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The Euler-Sombor Index of Trees

Xiangyu Ren[a], Guoqing Cao[a], Fangxia Wang[b], Min Zhou[c,*]

[a] *School of Mathematics and Statistics, Shanxi University, Taiyuan Shanxi, 030006, P. R. China*
[b] *School of Mathematics Sciences, Xinjiang Normal University, Urumqi Xinjiang, 830017, P. R. China*
[c] *The Pharmacy Department of Shanxi Provincial People's Hospital, Taiyuan Shanxi, 030012, P. R. China*

renxy@sxu.edu.cn, 202422211003@email.sxu.edu.cn, fangxwangxj@163.com, Zhoumin19880806@163.com

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Abstract

In 2024, the Euler-Sombor index (EU) was introduced based on the calculation of the circumference of an ellipse, which is defined as

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

In this paper, we present the maximum values of the Euler-Sombor index of trees with some given parameters, such as the matching number, the number of pendent vertices and the diameter.

1 Introduction

Recently, Gutman [20] introduced the Sombor index which has become very familiar in most of the recent studies. The Sombor index is defined

*Corresponding author.

as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2},$$

where $d(u)$ and $d(v)$ are the degrees of vertices u and v. In a short period, the Sombor index has been extensively studied in literatures [1, 3, 4, 7–14, 16–19, 23, 28–32, 34–38, 41, 45–47]. Then, Gutman et al. [22] subsequently introduced a geometric method for constructing degree-based topological indices, the Elliptic Sombor index (ESO), which is defined as

$$ESO(G) = \sum_{uv \in E(G)} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2}.$$

Espinal et al. [15] solved the extremal ESO problem for chemical graphs and trees with equal vertices. Tang et al. [43] analyzed the maximal value of the Elliptic Sombor index of trees with a given number of pendent vertices, diameter and matching number.

Soon after, Gutman [21] and Tang et al. [44] proposed another index with geometric significance, the Euler-Sombor index (EU), which is defined as

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

After this, the Euler-Sombor index demonstrates its vitality as a new index. Hu et al. [25] found tight bounds for the Euler-Sombor index of maximal outerplanar graphs. Kirana et al. [26, 27] investigated the Elliptic and Euler Sombor indices of the union and corona products of paths, cycles and complete graphs, and also analyzed the Euler-Sombor index of trees, unicyclic graphs and chemical graphs. Tang et al. [39, 40, 44] studied the extremal values of the Euler-Sombor index in unicyclic and bicyclic graphs, highlighting its chemical relevance for property prediction. Bansode et al. [5] discussed the Euler-Sombor index, eigenvalues, and energy of some graph classes. Albalahia et al. [2] identified optimal Euler-Sombor index graphs for tricyclic molecules of a given order. G. O. Kızılırmak [33] determined the extremal values of the Euler-Sombor index of tricyclic graphs. Tache et al. [42] determined, for all $x \geq 5$, the first, second, and third minimum and maximum unicyclic graphs of order n with respect to the Euler-Sombor index.

In this paper, we aim to give some sharp upper bounds for the Euler-Sombor index in trees, with some given parameters, such as the matching number, the number of pendent vertices and the diameter. The rest of the paper is organised as: In Section 2, some needed preliminary is given. In section 3, we present the main proof process. In Section 4, future research on the Euler-Sombor index is proposed.

2 Preliminaries

Throughout this paper, all graphs are assumed to be simple, undirected, and finite; see [6] for undefined terms. We denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, respectively. The number of vertices and the number of edges of G are called the *order* and the *size*, respectively. Let n and m represent the order and size of G, respectively. For a vertex $u \in V(G)$, the *degree* of u, denoted by $d_G(u)$, is the number of edges incident with u in G. The maximum degree of G, denoted by Δ, is $\max\{d_G(u) : u \in V(G)\}$. A vertex u with $d_G(u) = 1$ is a pendent vertex, and its incident edge is a pendent edge. The neighborhood of a vertex u is denoted by $N_G(u)$.

For $S \subseteq E(G)$, $G - S$ denotes the graph obtained from G by deleting the edges in S. $G + S$ denotes the graph obtained by adding the edges in S to G. If $S = \{uv\}$, we can write $G - uv$ and $G + uv$ instead of $G - S$ and $G + S$, respectively. For $v \in V(G)$, $G - v$ is the graph obtained from G by deleting v and all its incident edges.

Let P_n and S_n denote the path and star of order $n \geq 1$, respectively. An acyclic graph, by definition, contains no cycles. A connected acyclic graph is called a *tree*. A tree T is called to be a *caterpillar* if it becomes a path after deleting all pendent vertices.

A matching is a set of edges in a graph, no two of which share a vertex. Given a matching M, the endpoints of each edge in M are matched by M, and any vertex incident to an edge in M is covered by M. A perfect matching covers all vertices of a graph. A matching M is a maximum matching of G if no matching M' of G has a larger cardinality than M.The matching number of G, denoted $\alpha'(G)$, is the cardinality of a maximum matching. A vertex cover of a graph G is a set S of vertices such that every edge of G is incident to a vertex in S. The covering number, denoted by $\beta(G)$, is the minimum cardinality of a vertex cover of G. The well-known König's min-max theorem states that

$$\alpha'(G) = \beta(G)$$

for any bipartite graph G. The distance $d_G(u, v)$ between vertices u and v in G is the length of the shortest path connecting them. The diameter of a graph G is defined as $diam(G) = \max_{\{u,v\}\subseteq V(G)} d_G(u, v)$. The following lemmas are crucial for proving our main theorems.

