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Extremal Euler Sombor Index of Unicyclic Graphs with Fixed Diameter

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Abstract

Topological indices are numerical descriptors of graphs that are widely used in fields such as mathematical chemistry, network theory, and structural analysis. Among the recently introduced degree-based indices, the Euler Sombor index has gained significant attention due to its applicability.

The Euler Sombor index is defined as:

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G(x)^2 + d_G(y)^2 + d_G(x)d_G(y)},$$

where $d_G(x)$ denotes the degree of vertex x in the graph G, and the sum is taken over all edges of G.

In this study, we focus on the minimum value of the Euler Sombor index for the class of unicyclic graphs with a fixed diameter $d \geq 2$.

1 Introduction

This paper considers only finite,connected and undirected graphs. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. The degree of the vertex $x \in V$ is defined as the number of vertices adjacent to x, and it is denoted by $d_G(x)$. The set of all neighbors of vertex x is $N_G(x)$. If $d_G(x) = 1$, then x is called a pendent vertex of G. Let $d(u, v)$ denote the distance between two distinct vertices u and v in a graph G, which is defined as the number of edges in the shortest path connecting them. The diameter of a graph G, denoted by $d(G)$, is defined as the greatest distance between any pair of vertices in the graph, i.e.,

$$d(G) = \max\{d(u, v) \mid u, v \in V(G)\}.$$

A diametral path is the shortest path between two vertices u and v such that $d(u, v) = d(G)$. For simplicity, we denote the diameter $d(G)$ by d. For graph-theoretical notions and terminology used in the present paper, we refer the reader to [3].

A unicyclic graph is a connected graph containing exactly one cycle, and due to this unique cycle, such graphs offer a rich structure for studying degree-based invariants. The diameter constraint imposes a structural limitation, making the identification of extremal graphs both more interesting and more complex.

Topological indices characterize the molecular structure of a graph and are called numerical parameters used to estimate physicochemical information . The Euler Sombor index is introduced in [7, 21], where the Euler Sombor index is defined

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y) + (d_G(x)d_G(y))}. \tag{1}$$

For other studies in the literature related to Euler Sombor index and other Sombor related indices, see [1, 5, 8–12, 18, 22]. Especially in recent years, a lot of work has been done on the extremal value problem of Sombor and Euler Sombor indices. Cruz and Rada [4] investigated the extremal values of the classical Sombor index in unicyclic and bicyclic graphs, identifying the graph structures that attain the minimum and maximum values. Alidadi et al. [2] studied unicyclic graphs with fixed diameter and vertex number, focusing on those achieving the minimum Sombor index. They established lower bounds for graphs with diameter at least two and described the graph structures that meet those bounds. Liu et al. [15] presented a comprehensive review of the Sombor index, summarizing known extremal results and bounds across various graph classes.Their work systematically compiled lower and upper bounds of the index, along with extremal structures that attain these values. Dorjsembe and Horoldagva [6] examined the reduced Sombor index in bicyclic graphs, identifying graph structures that attain the extremal values and discussing the implications for graph theory.

Shooshtari et al. [19] focused on modified Sombor index in unicyclic graphs with fixed diameter, determining the lower bounds and characterizing the graphs that minimize this index. Zhang and Zhao [23] investigated the extremal values of the Sombor index in tricyclic graphs, identifying the graph structures that attain the minimum and maximum values. Khanra and Das [13] conducted a systematic study of the Euler Sombor index across trees, unicyclic, and chemical graphs, comparing how the index behaves in different graph classes and offering new theoretical insights. Kızılırmak [14] analyzed the Euler Sombor index in tricyclic graphs, identifying the structures that attain the maximum and minimum values of the index.

Su and Tang [20]investigated unicyclic and bicyclic graphs with respect to their Euler Sombor index, characterizing extremal graphs and elaborating on the structural implications. Liu [16] explored unicyclic graphs with fixed diameter and number of vertices that achieve the maximum Sombor index, offering a structural characterization of extremal graphs. Liu et al. [17] focused on tetracyclic (chemical) graphs, examining extremal values of the Sombor index and its relationships with graph parameters. Unicyclic graphs, which are connected graphs containing exactly one cycle, offer a rich structure for exploring degree-based invariants. The diameter constraint imposes a structural limitation, making the identification of extremal graphs both more interesting and more complex. In this paper, we aim to present the minimum Euler Sombor index for unicyclic graphs with the diameter $d \geq 2$.

