R}$, where $3 \leq \mathcal{R} \leq n$ is known as $\mathbb{U}_{n,\mathcal{R}}$. It is observed that, $\mathbb{U}_n = \bigcup_{\mathcal{R}=3}^{n} \mathbb{U}_{n,\mathcal{R}}$. The cycle comprising n vertices is denoted by C_n and unique unicyclic graph comprising n vertices and girth $n-1$ is denoted by $U^1_{n,n-1}$. It is evident that $\mathbb{U}_{n,n} = \{C_n\}$ and $\mathbb{U}_{n,n-1} = \left\{U^1_{n,n-1}\right\}$. In the subsequent arguments, we will specifically focus on the cases where $(3 \leq \mathcal{R} \leq n-2)$.

2 The maximum and second-maximum ABS index among $\mathbb{U}_{n,\mathcal{R}}$

In the following part, we will discuss the maximum and second-maximum ABS index for graphs that belong to the set $\mathbb{U}_{n,\mathcal{R}}$, as well as the extremal graphs that corresponds to those graphs. Additionally, we determine the extremal graph that has the second-maximum ABS index for graphs $\mathbb{U}_n$. First, we will examine two lemmas.

Lemma 1. (i) Define $f(p) = \sqrt{\frac{2}{b+p}} - \sqrt{\frac{2}{b+p-1}}$. Consider a integer $b \geq 3$ holds true for $p \geq 1$, the function f is decreasing.

(ii) Define $g(p) = 2\sqrt{\frac{p}{p+2}} + (p-4)\sqrt{\frac{p-1}{p+1}} - (p-3)\sqrt{\frac{p-2}{p}}$. For $p \geq 3$, the function g is increasing.

(iii) Define $h(p) = \sqrt{\frac{p}{p+2}} + (p-3)\sqrt{\frac{p-1}{p+1}} - (p-3)\sqrt{\frac{p-2}{p}}$. For $p \geq 3$, the function h is increasing.

(iv) Define $m(p) = -\sqrt{\frac{p-1}{p+1}} - 2\sqrt{\frac{1}{p(p+2)^3}} - (p-3)\sqrt{\frac{1}{(p-1)(p+1)^3}} +$

$\left(\frac{p+1}{p}\right)\sqrt{\frac{p-2}{p}}$. For $p \geq 3$, the function m is increasing.

(v) Define $o(p) = -2\sqrt{\frac{1}{(p-1)(p+1)^3}} - (p-4)\sqrt{\frac{1}{(p-2)p^3}} + \sqrt{\frac{p-2}{p^3}}$. For $p \geq 3$, the function o is decreasing.

Proof. (i) consider the derivative function

$$f'(p) = \sqrt{\frac{1}{(b+p)^3(b+p-2)}} - \sqrt{\frac{1}{(b+p-1)^3(b+p-3)}}$$

is negative-valued under the specified constraints.

(ii) consider the derivative function

$$g'(p)=2\sqrt{\frac{1}{p(p+2)^3}}+(p^2+p-5)\sqrt{\frac{1}{(p+1)^3(p-1)}}-(p^2-p-3)\sqrt{\frac{1}{p^3(p-2)}}$$

is positive-valued under the specified constraints.

(iii) consider the derivative function, for $p \geq 3$.

$$h'(p)=\sqrt{\frac{1}{p(p+2)^3}}+(p^2+p-4)\sqrt{\frac{1}{(p+1)^3(p-1)}}-(p^2-p-3)\sqrt{\frac{1}{p^3(p-2)}}$$

is positive-valued under the specified constraints.

(iv) Consider the derivative function,

$$\begin{aligned} m'(p) = &-2\sqrt{\frac{1}{(p-1)(p+1)^3}} + 2(2p+1)\sqrt{\frac{1}{p^3(p+2)^5}} \\ &+ (2p-1)(p-3)\sqrt{\frac{1}{(p-1)^3(p+1)^5}} + 3\sqrt{\frac{1}{p^5(p-2)}} \end{aligned}$$

is positive-valued under the specified constraints.

(v) Consider the derivative function,

$$o'(p) = 2(2p-1)\sqrt{\frac{1}{(p-1)^3(p+1)^5}} - \left(\frac{4p^2+6p}{p^2(p-2)}\right)\sqrt{\frac{1}{p^3(p-2)}}$$

is negative-valued under the specified constraints. ∎

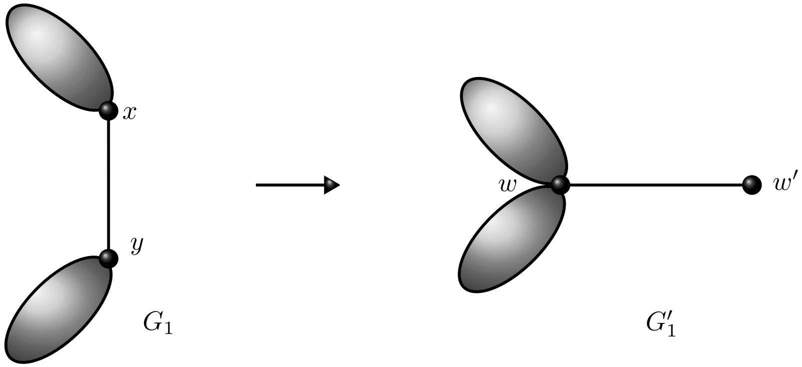

Figure 1. Transformation 1: Graphs G_1 and G'_1 in Lemma 2

The proof of Lemma 2.1 in [6] does not require the condition of degrees. Consequently, Lemma 2.1 can be refined to yield the following result.