Lemma 1. *the function*

$$f(x, y) = \sqrt{x^2 + y^2 + xy} - \sqrt{(x-1)^2 + y^2 + (x-1)y},$$

is increasing with respect to $x \in [1, +\infty)$ *and is decreasing with respect to* $y \in [1, +\infty)$*, respectively.*

Proof. Since $x \geqslant 1$, we have

$$(x + \frac{1}{2}y)^2[(x-1)^2 + y^2 + (x-1)y] - (x + \frac{1}{2}y - 1)^2(x^2 + y^2 + xy) > 0.$$

A simple calculation shows that

$$\frac{\partial f}{\partial x} = \frac{x + \frac{1}{2}y}{\sqrt{x^2 + y^2 + xy}} - \frac{x + \frac{1}{2}y - 1}{\sqrt{(x-1)^2 + y^2 + (x-1)y}} > 0.$$

As in the above proof process, we can obtain $\frac{\partial f}{\partial y} < 0$, when $y \geqslant 1$. By the elementary calculus, the result follows. ■

Lemma 2. *Let* T *be a tree with an edge* $e = uv$ *satisfying* $d_T(u) \geq 2$ *and* $d_T(v) \geq 2$. $H_1 = \{uw : w \in N_T(v) \setminus \{u\}\}, H_2 = \{vw : w \in N_T(v) \setminus \{v\}\}$. *If* $T' = T - H_1 + H_2$, *then*

$$EU(T') > EU(T).$$

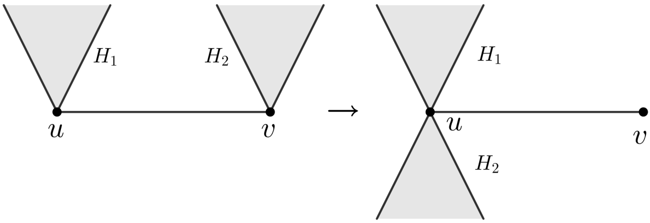

Figure 1. The transformation-I

Proof. Let $x = d_T(u) \geq 2$ and $y = d_T(v) \geq 2$. Since $x \geq 2$ and $y \geq 2$, we have

$$1 + (x + y - 1)^2 + x + y - 1 \geq x^2 + y^2 + xy.$$

Note that $d_{T'}(w) = d_T(w)$, $d_{T'}(u) = x + y - 1$ and $d_{T'}(v) = 1$, where $w \in N_{T'}(u) \cup N_{T'}(v) \setminus \{u, v\}$. Thus,

$$\begin{aligned}
&EU(T') - EU(T)\\
&= \sum_{w \in N_T(v) \setminus \{u\}} \sqrt{(x+y-1)^2 + d_{T'}^2(w) + (x+y-1)d_{T'}(w)}\\
&\quad - \sum_{w \in N_T(v) \setminus \{u\}} \sqrt{y^2 + d_T^2(w) + y d_T(w)}\\
&\quad + \sqrt{(x+y-1)^2 + 1 + x + y - 1} - \sqrt{x^2 + y^2 + xy}\\
&\quad + \sum_{w \in N_T(u) \setminus \{v\}} \sqrt{(x+y-1)^2 + d_{T'}^2(w) + (x+y-1)d_{T'}(w)}\\
&\quad - \sum_{w \in N_T(u) \setminus \{v\}} \sqrt{x^2 + d_T^2(w) + x d_T(w)}\\
&> 0.
\end{aligned}$$

The proof is completed. ■

Lemma 3. *Let P be the longest path of a tree T. Assume that u is an end vertex of P and $v \in V(P)$ is the vertex such that $d_P(v, u)$ is as small as possible, subject to $d_T(v) \geq 3$. Let u_1 be the unique neighbor of u*

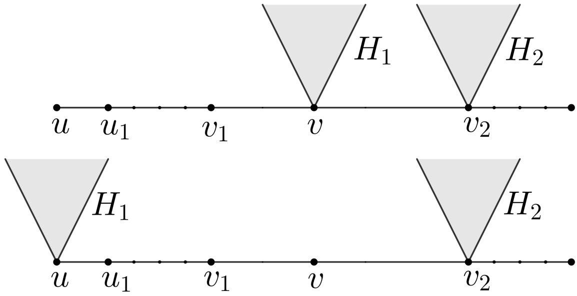

Figure 2. The transformation-II

on P and v_1 and v_2 be the two neighbors of v on P. $H_1 = \{vw : w \in N_T(v) \setminus \{v_1, v_2\}\}, H_2 = \{uw : w \in N_T(v) \setminus \{v_1, v_2\}\}$. If $T' = T - H_1 + H_2$, then

$$EU(T') < EU(T).$$

Proof. Let $x = d_T(v) \geq 3$. Next, we consider two cases based on the value of $d_P(v, u)$.

Case 1. $d_P(v, u) = 1$.