2 Main results

In this section, we will provide a lower bound for the Euler Sombor index on unicyclic graphs with diameter $d \geq 3$. First, we present two Lemmas that will be used in the theorems.

Lemma 1. *Let G be a unicyclic graph and let H be a diametral path in G. If there exists a pendent vertex $v \notin V(H)$, then one can construct a unicyclic subgraph $G^* \subset G$ such that $v \notin V(G^*)$, the diameter remains unchanged, and the Euler Sombor index satisfies $EU(G) > EU(G^*)$.*

Proof. Let H be a diametral path in G and suppose $v \in V(G)$ is a pendent vertex such that $v \notin V(H)$. Let u denote the closest vertex to v for which $d_G(u) \neq 2$. Define the subgraph $G^* \subset G$ to be the graph obtained by deleting the path connecting u and v from G. Let x be the neighbor of u along the deleted path (in case the path has only one edge, we set $x = v$). It is clear that G^* remains a unicyclic graph and preserves the diameter, i.e., $d(G^*) = d(G)$. We then obtain the following inequality:

$$\begin{aligned}
EU(G) - EU(G^*) \quad &\geq \quad \sqrt{d_u^2 + 1 + d_u} \\
&+ \sum_{y \in N(u) \setminus \{x\}} \sqrt{d_u^2 + d_y^2 + d_u d_y} \\
&- \sum_{y \in N(u) \setminus \{x\}} \sqrt{(d_u - 1)^2 + d_y^2 + (d_u - 1) d_y} \\
&> \quad 0.
\end{aligned}$$

This confirms that the Euler Sombor index of G strictly exceeds that of G^*, i.e., $EU(G) > EU(G^*)$. ■

Lemma 2 ([13]). *Among all unicyclic graphs with $n \geq 3$ vertices, the unique graph that attains the minimum Euler Sombor index is the cycle C_n, and the minimum value is $2\sqrt{3}n$.*

In light of Lemma 2, our focus in this paper will be on unicyclic graphs that contain at least one pendent vertex.

Theorem 1. *Let G be a unicyclic graph with n vertices and diameter d, where $d \geq 3$ and $n \geq d+2$. Then the following inequality holds:*

$$EU(G) \geq EU(H_1) = 3\sqrt{19} + \sqrt{7} + 2\sqrt{3}(n-4),$$

where H_1 denotes the graph constructed by attaching a path of length $2d - n + 1$ to a vertex of the cycle $C_{2n-2d-1}$.

Proof. Because G has at least one pendent vertex, we divide our discussion into three separate cases.

Case 1: G has exactly one pendent vertex.
The graph G consists of a path P with $m \geq 1$ edges and a cycle C_l with $l \geq 3$. The cycle C_l and path P share a common vertex of degree 3 in G. Thus,

$$EU(G) = EU(C_l) + EU(P). \tag{1}$$

First, let $m \geq 2$ and $l \geq 4$. The path P contains

- $(m-2)$ edges of d-coordinate $(2,2)$ for $i = 1, 2$,
- one $(1,2)$-edge,
- one $(2,3)$-edge.

Hence, we get

$$EU(P) = \sqrt{19} + \sqrt{7} + 2\sqrt{3}(m-2) \tag{2}$$

On the other hand, the cycle C_l contains $l-2$ edges of d-coordinate $(2,2)$ and two $(2,3)$-edges. Therefore, we have

$$EU(C_l) = 2\sqrt{19} + 2\sqrt{3}(l-2). \tag{3}$$

By substituting equations (2) and (3) into equation (1), we get

$$EU(G) = 3\sqrt{19} + \sqrt{7} + 2\sqrt{3}(m+l-4).$$

Since $n = l + m$, we obtain

$$EU(G) \geq 3\sqrt{19} + \sqrt{7} + 2\sqrt{3}(n-4) \geq EU(H_1).$$

If $m = 1$ and $l \geq 4$, then $n = l + 1$ and $EU(P) = \sqrt{13}$. Therefore,

$$\begin{aligned} EU(G) &= 2\sqrt{19} + \sqrt{13} + 2\sqrt{3}(l-2) \\ &\geq (n-3)\sqrt{8} + 2\sqrt{13} + \sqrt{10} \geq EU(H_1). \end{aligned}$$