Lemma 2. [6] *Let G_1 be a connected graph that is not trivial. Let x and y be vertices such that there exists an edge $xy \in E(G_1)$. Furthermore, $d_{G_1}(x), d_{G_1}(y) \geq 2$ and $N_{G_1}(x) \cap N_{G_1}(y) = \emptyset$. The graph G'_1 is constructed by merging the edge xy into a new vertex w and then attaching a new pendent edge ww' to w. (Refer to Figure 1 for a visual representation of these graphs). Then $ABS(G_1) < ABS(G'_1)$.*

The graph $\mathfrak{L}_{n,\mathcal{R}}$ is defined as a unicyclic graph comprising n vertices, where $n - \mathcal{R}$ pendent edges are connected to a single vertex x of the cycle $C_{\mathcal{R}}$, for all $3 \leq \mathcal{R} \leq n-2$. (Refer to Figure 2 for a visual representation of these graphs).

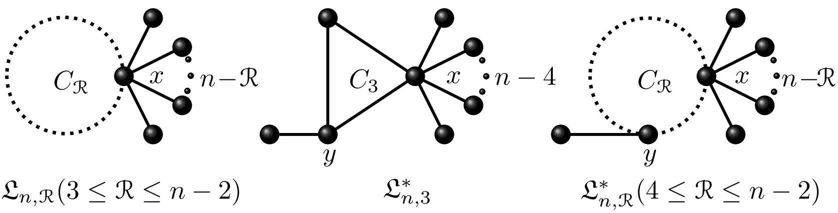

Figure 2. The graphs $\mathfrak{L}_{n,\mathcal{R}}$, $\mathfrak{L}^*_{n,3}$ and $\mathfrak{L}^*_{n,\mathcal{R}}$ $(3 \leq \mathcal{R} \leq n-2)$

Theorem 1. *Let $G \in \mathbb{U}_{n,\mathcal{R}}$. Then*

$$ABS(G) \leq (n-\mathcal{R})\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + 2\sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}}$$

with equality if and only if $G \cong \mathfrak{L}_{n,\mathcal{R}}$.

Proof. The theorem will be established through a mathematical induction process based on n. If $n = \mathcal{R}+2$ (*i.e.*, $\mathcal{R} = n-2$), then

$$\begin{aligned} ABS(\mathfrak{L}_{n,n-2}) &= 2\sqrt{\frac{3}{5}} + 2\sqrt{\frac{2}{3}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} \\ &= (n-\mathcal{R})\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + 2\sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}}, \end{aligned}$$

therefore the theorem is valid for $n = \mathcal{R}+2$. Hence, we proceed by assuming that $n > \mathcal{R}+2$ and the theorem is applicable for smaller n values. Additionally, for the sake of convenience, we make the assumption that G possesses the maximum ABS index compared to all other graphs within $\mathbb{U}_{n,\mathcal{R}}$.

Consider a pendent vertex $w \in V(G)$ and an edge $yw \in E(G)$. The vertex y has a degree $\psi \geq 2$, given that $G \in \mathbb{U}_{n,\mathcal{R}}$, which implies $\psi \leq n-\mathcal{R}+2$. We define $N(y) = \{w, y_1, ..., y_{\psi-1}\}$ where each y_i has a degree ψ_i (i.e., $d(y_i) = \psi_i$) for each $1 \leq i \leq \psi-1$. Notably, within the set $\{y_1, ..., y_{\psi-1}\}$, there is at least one vertex that has a degree greater than or equal to 2.

If only one vertex in $\{y_1, ..., y_{\psi-1}\}$, say x, has a degree greater than or equal to 2. According to Lemma 2, there is a graph $G' \in \mathbb{U}_{n,\mathcal{R}}$ such that $ABS(G) < ABS(G')$. However, this contradicts our initial assumption that G has the maximum ABS index among all graphs in $\mathbb{U}_{n,\mathcal{R}}$.

Hence, it follows that there must be at least 2 vertices in the set $\{y_1, ..., y_{\psi-1}\}$ that have a degree of at least 2 (and thus $\psi \geq 3$). Consequently, we define G'' as the graph obtained by deleting vertex w from G, i.e., $G'' := G - w$, it follows that $G'' \in \mathbb{U}_{n-1,\mathcal{R}}$.

Utilizing Lemma 1 (i) and (ii) along with the assumption from the induction hypothesis, we can derive the following.

$$
\begin{aligned}
ABS(G) &= ABS(G'') + \sum_{i=1}^{\psi-1}\left[\sqrt{\frac{\psi+\psi_i-2}{\psi+\psi_i}} - \sqrt{\frac{(\psi-1)+\psi_i-2}{(\psi-1)+\psi_i}}\right] + \sqrt{\frac{\psi-1}{\psi+1}} \\
&\le ABS(G'') + \sqrt{\frac{\psi-1}{\psi+1}} + 2\sqrt{\frac{\psi}{\psi+2}} - 2\sqrt{\frac{\psi-1}{\psi+}} + (\psi-3)\left[\sqrt{\frac{\psi-1}{\psi+1}} - \sqrt{\frac{\psi-2}{\psi}}\right] \\
&= ABS(G'') + 2\sqrt{\frac{\psi}{\psi+2}} + (\psi-4)\sqrt{\frac{\psi-1}{\psi+1}} - (\psi-3)\sqrt{\frac{\psi-2}{\psi}} \\
&\le (n-1-\mathcal{R})\sqrt{\frac{(n-1)-\mathcal{R}+1}{(n-1)-\mathcal{R}+3}} + 2\sqrt{\frac{(n-1)-\mathcal{R}+2}{(n-1)-\mathcal{R}+4}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} \\
&\quad + 2\sqrt{\frac{\psi}{\psi+2}} + (\psi-4)\sqrt{\frac{\psi-1}{\psi+1}} - (\psi-3)\sqrt{\frac{\psi-2}{\psi}} \\
&\le (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} \\
&\quad + 2\sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + (n-\mathcal{R}-2)\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} - (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} \\
&= (n-\mathcal{R})\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + 2\sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}}
\end{aligned}
$$