By the assumption, $u_1 = v$, $v_1 = u$, $d_{T'}(u) = x - 1$ and $d_{T'}(v) = 2$, $d_{T'}(v_2) = d_T(v_2)$. Since $x \geq 3$, we have

$$\sqrt{(x-1)^2 + 2^2 + 2(x-1)} < \sqrt{x^2 + 1 + x}.$$

Thus,

$$\begin{aligned}
&EU(T') - EU(T) \\
&= \sum_{w \in N_T(v) \setminus \{u, v_2\}} \sqrt{(x-1)^2 + d_{T'}^2(w) + (x-1)d_{T'}(w)} \\
&\quad - \sum_{w \in N_T(v) \setminus \{u, v_2\}} \sqrt{x^2 + d_T^2(w) + x d_T(w)} \\
&\quad + \sqrt{(x-1)^2 + 2^2 + 2(x-1)} - \sqrt{1 + x^2 + x} \\
&\quad + (\sqrt{2^2 + d_{T'}^2(v_2) + 2d_{T'}(v_2)} - \sqrt{x^2 + d_T^2(v_2) + 2d_T(v_2)})
\end{aligned}$$

$< 0.$

Case 2. $d_P(v,u) \geq 2$.

By Lemma 1 and $x \geq 3$, we have

$$(\sqrt{(x-1)^2+2^2+2(x-1)} - \sqrt{x^2+2^2+2x}) - (\sqrt{7}-2\sqrt{3}) < 0. \qquad (1)$$

Furthermore, since $d_P(v,u)$ is as small as possible, we have $d_{T'}(u_1) = d_T(u_1) = 2$ and $d_{T'}(v_1) = d_T(v_1) = 2$

Thus, by (1), we obtain

$$\begin{aligned}
&EU(T') - EU(T) \\
&= \sum_{w \in N_T(v)\setminus\{v_1,v_2\}} (\sqrt{(x-1)^2 + d_{T'}^2(w) + (x-1)d_{T'}(w)} \\
&\quad - \sum_{w \in N_T(v)\setminus\{v_1,v_2\}} \sqrt{x^2 + d_T^2(w) + xd_T(w)}) \\
&\quad + \sqrt{(x-1)^2+2^2+2(x-1)} - \sqrt{7} + 2\sqrt{3} - \sqrt{x^2+2^2+2x} \\
&\quad + \sqrt{2^2 + d_{T'}^2(v_2) + 2d_{T'}(v_2)} - \sqrt{x^2 + d_{T'}^2(v_2) + xd_{T'}(v_2)} \\
&< (\sqrt{(x-1)^2+2^2+2(x-1)} - \sqrt{x^2+2^2+2x}) - (\sqrt{7}-2\sqrt{3}) \\
&< 0.
\end{aligned}$$

This completes the proof. ■

Corollary. *For a tree T of order $n \geq 3$, we have*

$$2\sqrt{7} + 2(n-3)\sqrt{3} \leq EU(T) \leq (n-1)\sqrt{(n-1)^2+n}.$$

The left equality holds if and only if $T \cong P_n$ and the right equality holds if and only if $T \cong S_n$.

Proof. Let $T \ncong S_n$. Using a sequence of Transformation-I operations $(T_1, T_2, \ldots, T_t)$ as described in Lemma 2, T can be transformed into S_n through a sequence $T_1 = T, T_2, \ldots, T_t = S_n$ where $EU(T_{i+1}) > EU(T_i)$

for all $1 \leq i < t$. Therefore,

$$EU(T) < EU(S_n) = (n-1)\sqrt{(n-1)^2 + n}.$$

Conversely, if $T \ncong P_n$, Lemma 3 guarantees a sequence of Transformation-II operations $(T_1, T_2, \ldots, T_t)$ transforming T into P_n, where $T_1 = T$, $T_t = P_n$, and $EU(T_{i+1}) < EU(T_i)$ for all $i \in \{1, \ldots, t-1\}$. Thus,

$$2\sqrt{7} + 2(n-3)\sqrt{3} = EU(P_n) < EU(T).$$

Additionally, if $T \cong P_n$ or $T \cong S_n$, then equality holds on the right or left. ■

3 Main results

3.1 Trees with matching number α'

In this section, Let n, α' be integers such that $n \geq 2\alpha' \geq 2$. Let $T_{n,\alpha'}$ be the tree formed from the star $S_{n-\alpha'-1}$ by subdividing $\alpha' - 1$ of its pendent edges. Note that $\alpha'(T_{n,\alpha'}) = \alpha'$, and $T_{n,\alpha'}$ and $T_{2\alpha',\alpha'}$ are illustrated in Fig. 3. Simply, we obtain

$$\begin{aligned} EU(T_{n,\alpha'}) &= (n - 2\alpha' + 1)\sqrt{1 + (n-\alpha')^2 + n - \alpha'} \\ &\quad + (\alpha' - 1)(\sqrt{7} + \sqrt{2^2 + (n-\alpha')^2 + 2(n-\alpha')}). \end{aligned}$$

$T_{n,\alpha'}$ maximizes the Euler-Sombor index among trees of order n with matching number α'. Define a function:

$$\begin{aligned} k(n,\alpha') &= (n - 2\alpha' + 1)\sqrt{1 + (n-\alpha')^2 + n - \alpha'} \\ &\quad + (\alpha' - 1)(\sqrt{7} + \sqrt{2^2 + (n-\alpha')^2 + 2(n-\alpha')}). \end{aligned}$$

We begin with a useful lemma (see [24]).