If $l = 3$ and $m \geq 2$, then $n = m + 1$ and the Euler Sombor index reaches the minimum bound:

$$EU(G) = 3\sqrt{19} + \sqrt{7} + 2\sqrt{3}(n-4) = EU(H_1).$$

Case 2: The graph G has precisely two pendent vertices.
In this scenario, G comprises two distinct paths, P and P', with $m_1 \geq 1$ and $m_2 \geq 1$ edges, respectively. Furthermore, the graph includes a cycle C_l where $l \geq 3$. Without loss of generality, we may assume that C_l shares a vertex with P, and that P' is connected either to a vertex on C_l or to an internal vertex of P.

Subcase 2-1: We consider the case where P and P' do not intersect.

$$EU(G) = EU(C_l) + EU(P) + EU(P').$$

First, let $l \geq 4$ and $m_1, m_2 \geq 2$. Each path P and P' contains:

- $(m_i - 2)$ edges of d-coordinate $(2, 2)$ for $i = 1, 2$,
- one $(1, 2)$-edge,
- one $(2, 3)$-edge.

This gives:

$$EU(P) = \sqrt{19} + \sqrt{7} + 2\sqrt{3}(m_1 - 2), \quad EU(P') = \sqrt{19} + \sqrt{7} + 2\sqrt{3}(m_2 - 2).$$

If P and P' are connected to non-adjacent vertices of C_l, then the cycle contains four $(2, 3)$-edges and $(l - 4)$ edges of d-coordinate $(2, 2)$, so we have

$$EU(C_l) = 4\sqrt{19} + 2\sqrt{3}(l-4). \tag{4}$$

On the other hand, if P and P' are connected to adjacent vertices of C_l, then the cycle has:

- two $(2,3)$-edges,
- one $(3,3)$-edge,
- and $(l-3)$ edges of d-coordinate $(2,2)$.

Therefor we get

$$EU(C_l) = 2\sqrt{19} + 3\sqrt{3} + 2\sqrt{3}(l-3). \tag{5}$$

Since $n = l + m_1 + m_2$ and expression (4) is greater than (5), we deduce the following inequalities:

$$\begin{aligned} EU(G) &\geq 2\sqrt{3}(l + m_1 + m_2 - 7) + 4\sqrt{19} + 2\sqrt{7} + 3\sqrt{3} \\ &\geq 2\sqrt{3}(n-7) + 4\sqrt{19} + 2\sqrt{7} + 3\sqrt{3} \\ &\geq 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

If $l \geq 4$ and $m_1 = 1$, $m_2 \geq 2$, then $n = l + m_2 + 1$, and $EU(P) = \sqrt{13}$ and

$$EU(P') = 2\sqrt{3}(m_2 - 2) + \sqrt{19} + \sqrt{7}.$$

Hence we have

$$\begin{aligned} EU(G) &\geq 2\sqrt{3}(l + m_2 - 5) + 3\sqrt{19} + 3\sqrt{3} + \sqrt{13} + \sqrt{7} \\ &\geq 2\sqrt{3}(n-6) + 3\sqrt{19} + 3\sqrt{3} + \sqrt{13} + \sqrt{7} \\ &\geq 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

If $l \geq 4$ and $m_1 = m_2 = 1$, then $n = l + 2$ and $EU(P) + EU(P') = 2\sqrt{13}$. This implies that

$$\begin{aligned} EU(G) &\geq 2\sqrt{3}(l-3) + 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3} \\ &\geq 2\sqrt{3}(n-5) + 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3} \\ &\geq 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

When $l = 3$ and $m_1, m_2 \geq 2$, then $n = m_1 + m_2 + 3$ and $EU(C_l) = 2\sqrt{19} + 3\sqrt{3}$. Thus,

$$\begin{aligned} EU(G) &= 2\sqrt{3}(m_1 + m_2 - 4) + 4\sqrt{19} + 3\sqrt{3} + 2\sqrt{7} \\ &= 2\sqrt{3}(n-7) + 4\sqrt{19} + 3\sqrt{3} + 2\sqrt{7} \\ &\geq 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

If $l = 3$ and $m_1 = m_2 = 1$, then $d = 3$ and $n = 5$. We get

$$\begin{aligned} EU(G) &= 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3} \\ &> 2\sqrt{3} + 3\sqrt{19} + \sqrt{7} \\ &= 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