equality occurs if and only if G'' is isomorphic to $\mathfrak{L}_{n-1,\mathcal{R}}$, where $d(y) = \psi = n - \mathcal{R} + 2$, with exactly 2 vertices in $\{y_1, ..., y_{\psi-1}\}$ have a degree of 2, while the remaining $(\psi - 3)$ vertices in $\{y_1, ..., y_{\psi-1}\}$ have a degree of 1. In other words, $G \cong \mathfrak{L}_{n,\mathcal{R}}$. Therefore, $\mathfrak{L}_{n,\mathcal{R}}$ is the unique graph within the set $\mathbb{U}_n$ that exhibits the maximum ABS index among all graphs. ■

To determine the second-maximum ABS index among the graphs in $\mathbb{U}_{n,\mathcal{R}}$, we will need to utilize some additional lemmas.

Transformation 2: Let H be a connected graph that is nontrivial. Let $xy \in E(H)$ such that $d_H(x) = d_H(y) = 2$. Additionally, the other neighbors of x and y have a degree of at least 2 in H. Now create a new graph G_2 from H by connecting $\mu - 2$ pendent edges to x and $\psi - 2$ pendent edges to y (where $\mu \geq \psi \geq 3$), respectively. Additionally, create another graph G_2' from H by connecting the $\mu + \psi - 4$ pendent edges to x. (Refer to Figure 3 for a visual representation of these graphs.)

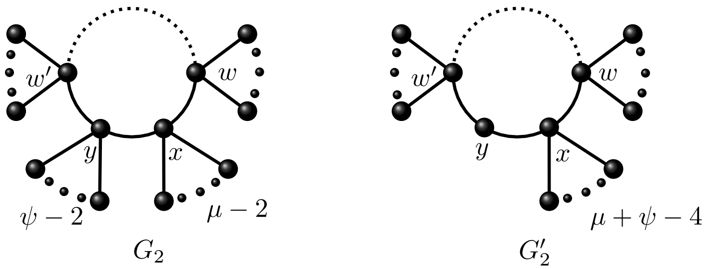

Figure 3. Illustration of Transformation 2

Lemma 3. *Let G_2 and G_2' be the graphs in Figure 3. Then $ABS(G_2) < ABS(G_2')$.*

Proof. Consider that $N_H(x) = \{y, w\}$ where $d_H(w) = s \geq 2$, and $N_H(y) = \{x, w'\}$ where $d_H(w') = t \geq 2$. Given that $(\mu \geq \psi \geq 3)$, then

$$
\begin{aligned}
&ABS(G_2) - ABS(G_2') \\
&= \left((\mu-2)\sqrt{\frac{\mu-1}{\mu+1}} + \sqrt{\frac{\mu+s-2}{\mu+s}} + (\psi-2)\sqrt{\frac{\psi-1}{\psi+1}} + \sqrt{\frac{\psi+t-2}{\psi+t}}\right) \\
&\quad - \left((\mu-2)\sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}} + \sqrt{\frac{\mu+\psi+s-4}{\mu+\psi+s-2}} + (\psi-2)\sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}} + \sqrt{\frac{t}{2+t}}\right) \\
&= (\mu-2)\left(\sqrt{\frac{\mu-1}{\mu+1}} - \sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}}\right) + \left(\sqrt{\frac{\mu+s-2}{\mu+s}} - \sqrt{\frac{\mu+\psi+s-4}{\mu+\psi+s-2}}\right) \\
&\quad + (\psi-2)\left(\sqrt{\frac{\psi-1}{\psi+1}} - \sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}}\right) + \left(\sqrt{\frac{\psi+t-2}{\psi+t}} - \sqrt{\frac{t}{2+t}}\right) \\
&\leq (\psi-2)\left(\sqrt{\frac{\psi-1}{\psi+1}} - \sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}}\right) + \left(\sqrt{\frac{\psi}{\psi+2}} - \sqrt{\frac{1}{2}}\right)
\end{aligned}
$$

if $\mu = \psi = 3$, then

$$ABS(G_2) - ABS(G_2') = \left(\sqrt{\frac{1}{2}} - \sqrt{\frac{3}{5}}\right) + \left(\sqrt{\frac{3}{5}} - \sqrt{\frac{1}{2}}\right) = 0$$

So we can consider that $\mu \geq 4$ and $\psi \geq 3$. Hence

$$\begin{aligned} &ABS(G_2) - ABS(G_2') \\ &\quad \leq (\psi-2)\left(\sqrt{\frac{\psi-1}{\psi+1}} - \sqrt{\frac{\mu+\psi-3}{\mu+\psi-1}}\right) + \left(\sqrt{\frac{\psi}{\psi+2}} - \sqrt{\frac{1}{2}}\right) < 0 \end{aligned}$$