Lemma 4. *[24] If T is a tree of order $n > 2\alpha'$ with $\alpha'(T) = \alpha'$, then there is a maximum matching M and a pendent vertex u such that M does not saturate u.*

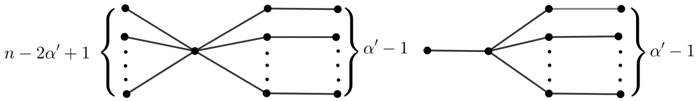

Figure 3. The tree $T_{n,\alpha'}$ (left) and $T_{2\alpha',\alpha'}$ (right).

We extend the previous result to the Euler-Sombor index as follows.

Lemma 5. *[7] Assume that T^* has the maximum Sombor index among all trees of order n with matching number α'. If M is a maximum matching with $|M| = \alpha'$, then*

(1) e is a pendent edge of T^* for each $e \in M$;

(2) u is saturated by M for each non-pendent vertex.

Lemma 6. *Assume that T^* has the maximum the Euler-Sombor index among trees of order n with matching number α'. If M is a maximum matching with $|M| = \alpha'$, then*

(1) e is a pendent edge of T^* if $e \in M$;

(2) u is saturated by M if $d(u) \geq 2$.

Proof. To prove (1), let $e = uv \in M$. Assume e is not a pendent edge of T^*, implying $d_{T^*}(u) \geq 2$ and $d_{T^*}(v) \geq 2$. Define $T' = T^* - \{vw : w \in N_{T^*}(v) \setminus \{u\}\} + \{uw : w \in N_{T^*}(v) \setminus \{u\}\}$.

Claim 1. $\alpha'(T') = \alpha'(T^*)$.

Because M is a matching of T', $\alpha'(T') \geq |M| = \alpha'(T^*)$. We now show that $\alpha'(T') \leq \alpha'(T^*)$. We can use Kőnig's theorem to prove $\beta(T') \leq \beta(T^*)$. Let S be a minimum cover of T^*. Then S' covers T', where

$$S' = \begin{cases} S \setminus \{v\}, & u \in S \\ (S \setminus \{v\}) \cup \{u\}, & \text{otherwise.} \end{cases}$$

Thus,

$$\alpha'(T^*) = \beta(T^*) = |S| \geq |S'| \geq \beta(T') = \alpha'(T').$$

This proves the claim.

This contradicts the choice of T^* because Lemma 2 implies $EU(T') > EU(T^*)$.

Then, we prove (2). If $T^* \cong S_n$, the claim is immediate. Otherwise, assume $T^* \ncong S_n$ and contains a non-M-saturated vertex u with $d_{T^*}(u) \geq 2$. Because T^* is not a star, u has a neighbor v with $d_{T^*}(v) \geq 2$. By the maximality of M, v is M-saturated. Let $T'' = T^* - \{vw : w \in N_{T^*}(v) \backslash \{u\}\} + \{uw : w \in N_{T^*}(v) \backslash \{u\}\}$. By Claim 1, we have $\alpha'(T'') = \alpha'$. Thus, $EU(T'') > EU(T^*)$ by Lemma 2, contradicting the maximality of T^*. ■

Recall that $S \subseteq V(G)$ is an independent set of a graph G if no two vertices in S are adjacent. The Independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G. The well-known Gallai identity says that

$$\alpha(G) + \beta(G) = n$$

for any graph G.

Next, we establish a sharp upper bound for the Euler-Sombor index $EU(T)$ of a tree T with a specified matching number.

Theorem 1. *If T is a tree of order $n > 2\alpha'$, which has matching number α',then $EU(T) \leq k(n, \alpha')$, with equality if and only if $T \cong T_{n,\alpha'}$.*

Proof. Assume that T has the maximum Euler-Sombor index EU among all trees with order n with matching number α'. If $n = 3$, then $T \cong P_3 \cong T_{3,1}$. Therefore, the result is immediate. Then, let $n \geq 4$ and assume the result holds for all trees of order between $2\alpha' + 1$ and $n - 1$. We proceed by induction on n.

Let M be a matching of T with $|M| = \alpha'$. By Lemma 4, a pendent vertex u exists such that $T - u$ has a matching M with size α'. Let v be the unique neighbor of u. Set $x = d_T(v)$ and $N_T(v) \setminus \{u\} = \{v_1, v_2, ..., v_{x-1}\}$. Let $T' = T - u$. Then T' is a tree of order $n - 1$ with matching number α'.