Finally, when $l = 3$ and $m_1 = 1$, $m_2 \geq 2$, then $n = m_2 + 4$. Thus, it can be concluded that

$$\begin{aligned} EU(G) &\geq 2\sqrt{3}(m_2 - 2) + \sqrt{7} + \sqrt{13} + 3\sqrt{19} + 3\sqrt{3} \\ &= 2\sqrt{3}(n-6) + 3\sqrt{19} + \sqrt{7} \\ &\geq 2\sqrt{3}(n-4) + 3\sqrt{19} + \sqrt{7}. \end{aligned}$$

Subcase 2-2: Suppose $P \cap P' \neq \emptyset$, and there exists a diametral path H that contains both pendent vertices of the graph G. Under this condition, the inclusion $H \subseteq P \cup P'$ holds.

If both paths P and P' are attached to a vertex of the cycle C_l, then an interior vertex on H, say u, must have degree 4 in the graph G. Alternatively, if P' is joined to an internal vertex of P, then one of the interior vertices on H, labeled u, has degree 3 in G.

Now, assuming that u is not adjacent to any pendent vertex along H, the following inequalities can be derived:

$$EU(H) \geq 2\sqrt{3}(d-4) + 2\sqrt{7} + 2\sqrt{19}$$

or

$$EU(H) \geq 2\sqrt{3}(d-4) + 6\sqrt{7}.$$

It follows that the lesser value provides a valid lower bound for the Euler Sombor index of the graph G:

$$EU(H) \geq 2\sqrt{3}(d-4) + 2\sqrt{7} + 2\sqrt{19}. \tag{6}$$

Let u be the neighbor of a pendent vertex of H. Then we get

$$EU(H) \geq 2\sqrt{3}(d-3) + 3\sqrt{7} + \sqrt{21}. \tag{7}$$

Observe that in this particular case, the condition $d \geq 4$ holds. Consequently, inequality (6) provides a tighter lower bound than inequality (7), leading to the conclusion that:

$$EU(H) \geq 2\sqrt{3}(d-4) + 2\sqrt{7} + 2\sqrt{19}.$$

Furthermore, the graph G contains the cycle C_l, which includes a vertex of degree 3 or 4. Hence, we get

$$EU(C_l) \geq 2\sqrt{3}(l-2) + 2\sqrt{19}$$

or

$$EU(C_l) \geq 2\sqrt{3}(l-2) + 4\sqrt{7} > 2\sqrt{3}(l-2) + 2\sqrt{19}.$$

Since $n = l + d$, we obtain

$$\begin{aligned} EU(G) &\geq EU(C_l) + EU(H) \\ &\geq 2\sqrt{3}(d+l-6) + 4\sqrt{19} + 2\sqrt{7} \\ &= 2\sqrt{3}(n-6) + 4\sqrt{19} + 2\sqrt{7} \\ &\geq EU(H_1). \end{aligned}$$

Subcase 2-3: Assume that $P \cap P' \neq \emptyset$ and there exists a diametral path H in G that contains exactly one pendent vertex. In this situation, since the other pendent vertex does not lie on H, Lemma 1 ensures the existence of a unicyclic subgraph $G^* \subset G$ such that only one pendent vertex lies on

H, and the following conditions are satisfied:

$$d(G^*) = d(G) \quad \text{and} \quad EU(G) > EU(G^*).$$

Based on Case 1, the following inequality then holds:

$$EU(G^*) \geq EU(H_1).$$

Case 3: Let the graph G have no fewer than three pendent vertices.

Assume that H is a diametral path in G. It is evident that H can include at most two of the pendent vertices. Given that G has $m \geq 3$ pendent vertices, it follows that at least $m-2$ of them are not situated on H.

According to Lemma 1, one can construct a unicyclic subgraph $G^* \subset G$ which retains only those pendent vertices that lie on H, and satisfies the following properties:

$$d(G^*) = d(G) \quad \text{and} \quad EU(G) > EU(G^*).$$

By applying the same reasoning as in Case 1, we obtain:

$$EU(G^*) \geq EU(H_1). \qquad \blacksquare$$

Theorem 2. *Let G be a unicyclic graph with diameter $d \geq 3$ and $n \geq 2d$. Then we have*

$$EU(G) \geq EU(H_2) = 2\sqrt{3}(n-3) + 2\sqrt{19} + \sqrt{13},$$

where H_2 is the graph obtained by attaching a pendent vertex to one vertex of the cycle C_{2d-1}.