Therefore, $ABS(G_2) - ABS(G_2') < 0$

■

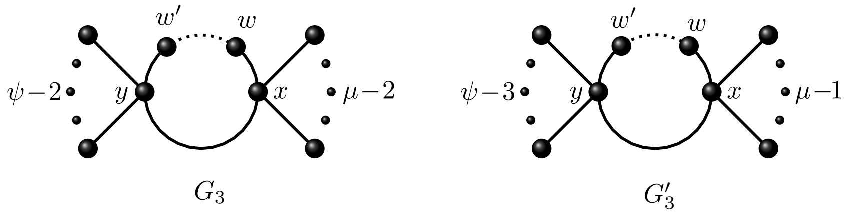

Figure 4. Illustration of Transformation 3

Transformation 3: Let H be a connected graph that is nontrivial. Let $xy \in E(H)$ such that $d_H(x) = d_H(y) = 2$, and their remaining neighbors in H also have a degree of 2 . Now create a new graph G_3 from H by connecting $\mu - 2$ pendent edges to x and $\psi - 2$ pendent edges to y, where $(\mu \geq \psi \geq 3)$, respectively. Additionally, create another graph G_3' from H by connecting $\mu - 1$ and $\psi - 3$ pendent edges to x and y, respectively. (Refer to Figure 4 for a visual representation of these graphs.)

Lemma 4. *Let G_3 and G_3' be the graphs in Figure 4. Then $ABS(G_3) < ABS(G_3')$.*

Proof. Using Lemma 1 (iii), for $p \geq 3$ the function $h(p)$ is increasing function. Given that $\mu \geq \psi \geq 3$, then

$$
\begin{aligned}
&ABS(G_3) - ABS(G'_3)\\
&\quad= \left((\mu-2)\sqrt{\frac{\mu-1}{\mu+1}} + \sqrt{\frac{\mu}{\mu+2}} + (\psi-2)\sqrt{\frac{\psi-1}{\psi+1}} + \sqrt{\frac{\psi}{\psi+2}}\right)\\
&\qquad - \left((\mu-1)\sqrt{\frac{\mu}{\mu+2}} + \sqrt{\frac{\mu+1}{\mu+3}} + (\psi-3)\sqrt{\frac{\psi-2}{\psi}} + \sqrt{\frac{\psi-1}{\psi+1}}\right)\\
&\quad= \left(\sqrt{\frac{\psi}{\psi+2}} + (\psi-3)\sqrt{\frac{\psi-1}{\psi+1}} - (\psi-3)\sqrt{\frac{\psi-2}{\psi}}\right)\\
&\qquad - \left(\sqrt{\frac{\mu+1}{\mu+3}} + (\mu-2)\sqrt{\frac{\mu}{\mu+2}} - (\mu-2)\sqrt{\frac{\mu-1}{\mu+1}}\right)\\
&\quad= h(\psi) - h(\mu+1) < 0
\end{aligned}
$$

Therefore $ABS(G_3) - ABS(G'_3) < 0$. ■

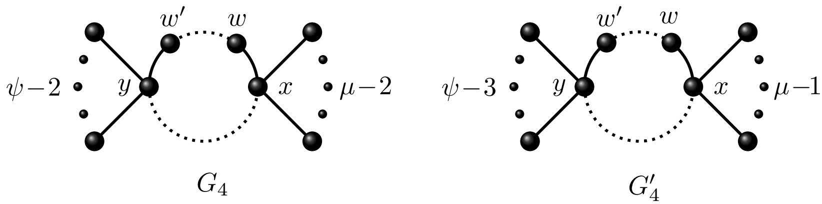

Figure 5. Illustration of Transformation 4

Transformation 4: Let H be a connected graph that is nontrivial. Let two distinct vertices x and y have a degree of 2 and such that $xy \notin E(H)$. Additionally, the remaining neighbors of x and y in H also have a degree of 2. Now create a new graph G_4 from H by connecting $\mu - 2$ pendent edges to x and $\psi - 2$ pendent edges to y where $(\mu \geq \psi \geq 3)$, respectively. Additionally, create another graph G'_4 from H by connecting $\mu-1$ and $\psi-3$ pendent edges to x and y. (Refer to Figure 5 for a visual representation of these graphs.)

Lemma 5. *Let G_4 and G'_4 be the graphs in Figure 5. Then $ABS(G_4) < ABS(G'_4)$.*

Proof. Using Lemma 1 (ii), for $p \geq 3$ the funtion $g(p)$ is increasing. Since $\mu \geq \psi \geq 3$, then

$$
\begin{aligned}
&ABS(G_4) - ABS(G_4')\\
&\quad = \left((\mu-2)\sqrt{\frac{\mu-1}{\mu+1}}+2\sqrt{\frac{\mu}{\mu+2}}+(\psi-2)\sqrt{\frac{\psi-1}{\psi+1}}+2\sqrt{\frac{\psi}{\psi+2}}\right)\\
&\qquad -\left((\mu-1)\sqrt{\frac{\mu}{\mu+2}}+2\sqrt{\frac{\mu+1}{\mu+3}}+(\psi-3)\sqrt{\frac{\psi-2}{\psi}}+2\sqrt{\frac{\psi-1}{\psi+1}}\right)\\
&\quad = \left(2\sqrt{\frac{\psi}{\psi+2}}+(\psi-4)\sqrt{\frac{\psi-1}{\psi+1}}-(\psi-3)\sqrt{\frac{\psi-2}{\psi}}\right)\\
&\qquad -\left(2\sqrt{\frac{\mu+1}{\mu+3}}+(\mu-3)\sqrt{\frac{\mu}{\mu+2}}-(\mu-2)\sqrt{\frac{\mu-1}{\mu+1}}\right)\\
&\quad = g(\psi) - g(\mu+1) < 0
\end{aligned}
$$