By Gallai's identity and König's theorem, we can know that $x = d_T(v) \leq \alpha(T) = n - \beta(T) = n - \alpha'(T) = n - \alpha'$. Let t be the number of pendent vertices in $N_{T'}(v)$ in T', say $v_1, \ldots, v_t$. Lemma 6 implies that T has $n - \alpha'$ pendent vertices. Therefore, $t + 1 \leq n - \alpha' - (\alpha' - 1) = n - 2\alpha' + 1$, which simplifies to $t \le n - 2\alpha'$. By induction hypothesis,

$$EU(T') \le (n-2\alpha')\sqrt{1+(n-1-\alpha')^2+n-1-\alpha'} \\ + (\alpha'-1)(\sqrt{7}+\sqrt{2^2+(n-1-\alpha')^2+2(n-1-\alpha')}).$$

Since $d_T(v_i) = d_{T'}(v_i) \ge 2$ for $i \in \{t+1, \dots, x-1\}$, by Lemma 1,

$$\sum_{i=t+1}^{x-1} (\sqrt{x^2+d_T(v_i)^2+xd_T(v_i)} - \sqrt{(x-1)^2+d_{T'}(v_i)^2+(x-1)d_{T'}(v_i)}) \\ = \sum_{i=t+1}^{x-1} f(x, d_T(v_i)) \ \le \ \sum_{i=t+1}^{x-1} f(x,2).$$

In addition, by $x \le n-\alpha'$, $t \le n-2\alpha'$ and $d_T(v_i) = d_{T'}(v_i) \ge 2$ we have

$$\begin{aligned} &EU(T) \\ &= EU(T') + \sqrt{x^2+1+x} + \sum_{i=1}^{t} f(x,1) + \sum_{i=t+1}^{x-1} f(x, d_T(v_i)) \\ &\le EU(T') + \sqrt{x^2+1+x} + \sum_{i=1}^{t} f(x,1) + \sum_{i=t+1}^{x-1} f(x,2) \\ &\le EU(T') + \sqrt{(n-\alpha')^2+1+(n-\alpha')} + tf(n-\alpha',1) \\ &\quad + (n-\alpha'-t-1)f(n-\alpha',2) \\ &\le EU(T') + \sqrt{(n-\alpha')^2+1+(n-\alpha')} + (n-\alpha'-1)f(n-\alpha',2) \\ &\quad + (n-2\alpha')(f(n-\alpha',1)-f(n-\alpha',2)) \\ &\le (n-2\alpha'+1)\sqrt{1+(n-\alpha')^2+(n-\alpha')} \\ &\quad + (\alpha'-1)(\sqrt{7}+\sqrt{2^2+(n-\alpha')^2+2(n-\alpha')}). \end{aligned}$$

A simple calculation shows that $EU(T_{n,\alpha'}) = k(n,\alpha')$. Conversely, if $EU(T_{n,\alpha'}) = k(n,\alpha')$, the preceding expression becomes an equality, implying $x = d_T(u) = n-\alpha'$, $t = n-2\alpha'$, and $d_{T'}(v_i) = 2$ for $n-2\alpha'+1 \le i \le n-\alpha'-1$. Hence, $T \cong T_{n,\alpha'}$. ■

3.2 Trees with p pendent vertices

In this section, let $T_{n,p}$ be the tree obtained from the star S_{p+1} (where $n \geq p \geq 2$) by subdividing one pendent edge $n-p-1$ times (see Fig. 4). Obviously, $T_{n,p}$ has exactly p pendent vertices. By a simple computation, we have

$$EU(T_{n,p}) = (p-1)\sqrt{1+p^2+p} + \sqrt{2^2+p^2+2p} + 2(n-p-2)\sqrt{3} + \sqrt{7}.$$

$T_{n,p}$ maximizes the Euler-Sombor index among trees of order n with p pendent vertices. For clarity, define a function:

$$g(n,p) = (p-1)\sqrt{1+p^2+p} + \sqrt{2^2+p^2+2p} + 2(n-p-2)\sqrt{3} + \sqrt{7}.$$

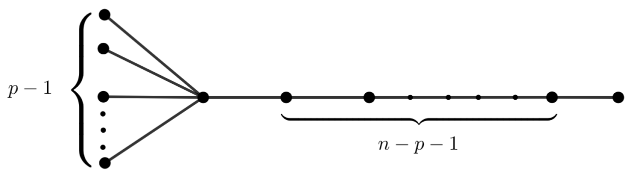

Figure 4. The tree $T_{n,p}$.

Theorem 2. *If T is a tree of order n with p pendent vertices, then $EU(T) \leq g(n,p)$, with equality if and only if $T \cong T_{n,p}$.*

Proof. Let T be a tree of order n with p pendent vertices. The result follows immediately, since $T \cong P_n \cong T_{n,2}$ when $p=2$, and $T \cong S_n \cong T_{n,n-1}$ when $p = n-1$.

Next, let $3 \leq p \leq n-2$. Assume the result holds for all trees T' of order n' with p' pendent vertices, where $n'+p' < n+p$. Let u be a leaf of T, and let v be its neighbor. Let $x = d_T(v)$. Note that $2 \leq x \leq p$. Let $N_T(v) \setminus \{u\} = \{v_1, v_2, ..., v_{x-1}\}$. Because T is not a star, v has a neighbor v_1 with degree at least two. Let $T' = T - u$. We now consider two cases based on $d_T(v)$.

Case 1. $d_T(v) = 2$.