Proof. Suppose that G has exactly one pendent vertex. In this case, we have $n = 2d$, and it follows that $G \cong H_2$, which verifies the statement.

Next, consider the situation where G has precisely two pendent vertices. Under this condition, the number of vertices satisfies $n = 2d$ or $n = 2d+1$.

In the case of $n = 2d$, there exists a diametral path H in G that includes only one of the pendent vertices. Applying Lemma 1, we obtain a unicyclic subgraph $G^* \subset G$ that retains only the pendent vertex lying on H, and satisfies the following:

$$d(G^*) = d(G) \quad \text{and} \quad EU(G) > EU(G^*).$$

Hence, we conclude:

$$EU(G) > EU(G^*) \geq EU(H_2) = 2\sqrt{3}(n-3) + 2\sqrt{19} + \sqrt{13}.$$

If $n = 2d+1$, then G is the graph obtained by connecting two pendent vertices to the cycle C_{2d-1}.

If both pendent vertices are connected to the same vertex of G, we get

$$\begin{aligned} EU(G) &= (2d-3)\sqrt{8} + 2\sqrt{20} + 2\sqrt{17} \\ &= 2\sqrt{3}(n-4) + 4\sqrt{7} + 2\sqrt{17} \\ &\geq 2\sqrt{3}(n-3) + 2\sqrt{19} + \sqrt{13} \\ &= EU(H_2). \end{aligned}$$

If the pendent vertices are connected to two different vertices of G, we have

$$EU(G) = 2\sqrt{3}(2d-4) + 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3}$$

or

$$EU(G) = 2\sqrt{3}(2d-5) + 4\sqrt{19} + 2\sqrt{13}.$$

Hence,

$$\begin{aligned} EU(G) &\geq 2\sqrt{3}(2d-4) + 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3} \\ &= 2\sqrt{3}(n-5) + 2\sqrt{19} + 2\sqrt{13} + 3\sqrt{3} \\ &\geq 2\sqrt{3}(n-3) + 2\sqrt{19} + \sqrt{13} \\ &= EU(H_2). \end{aligned}$$

Alternatively, if G has three or more pendent vertices, then at most two of these can be located on the diametral path H. Applying Lemma 1, one can find a unicyclic subgraph $G^* \subset G$ that includes only the pendent vertices that lie on H, satisfying the following:

$$d(G^*) = d(G) \quad \text{and} \quad EU(G) > EU(G^*).$$

As a result, the following inequality holds:

$$EU(G) > EU(G^*) \geq EU(H_2). \qquad \blacksquare$$

Theorem 3. *Let G be a unicyclic graph with diameter $d = 2$, then:*

$$EU(G) \geq 4\sqrt{3}d.$$

Proof. The unicyclic graphs G with diameter $d = 2$ fall into one of the following categories: they are either the cycle C_4 with Sombor index $EU(G) = 8\sqrt{3}$, the cycle C_5 with $EU(G) = 10\sqrt{3}$, or a graph obtained by attaching at least one pendent vertex to a vertex of the cycle C_3.

Let $V(C_3) = \{x_1, x_2, x_3\}$, and suppose that $y_1, y_2, \ldots, y_k$ are pendent vertices connected to x_1. In this case, the path $x_2x_1y_1$ forms a diametral path in G.

By Lemma 1, there exists a unicyclic subgraph $G^* \subset G$ which contains only the pendent vertex y_1, and satisfies:

$$d(G^*) = d(G) \quad \text{and} \quad EU(G) > EU(G^*).$$

It follows that:

$$EU(G^*) = 2\sqrt{3} + 2\sqrt{19} + \sqrt{7}.$$

Therefore, we obtain:

$$EU(G) \geq 8\sqrt{3} = 4\sqrt{3}d. \qquad \blacksquare$$

3 Conclusions

In this study, the minimum Euler Sombor index for unicyclic graphs with diameter $d \geq 2$ was presented. The results contribute to the characterization of structures that minimize the Euler Sombor index among such graphs. In future work, some lower and upper bounds on the Euler Sombor index of a graph G in terms of various graph parameters (such as clique number, chromatic number, number of pendant vertices, etc.) are intended to be established, and the extremal graphs that attain these bounds will be characterized.

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