Therefore $ABS(G_4) - ABS({G_4}') < 0$. ■

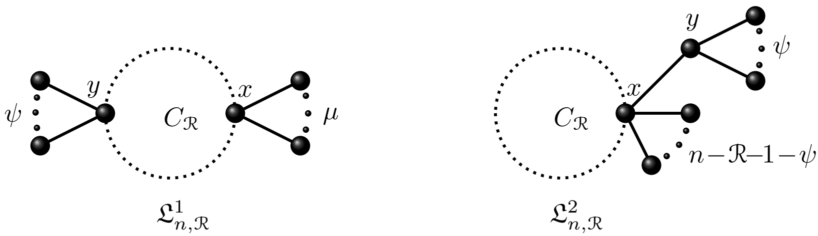

Figure 6. The graph sets $\mathfrak{L}^1_{n,\mathcal{R}}$ and $\mathfrak{L}^2_{n,\mathcal{R}}$ $(3 \leq \mathcal{R} \leq n-2)$

We consider two sets of unicyclic graphs comprising n vertices for $3 \leq \mathcal{R} \leq n-2$. The first set denoted by $\mathfrak{L}^1_{n,\mathcal{R}}$ consists of graphs created by attaching μ and ψ pendent edges ($\mu \geq \psi \geq 1$ *and* $\mu + \psi = n - \mathcal{R}$) to two distinct vertices x and y of the cycle $C_{\mathcal{R}}$, respectively. The second set of unicyclic graphs comprising n vertices denoted by $\mathfrak{L}^2_{n,\mathcal{R}}$ created by connecting ψ pendent edges ($1 \leq \psi \leq n - \mathcal{R} - 1$) to a pendent vertex y of $\mathfrak{L}_{n-\psi,\mathcal{R}}$ (as shown in Figure 6 for visual representation).

Lemma 6. *Let* $G \in \mathfrak{L}^1_{n,\mathcal{R}}$
(i) If $\mathcal{R} = 3$, then

$$ABS(G) \leq (n-4)\sqrt{\frac{n-3}{n-1}} + \sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n-2}{n}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}}$$

maintains equality holds if and only if $\psi = 1$.
(ii) If $4 \leq \mathcal{R} \leq n-2$, then

$$ABS(G) \leq (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-3)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}}$$

maintains equality holds if and only if $xy \notin E(G)$ and $\psi = 1$.

Proof. Assuming $\mathcal{R} = 3$, it follows that $xy \in E(G)$. we can apply Lemma 4 to draw the following conclusion:

$$ABS(G) \leq (n-4)\sqrt{\frac{n-3}{n-1}} + \sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n-2}{n}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}}$$

maintains equality if and only if $\mu = n-4$ and $\psi = 1$. Hence proved (i).

Assume for the moment that $4 \leq \mathcal{R} \leq n-2$. If $xy \in E(G)$, then by Lemma 4, we derive the following conclusion:

$$ABS(G) \leq (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + \sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + \sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}}$$

maintains equality if and only if $\mu = n - \mathcal{R} - 1$and $\psi = 1$ both hold true.

If $xy \notin E(G)$, then we can apply Lemma 5 to derive the following conclusion:

$$ABS(G) \leq (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-3)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}}$$

maintains equality if and only if $\mu = n - \mathcal{R} - 1$and $\psi = 1$ both hold true.

Since

$$\left((n-\mathfrak{R}-1)\sqrt{\frac{n-\mathfrak{R}}{n-\mathfrak{R}+2}}+\sqrt{\frac{n-\mathfrak{R}+1}{n-\mathfrak{R}+3}}+\sqrt{\frac{n-\mathfrak{R}+2}{n-\mathfrak{R}+4}}+(\mathfrak{R}-2)\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}\right)$$
$$-\left((n-\mathfrak{R}-1)\sqrt{\frac{n-\mathfrak{R}}{n-\mathfrak{R}+2}}+2\sqrt{\frac{n-\mathfrak{R}+1}{n-\mathfrak{R}+3}}+(\mathfrak{R}-3)\sqrt{\frac{1}{2}}+2\sqrt{\frac{3}{5}}\right)$$
$$=\sqrt{\frac{n-\mathfrak{R}+2}{n-\mathfrak{R}+4}}-\sqrt{\frac{n-\mathfrak{R}+1}{n-\mathfrak{R}+3}}+\sqrt{\frac{1}{2}}-\sqrt{\frac{3}{5}}$$
$$\leq\sqrt{\frac{4}{6}}-\sqrt{\frac{3}{5}}+\sqrt{\frac{1}{2}}-\sqrt{\frac{3}{5}}<0$$

therefore (ii) holds. ■

Lemma 7. *Consider $G\in\mathfrak{L}^2_{n,\mathfrak{R}}$. Then*

$$ABS(G)\leq(n-\mathfrak{R}-2)\sqrt{\frac{n-\mathfrak{R}}{n-\mathfrak{R}+2}}+3\sqrt{\frac{n-\mathfrak{R}+1}{n-\mathfrak{R}+3}}+(\mathfrak{R}-2)\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}.$$

maintains equality holds if and only if $\psi=1$.