By the assumption, $d_{T'}(v) = 1$, and thus, T' is a tree of order $n-1$ with p pendent vertices. By the induction hypothesis,

$$EU(T') \leq (p-1)\sqrt{1+p^2+p} + \sqrt{2^2+p^2+2p} + 2(n-p-3)\sqrt{3} + \sqrt{7}.$$

In addition, since $d_{T'}(v_1) = d_T(v_1) \geq 2$, by Lemma 1,

$$\sqrt{2^2 + d_{T'}^2(v_1) + 2d_{T'}(v_1)} - \sqrt{1 + d_{T'}^2(v_1) + d_{T'}(v_1)} \leq 2\sqrt{3} - \sqrt{7}.$$

Thus,

$$\begin{aligned}
&EU(T) \\
&= EU(T') + \sqrt{7} + \sqrt{2^2 + d_T^2(v_1) + 2d_T(v_1)} - \sqrt{1 + d_{T'}^2(v_1) + d_{T'}(v_1)} \\
&\leq EU(T') + \sqrt{7} + 2\sqrt{3} - \sqrt{7} \\
&\leq (p-1)\sqrt{1+p^2+p} + \sqrt{2^2+p^2+2p} + 2(n-p-3)\sqrt{3} \\
&\quad + \sqrt{7} + \sqrt{7} + 2\sqrt{3} - \sqrt{7} \\
&= (p-1)\sqrt{1+p^2+p} + \sqrt{2^2+p^2+2p} + 2(n-p-2)\sqrt{3} + \sqrt{7}.
\end{aligned}$$

Case 2. $d_T(v) \geq 3$.

Since $d_{T'}(v) = d_T(v) - 1 \geq 2$, T' is a tree of order $n-1$ with $p-1$ pendent vertices. Therefore, by the induction hypothesis,

$$\begin{aligned}
EU(T') \leq\ & (p-2)\sqrt{1+(p-1)^2+p-1} + \sqrt{2^2+(p-1)^2+2(p-1)} \\
& + 2(n-p-2)\sqrt{3} + \sqrt{7}.
\end{aligned}$$

Since $d_{T'}(v_1) = d_T(v_1) \geq 2$ and $d_{T'}(v_i) = d_T(v_i) \geq 1$ for each $i \in \{2, 3\ldots, x-1\}$, by Lemma 1,

$$\sqrt{2^2 + d_{T'}^2(v_1) + 2d_{T'}(v_1)} - \sqrt{1 + d_{T'}^2(v_1) + d_{T'}(v_1)} \leq 2\sqrt{3} - \sqrt{7},$$

and

$$\sum_{i=2}^{x-1} \sqrt{d_T^2(v) + d_T^2(v_i) + d_T(v)d_T(v_i)} - \sqrt{d_{T'}^2(v) + d_{T'}^2(v_i) + d_{T'}(v)d_{T'}(v_i)}$$

$$\leq \sum_{i=2}^{x-1}(\sqrt{x^2+1+x}-\sqrt{(x-1)^2+1+x-1}) = (x-2)f(x,1).$$

In addition, since $x \leq p$, by Lemma 1,

$$\sqrt{x^2+1+x}+(x-2)f(x,1)+f(x,2) \leq \sqrt{p^2+1+p}+(p-2)f(p,1)+f(p,2).$$

Thus,

$$\begin{aligned}
&EU(T)\\
&= EU(T') + \sqrt{x^2+1+x} + \sum_{i=1}^{x-1}(\sqrt{x^2+d_T^2(v_i)+xd_T(v_i)}\\
&\quad - \sqrt{(x-1)^2+d_{T'}^2(v_i)+(x-1)d_{T'}(v_i)})\\
&\leq EU(T') + \sqrt{x^2+1+x}+(x-2)f(x,1)+f(x,2)\\
&\leq EU(T') + \sqrt{p^2+1+p}+(p-2)f(p,1)+f(p,2)\\
&\leq (p-1)\sqrt{1+p^2+p}+\sqrt{2^2+p^2+2p}+2(n-p-2)\sqrt{3}+\sqrt{7}.
\end{aligned}$$

A simple calculation shows that $EU(T_{n,p}) = g(n,p)$. On the other hand, if $EU(T_{n,p}) = g(n,p)$, then all the equalities hold in the above expression. Thus, $x = d_T(v) = p$, $d_T(v_1) = 2$ and $d_T(v_i) = 1$ for $2 \leq i \leq x-1$, suggesting $T \cong T_{n,p}$. ■

3.3 Trees with diameter *d*

In this section, let n and d be integers such that $n \geq d+1 \geq 2$. Define $T_{n,d}^i$ as the tree formed by attaching $n-d-1$ edges to vertex u_i of the path $P = u_0u_1\cdots u_d$ (see Fig. 5). Note that $diam(T_{n,d}^i) = d$ and $T_{n,d}^i \cong T_{n,d}^{d-i}$ for all i.