Proof. Since there is only one graph in $\mathfrak{L}^2_{n,\mathfrak{R}}$, the statement of the lemma is trivially true for the case $n-\mathfrak{R}=2$. Therefore, we proceed by assuming that $n-\mathfrak{R}\geq 3$. Let

$$f(\psi)=ABS(G)=\sqrt{\frac{n-\mathfrak{R}+1}{n-\mathfrak{R}+3}}+2\sqrt{\frac{n-\mathfrak{R}+2-\psi}{n-\mathfrak{R}+4-\psi}}+(\mathfrak{R}-2)\sqrt{\frac{1}{2}}$$
$$+(n-\mathfrak{R}-1-\psi)\sqrt{\frac{n-\mathfrak{R}+1-\psi}{n-\mathfrak{R}+3-\psi}}+\psi\sqrt{\frac{\psi}{\psi+2}}$$

where $1\leq\psi\leq n-\mathfrak{R}-1$. Hence

$$f'(\psi)=-\sqrt{\frac{n-\mathfrak{R}+1-\psi}{n-\mathfrak{R}+3-\psi}}-2\sqrt{\frac{1}{(n-\mathfrak{R}+2-\psi)(n-\mathfrak{R}+4-\psi)^3}}$$
$$-(n-\mathfrak{R}-1-\psi)\sqrt{\frac{1}{(n-\mathfrak{R}+1-\psi)(n-\mathfrak{R}+3-\psi)^3}}+\sqrt{\frac{\psi}{\psi+2}}+\sqrt{\frac{\psi}{(\psi+2)^3}}$$

If $1\leq\psi\leq\left\lfloor\frac{n-\mathfrak{R}}{2}\right\rfloor$ (i.e., $\psi+2\leq n-\mathfrak{R}+2-\psi$), then by Lemma 1 (iv) (with $p = n - \mathcal{R} + 2 - \psi \geq 3$), leading to:

$$\begin{aligned}
f'(\psi) = &-\sqrt{\frac{n-\mathcal{R}+1-\psi}{n-\mathcal{R}+3-\psi}} - 2\sqrt{\frac{1}{(n-\mathcal{R}+2-\psi)(n-\mathcal{R}+4-\psi)^3}} \\
&-(n-\mathcal{R}-1-\psi)\sqrt{\frac{1}{(n-\mathcal{R}+1-\psi)(n-\mathcal{R}+3-\psi)^3}} + \left(\frac{\psi+3}{\psi+2}\right)\sqrt{\frac{\psi}{\psi+2}} \\
&+\left(\frac{n-\mathcal{R}+3-\psi}{n-\mathcal{R}+2-\psi}\right)\sqrt{\frac{n-\mathcal{R}-\psi}{n-\mathcal{R}+2-\psi}} - \left(\frac{n-\mathcal{R}+3-\psi}{n-\mathcal{R}+2-\psi}\right)\sqrt{\frac{n-\mathcal{R}-\psi}{n-\mathcal{R}+2-\psi}}
\end{aligned}$$

$$\begin{aligned}
&\leq -\sqrt{\frac{n-\mathcal{R}+1-\psi}{n-\mathcal{R}+3-\psi}} - 2\sqrt{\frac{1}{(n-\mathcal{R}+2-\psi)(n-\mathcal{R}+4-\psi)^3}} \\
&\quad -(n-\mathcal{R}-1-\psi)\sqrt{\frac{1}{(n-\mathcal{R}+1-\psi)(n-\mathcal{R}+3-\psi)^3}} \\
&\quad +\left(\frac{n-\mathcal{R}+3-\psi}{n-\mathcal{R}+2-\psi}\right)\sqrt{\frac{n-\mathcal{R}-\psi}{n-\mathcal{R}+2-\psi}} \\
&= -\sqrt{\frac{p-1}{p+1}} - 2\sqrt{\frac{1}{p(p+2)^3}} - (p-3)\sqrt{\frac{1}{(p-1)(p+1)^3}} + \left(\frac{p+1}{p}\right)\sqrt{\frac{p-2}{p}} < 0.
\end{aligned}$$

If $\left\lfloor \frac{n-\mathcal{R}+1}{2} \right\rfloor \leq \psi \leq n - \mathcal{R} - 1$ (i.e., $\psi + 2 \geq n - \mathcal{R} + 3 - \psi$), then by Lemma 1 (v) (with $p = n - \mathcal{R} + 3 - \psi$), leading to:

$$\begin{aligned}
f'(\psi) = &-\sqrt{\frac{\psi}{\psi+2}} - 2\sqrt{\frac{1}{(n-\mathcal{R}+2-\psi)(n-\mathcal{R}+4-\psi)^3}} \\
&-(n-\mathcal{R}-1-\psi)\sqrt{\frac{1}{(n-\mathcal{R}+1-\psi)(n-\mathcal{R}+3-\psi)^3}} + \sqrt{\frac{\psi}{\psi+2}} + \sqrt{\frac{\psi}{(\psi+2)^3}} \\
&\geq -2\sqrt{\frac{1}{(p-1)(p+1)^3}} - (p-4)\sqrt{\frac{1}{(p-2)p^3}} + \sqrt{\frac{p-2}{p^3}} > 0.
\end{aligned}$$

This suggests that for $1 \leq \psi \leq \left\lfloor \frac{n-\mathcal{R}}{2} \right\rfloor$, $f(\psi)$ is increasing and for $\left\lfloor \frac{n-\mathcal{R}+1}{2} \right\rfloor \leq \psi \leq n-\mathcal{R}-1$, $f(\psi)$ is decreasing. Therefore the maximum value of $f(\psi)$ is $max\{f(1), f(n-\mathcal{R}-1)\}$. Since $n-\mathcal{R}\geq 3$, we have

$$
\begin{aligned}
& f(n-\mathcal{R}-1)-f(1)\\
&=\left((n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}-1}{n-\mathcal{R}+1}}+\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}}+(\mathcal{R}-2)\sqrt{\frac{1}{2}}+2\sqrt{\frac{3}{5}}\right)\\
&\quad-\left((n-\mathcal{R}-2)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}}+\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}}+2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}}+(\mathcal{R}-2)\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}\right)\\
&=(n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}-1}{n-\mathcal{R}-1+2}}-2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}}-(n-\mathcal{R}-2)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}}+2\sqrt{\frac{3}{5}}-\sqrt{\frac{1}{3}}\\
&\leq(3-1)\sqrt{\frac{3-1}{3+1}}-2\sqrt{\frac{3+1}{3+3}}-(3-2)\sqrt{\frac{3}{3+2}}+2\sqrt{\frac{3}{5}}-\sqrt{\frac{1}{3}}<0.
\end{aligned}
$$

As a result, the conclusion of Lemma 7 is valid. ■

The collection of unicyclic graph comprising n vertices denoted by $\mathfrak{L}^*_{n,3}$ created by connecting $n-4$ and one pendent edge to 2 adjacent vertices x and y of a triangle. For $4\leq\mathcal{R}\leq n-2$, the collection of unicyclic graph comprising n vertices denoted by $\mathfrak{L}^*_{n,\mathcal{R}}$ created by connecting $n-\mathcal{R}-1$ and one pendent edge to 2 non-adjacent vertices x and y of the cycle $C_{\mathcal{R}}$ (as shown in figure 2 for visual representation). It is important to note that $\mathfrak{L}^*_{n,\mathcal{R}}\subseteq\mathfrak{L}^1_{n,\mathcal{R}}$ and there are $\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor-1$ unicyclic graphs in $\mathfrak{L}^*_{n,\mathcal{R}}$ for each $4\leq\mathcal{R}\leq n-2$. We can now determine the second-maximum ABS index among all graphs in $\mathbb{U}_{n,\mathcal{R}}$.

Theorem 2. *Let $G\in\mathbb{U}_{n,\mathcal{R}}$ and $G\ncong\mathfrak{L}_{n,\mathcal{R}}$.*
(i) If $\mathcal{R}=3$, then

$$ABS(G)\leq(n-4)\sqrt{\frac{n-3}{n-1}}+\sqrt{\frac{n-1}{n+1}}+\sqrt{\frac{n-2}{n}}+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}$$

maintains equality if and only if $G\cong\mathfrak{L}^*_{n,3}$.
(ii)If $4\leq\mathcal{R}\leq n-2$, then

$$ABS(G)\leq(n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}}+2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}}+(\mathcal{R}-3)\sqrt{\frac{1}{2}}+2\sqrt{\frac{3}{5}}$$

maintains equality if and only if $G\cong\mathfrak{L}^*_{n,\mathcal{R}}$.

Proof. Suppose G is a graph with a unique cycle C, where $C := y_1y_2\ldots y_{\mathcal{R}}y_1$.

Then there exists at least one vertex among $\{y_1, y_2, ..., y_{\mathcal{R}}\}$ have a degree of at least 3. For the sake of the proof, let

$$P := (n-4)\sqrt{\frac{n-3}{n-1}} + \sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n-2}{n}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}},$$

for $\mathcal{R} = 3$, or

$$P := (n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-3)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}},$$

for $4 \leq \mathcal{R} \leq n-2$.
Similarly,

$$Q := (n-\mathcal{R}-2)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 3\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}.$$

If G has at least 3 vertices among $\{y_1, y_2, ..., y_{\mathcal{R}}\}$ have a degree of at least 3. Then by using Lemmas 2, 3, (or 5), and 6, a graph $G_1 \in \mathfrak{L}^1_{n,\mathcal{R}}$ can be found such that $ABS(G) < ABS(G_1) \leq P$.

If G has exactly 2 vertices among $\{y_1, y_2, ..., y_{\mathcal{R}}\}$ have a degree of at least 3. Then by using Lemmas 2, 4, (or 5) and 6, a graph $G_2 \in \mathfrak{L}^1_{n,\mathcal{R}}$ can be found such that $ABS(G) < ABS(G_2) \leq P$ maintains equality if and only if $G \in \mathfrak{L}^*_{n,\mathcal{R}}$.

Hence, it can be assumed that there is exactly one vertex among $\{y_1, y_2, ..., y_{\mathcal{R}}\}$ have a degree of at least 3. As $G \ncong \mathfrak{L}_{n,\mathcal{R}}$, there must be at least one non-pendent vertex that is located outside C. If, according to Lemma 2 and 7, there exists at least two non-pendent vertex that is located outside C, then allows to finding a graph $G_3 \cong \mathfrak{L}^2_{n,\mathcal{R}}$ such that $ABS(G) < ABS(G_3) \leq Q$. Assuming there is only one non-pendent vertex outside of C, implying that $G \cong \mathfrak{L}^2_{n,\mathcal{R}}$. By using Lemma 7, we can deduce that $ABS(G) \leq Q$, with equality holding if and only if $\psi = 1$.

Comparing the values of P and Q is essential for proving the theorem.