Simply, we have

$$\begin{aligned}
EU(T_{n,d}^1) &= \sqrt{7}+\sqrt{2^2+(n-d+1)^2+2(n-d+1)}\\
&\quad + (n-d)\sqrt{1+(n-d+1)^2+n-d+1}+2(d-3)\sqrt{3}.
\end{aligned}$$

Our goal is to prove that $T_{n,d}^1$ maximizes the EU value among all trees of order n and diameter d. To this end, we define the following function:

$$h(n,d) = \sqrt{7} + \sqrt{2^2 + (n-d+1)^2 + 2(n-d+1)} + (n-d)\sqrt{1 + (n-d+1)^2 + n - d + 1} + 2(d-3)\sqrt{3}.$$

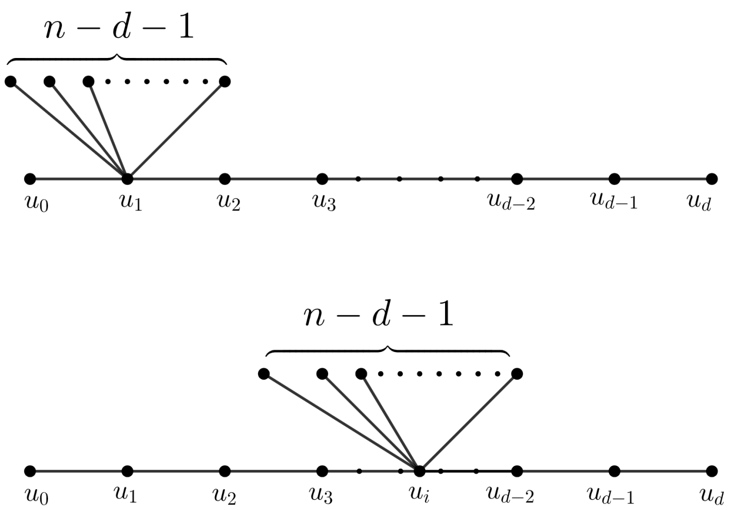

Figure 5. The tree $T^1_{n,d}$ (above) and $T^i_{n,d}$ (below).

Lemma 7. *If T^* has the maximum the the Euler-Sombor index among all trees of order n with diameter d, then T^* is a caterpillar.*

Proof. Assume for contradiction that T^* is not a caterpillar. Let $P = u_0u_1\cdots u_d$ be a diametrical path of T^*. Since T^* is not a caterpillar, there exists an integer i between 2 and $d-2$ such that u_i has a neighbor $v \notin \{u_{i-1}, u_{i+1}\}$ with $d_{T^*}(v) \geq 2$. Thus, the edge vu_i satisfies $d_{T^*}(v) \geq 2$ and $d_{T^*}(u_i) \geq 2$. Let $T' = T^* - \{vw : w \in N_{T^*}(v) \setminus \{u_i\}\} + \{u_iw : w \in N_{T^*}(v) \setminus \{u_i\}\}$. One can see that T' is also a tree of order n with diameter d. This contradicts the maximality of T^*, since Lemma 2 implies $EU(T') > EU(T^*)$. ■

Lemma 8. *Let T^* be a caterpillar graph of order n and diameter d. If it has two vertices of degree greater than 2, then its Euler-Sombor index EU* *cannot attain the maximum among all trees of order n with diameter d.*

Proof. Let $P = u_0u_1\cdots u_d$ be a diametrical path of T^*. Assume, for contradiction, that there exist $u_i, u_j \in V(P)$ with $i < j$, $d_{T^*}(u_i) = \Delta \geq 3$, and $d_{T^*}(u_j) \geq 3$, where $i \in \{1, 2, ..., d-1\}$. Choose u_i and u_j such that $d_T(u_i, u_j)$ is minimized. Let $x = d_{T^*}(u_i)$ and $y = d_{T^*}(u_j)$. We consider two cases based on whether $d_T(u_i, u_j) = 1$.

Case 1. $d_T(u_i, u_j) = 1$.

Because $i < j$, we have $u_j = u_{i+1}$. Define $T' = T^* - u_{i+1}w_1 + u_iw_1$, where $w_1 \in N_{T^*}(u_{i+1}) \setminus \{u_i, u_{i+2}\}$. Then $\mathrm{diam}(T') = d$.

Now, we show $EU(T') > EU(T^*)$. Since $x \geq y \geq 3$, we have

$$\sqrt{(x+1)^2 + (y-1)^2 + (x+1)(y-1)} > \sqrt{x^2 + y^2 + xy}.$$

By Lemma 7, we have $d_{T'}(w) = d_{T^*}(w) = 1$, where $w \in N_{T^*}(u_i) \cup N_{T^*}(u_{i+1}) \setminus \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$.

Since $x \geq y$,

$$\begin{aligned}&(x-2)(\sqrt{(x+1)^2 + 1 + x + 1} - \sqrt{x^2 + 1 + x})\\ &\quad > (y-3)(\sqrt{y^2 + 1 + y} - \sqrt{(y-1)^2 + 1 + y - 1}).\end{aligned}$$

and

$$\sqrt{(x+1)^2 + x + 2} - \sqrt{y^2 + 1 + y} \geq \sqrt{(y+1)^2 + y + 2} - \sqrt{y^2 + 1 + y}.$$

Since $d_{T'}(u_{i+2}) = d_T(u_{i+2}) \geq 1$, by Lemma 1,

$$\begin{aligned}&\sqrt{y^2 + d_{T'}^2(u_{i+2}) + yd_{T'}(u_{i+2})} - \sqrt{(y-1)^2 + d_{T^*}^2(u_{i+2}) + (y-1)d_{T^*}(u_{i+2})}\\ &= f(y, d_{T'}(u_{i+2}))\\ &\leq f(y, 1) = \sqrt{y^2 + 1 + y} - \sqrt{(y-1)^2 + 1 + y - 1}.\end{aligned}$$

In addition, since $y + 1 > y$, by Lemma 1,

$$(\sqrt{(y+1)^2 + y + 2} - \sqrt{y^2 + 1 + y}) - (\sqrt{y^2 + 1 + y} - \sqrt{(y-1)^2 + y}) > 0.$$

Thus,

$$EU(T') - EU(T^*)$$

$$= (x-2)f(x+1,1) - (y-3)f(y,1) + \sqrt{(x+1)^2+x+2} - \sqrt{y^2+1+y}$$

$$+ f(x+1, d_{T'}(u_{i-1})) - f(y, d_{T'}(u_{i+2}))$$

$$+ (\sqrt{(x+1)^2+(y-1)^2+(x+1)(y-1)} - \sqrt{x^2+y^2+xy})$$

$$> (\sqrt{(x+1)^2+x+2} - \sqrt{y^2+1+y}) - f(y, d_{T'}(u_{i+2}))$$

$$> f(y+1,1) - f(y, d_{T'}(u_{i+2}))$$

$$> f(y+1,1) - f(y,1)$$

$$> 0.$$

Case 2. $d_T(u_i, u_j) \geq 2$.

Since $j \geq i+2$, let $T' = T^* - u_j w_1 + u_i w_1$, where $w_1 \in N(u_j) \setminus \{u_{j-1}, u_{j+1}\}$. Then $diam(T') = d$.

Now we show $EU(T') > EU(T^*)$.

By the choice of u_i and u_j, $d_{T^*}(u_{i+1}) = d_{T'}(u_{i+1}) = 2$ and $d_{T^*}(u_{j-1}) = d_{T'}(u_{j-1}) = 2$.

Since $d_{T'}(u_{j+1}) = d_T(u_{j+1}) \geq 1$ and $x \geq y \geq 3$, by Lemma 1,

$$\sqrt{y^2 + d_{T'}^2(u_{j+1}) + y d_{T'}(u_{j+1})} - \sqrt{(y-1)^2 + d_{T^*}^2(u_{j+1}) + (y-1) d_{T^*}(u_{j+1})}$$

$$\leq \sqrt{y^2+1+y} - \sqrt{(y-1)^2+y}$$

$$\leq \sqrt{(y+1)^2+2+y} - \sqrt{y^2+1+y} = f(y+1,1).$$

Thus,

$$EU(T') - EU(T^*)$$

$$= (x-2)f(x+1,1) - (y-3)f(y,1)$$

$$+ (\sqrt{(x+1)^2+1+x+1} - \sqrt{y^2+1+y})$$

$$+ f(x+1, d_{T'}(u_{i-1})) + f(x+1, 2) - f(y, 2) - f(y, d_{T'}(u_{j+1}))$$

$$> f(y+1, 1) + f(x+1, 2) - f(y+1, 2) - f(y+1, 1)$$

$$> 0.$$

Therefore, the proof is completed. ■

Theorem 3. *Let n and d be two integers with $2 \leq d \leq n-2$. If T is a tree of order n with diameter d, then $EU(T) \leq h(n, d)$, with equality if and only if $T \cong EU(T^1_{n,d})$.*

Proof. Suppose T has the maximum Euler-Sombor index among all trees of order n with diameter d. Lemmas 7 and 8 imply $T \cong T^i_{n,d}$ for some $i \in \{1, 2, \ldots, d-1\}$. Simply put, if $2 \leq i \leq d-2$, then

$$\begin{aligned} EU(T^i_{n,d}) = \quad & 2\sqrt{7} + 2\sqrt{2^2 + (n-d+1)^2 + 2(n-d+1)} + 2(d-4)\sqrt{3} \\ & + (n-d-1)\sqrt{(n-d+1)^2 + 1 + (n-d+1)}, \end{aligned}$$

and

$$\begin{aligned} EU(T^1_{n,d}) = \quad & \sqrt{7} + \sqrt{2^2 + (n-d+1)^2 + 2(n-d+1)} \\ & + (n-d)\sqrt{1^2 + (n-d+1)^2 + n-d+1} + 2(d-3)\sqrt{3}. \end{aligned}$$

Since $n-d+1 > 2$, by Lemma 1, we have

$$\begin{aligned} EU(T^1_{n,d}) - EU(T^i_{n,d}) &= (2\sqrt{3} - \sqrt{7}) \\ &\quad - (\sqrt{2^2 + (n-d+1)^2 + 2(n-d+1)} \\ &\quad - \sqrt{(n-d+1)^2 + n-d+2}) \\ &= f(2, 2) - f(2, n-d+1) \\ &> 0. \end{aligned}$$

Thus, $T \cong T^1_{n,d}$, completing the proof. ■

4 Discussion

In this paper, we give the sharp bounds for the Euler-Sombor index among all trees of order n with the matching number, the number of pendent vertices and the diameter, respectively. Naturally, the Euler-Sombor index is also meaningful for extremal problems in other special graph classes, especially for chemical graphs. For further study, the extremal problems for chemical trees are interesting.

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