If $\mathcal{R} = 3$, then

$$
\begin{aligned}
Q - P &= \left((n-5)\sqrt{\frac{n-3}{n-1}} + 3\sqrt{\frac{n-2}{n}} + (1)\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}\right) \\
&\quad - \left((n-4)\sqrt{\frac{n-3}{n-1}} + \sqrt{\frac{n-2}{n}} + \sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}}\right) \\
&= -\sqrt{\frac{n-3}{n-1}} + 2\sqrt{\frac{n-2}{n}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{n-1}{n+1}} - \sqrt{\frac{3}{5}} \\
&\leq -\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{3}{5}} < 0.
\end{aligned}
$$

So (i) holds.
If $4 \leq \mathcal{R} \leq n-2$, then

$$
\begin{aligned}
Q - P &= \left((n-\mathcal{R}-2)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 3\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-2)\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}\right) \\
&\quad - \left((n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-3)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}}\right) \\
&= -\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + \sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}} + -2\sqrt{\frac{3}{5}} \\
&\leq -\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{1}{3}} - 2\sqrt{\frac{3}{5}} < 0.
\end{aligned}
$$

This suggests (ii), and as a result, the proof confirming the theorem has been explored. ∎

The following corollary can be drawn by using Theorem 1 and Theorem 2 respectively.

Corollary 1. *Let $\mathfrak{L}_{n,\mathcal{R}}$, and $\mathfrak{L}^*_{n,\mathcal{R}}$ be the graphs defined as above.*
(i) If $n = 5$, then

$$ABS(\mathfrak{L}_{5,3}) > ABS(\mathfrak{L}^*_{5,3}) > ABS(u_{5,4}) > ABS(u_{5,3}) > ABS(C_5)$$

(ii) If $n \geq 6$, then

$$ABS(\mathfrak{L}_{n,3}) > ABS(\mathfrak{L}^*_{n,3}) > ABS(\mathfrak{L}_{n,4}) > ABS(\mathfrak{L}^*_{n,4}) > \ldots. > ABS(\mathfrak{L}_{n,n-2}) \\ > ABS(\mathfrak{L}^*_{n,n-2}) > ABS(u_{n,n-1}) > ABS(C_n)$$

Proof. Since

$$ABS(\mathfrak{L}_{5,3}) = 3.8893 > ABS(\mathfrak{L}^*_{5,3}) = 3.7799 > ABS(u_{5,4}) > ABS(u_{5,3}) \\ > ABS(C_5)$$

This verifies that (i) valid.
Now we prove (ii). We utilize Theorem 1 and Theorem 2, to estabish that $ABS(\mathfrak{L}_{n,\mathcal{R}}) > ABS(\mathfrak{L}^*_{n,\mathcal{R}})$ for each $3 \leq \mathcal{R} \leq n-2$. Since

$$ABS(\mathfrak{L}^*_{n,n-2}) = (\mathcal{R}-2)\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{5}}$$
$$ABS(U_{n,n-1}) = (n-\mathcal{R}+3)\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{5}}$$

and $ABS(C_n) = n\sqrt{\frac{1}{2}}$ We can see that, $ABS(\mathfrak{L}^*_{n,n-2}) > ABS(u^*_{n,n-1}) > ABS(C_n)$
To establish (ii), we need to prove that $ABS(\mathfrak{L}^*_{n,\mathcal{R}}) > ABS(\mathfrak{L}_{n,\mathcal{R}+1})$ for each value of $\mathcal{R}$, where $3 \leq \mathcal{R} \leq n-3$.
If $\mathcal{R} = 3$, then utilize Theorem 2(i)

$$\begin{aligned} &ABS(\mathfrak{L}^*_{n,3}) - ABS(\mathfrak{L}_{n,4}) \\ &\quad = \left((n-4)\sqrt{\frac{n-3}{n-1}} + \sqrt{\frac{n-1}{n+1}} + \sqrt{\frac{n-2}{n}} + \sqrt{\frac{2}{4}} + \sqrt{\frac{3}{5}}\right) \\ &\quad - \left((n-4)\sqrt{\frac{n-3}{n-1}} + 2\sqrt{\frac{n-2}{n}} + 2\sqrt{\frac{1}{2}}\right) \\ &\quad = \sqrt{\frac{n-1}{n+1}} - \sqrt{\frac{n-2}{n}} + \sqrt{\frac{3}{5}} - \sqrt{\frac{1}{2}} > 0 \end{aligned}$$

Therefore, $ABS(\mathfrak{L}^*_{n,3}) > ABS(\mathfrak{L}_{n,4})$. For each $4 \leq \mathcal{R} \leq n-3$ (and hence

$n \geq 7$), we have

$$\begin{aligned} &ABS(\mathfrak{L}^*_{n,\mathcal{R}}) - ABS(\mathfrak{L}_{n,\mathcal{R}+1}) \\ &\quad = \left((n-\mathcal{R}-1)\sqrt{\frac{n-\mathcal{R}}{n-\mathcal{R}+2}} + 2\sqrt{\frac{n-\mathcal{R}+1}{n-\mathcal{R}+3}} + (\mathcal{R}-3)\sqrt{\frac{2}{4}} + 2\sqrt{\frac{3}{5}} \right) \\ &\qquad - \left((n-\mathcal{R}+1)\sqrt{\frac{n-\mathcal{R}+2}{n-\mathcal{R}+4}} + 2\sqrt{\frac{n-\mathcal{R}+3}{n-\mathcal{R}+5}} + (\mathcal{R}-1)\sqrt{\frac{1}{2}} \right) \\ &\quad > 0 \end{aligned}$$

So the assertion of the corollary holds. ■

Hence, the only graph in $\mathbb{U}_n$ with the maximum ABS index is the $\mathfrak{L}_{n,3}$ graph. The only graph in $\mathbb{U}_n$with the second-maximum ABS is $\mathfrak{L}^*_{n,3}$ graph; this is demonstrated by Theorem 1, 2 and Corollary 1.

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