MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **94** (2025) 701–724
ISSN: 0340–6253
doi: 10.46793/match.94-3.02525

Sombor-Type Indices for Certain Interconnection Networks

Zhen Lin*

School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai, China

lnlinzhen@163.com

(Received February 8, 2025)

Editor's note

This paper was created in 2021, but not published until now. Since then, major progress happened in the theory of Sombor-type topological indices. It is remarkable that the present author anticipated many aspects of the theory that emerged in the meantime: exponential and multiplicative Sombor-type indices, elliptic Sombor index, etc. Some other variants of the Sombor index are put forward here, that so far eluded the attention of colleagues. For these reasons, the present paper is both interesting as a historic document, and a motivation for further studies.

Abstract

The Sombor-type indices, recently introduced by Gutman, are novel the vertex-degree-based topological indices. Based on these, we study the Sombor-type indices and other new vertex-degree-based topological indices for butterfly networks, augmented butterfly networks, enhanced butterfly networks, Benes networks, mesh derived networks and the Optical Transpose Interconnection System (OTIS) networks, which provide a new kind of prediction indices for these networks.

*Corresponding author.

1 Introduction

Cheminformatics is a new interdiscipline composed of chemistry, mathematics and information science, which contributes a major role in the field of chemical sciences by implementing graph theory to mathematical modeling of chemical occurrence. In cheminformatics study, the topological indices play a significant role in predicting the biological activities and properties of chemical compounds due to the fact that the numerical characteristics of topological indices reflect certain physico-chemical properties of chemical compounds, such as boiling point, acentric factor, stability, strain energy etc. A large number of topological indices have been studied in the models of Quantitative structure-activity relationships (QSAR) and structure-property relationships (QSPR), such as Wiener index, Randić index, Zagreb index, ABC index and so on.

Let G be a simple undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. Denote by d_v and $\overline{d}$ the degree of vertex v and the average vertex degree of the graph G, respectively. In cheminformatics, several dozens of vertex-degree-based topological indices have been introduced and extensively studied [31]. Their general formula is

$$DBI(G) = \sum_{uv \in E(G)} f(d_u, d_v),$$

where $f(d_u, d_v)$ is a function of d_u and d_v.

In 1972, the first and second Zagreb indices of a graph G, introduced by Gutman and Trinajstić [17], are respectively defined as

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{v \in V(G)} d_v^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_u d_v,$$

which characterize the degree of branching in molecular carbon-atom skeleton and are regarded as powerful molecular structure-descriptors [5, 31]. Moreover, many mathematical properties such as lower and upper bounds involving other important graphical invariants are studied extensively and deeply [3].

In 2021, the Sombor-type indices of a graph G are respectively defined as

$$
\begin{aligned}
SO(G) &= \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}, \\
SO_{red}(G) &= \sum_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}, \\
SO_{avg}(G) &= \sum_{uv\in E(G)} \sqrt{(d_u-\overline{d})^2+(d_v-\overline{d})^2},
\end{aligned}
$$

which are the novel vertex-degree-based topological indices proposed by Gutman [14]. Recently, researches showed that the Sombor-type indices play an important role in theoretical chemistry. Deng et al. [12] showed that the Sombor index can help to predict these physico-chemical properties of octane isomers and confirmed suitability of the Sombor index in QSPR analysis. Redžepović [28] showed that the Sombor index may be used successfully on modeling thermodynamic properties of compounds due to the fact that the Sombor index has satisfactory prediction potential in modeling entropy and enthalpy of vaporization of alkanes. And statistical data indicate that the reduced Sombor index preforms with slightly better predictive potential. In addition, much work has been done to study the extremal graph according to the Sombor-type indices, we refer the reader to [2,6,13,22] and the references therein. On the other hand, the Sombor-type indices have been an interesting topic in mathematical literature and have been studied extensively, one may refer to [7,8,11,23,27,30,32] and the references therein.

Let $K = \{(i,j) \in N \times N : 1 \le i \le j \le n-1\}$ and $m_{i,j}(G)$ be the number of edges in G joining vertices of degree i and j. Then the vertex-degree-based topological index is defined for any set of numbers $\{\varphi_{i,j}\}_{(i,j)\in K}$ as

$$\varphi(G) = \sum_{(i,j)\in K} m_{i,j}(G)\varphi_{i,j}.$$

If $\varphi_{i,j} = i+j$, $\varphi_{i,j} = ij$ and $\varphi_{i,j} = \sqrt{i^2+j^2}$, then we recover the first Zagreb index, the second Zagreb index and Sombor index, respectively. The discrimination ability of topological indices is an important aspect in the study of topological indices [9,10,20]. In view of this, the exponential of a vertex-degree-based topological index was introduced in [29]. Inspired by this, the Estrada-Sombor-type indices are defined as

$$\begin{aligned} e^{SO(G)} &= \sum_{uv\in E(G)} e^{\sqrt{d_u^2+d_v^2}} = \sum_{(i,j)\in K} m_{i,j}(G)e^{\sqrt{i^2+j^2}}, \\ e^{SO_{red}(G)} &= \sum_{uv\in E(G)} e^{\sqrt{(d_u-1)^2+(d_v-1)^2}} = \sum_{(i,j)\in K} m_{i,j}(G)e^{\sqrt{(i-1)^2+(j-1)^2}}, \\ e^{SO_{avg}(G)} &= \sum_{uv\in E(G)} e^{\sqrt{(d_u-\overline{d})^2+(d_v-\overline{d})^2}} = \sum_{(i,j)\in K} m_{i,j}(G)e^{\sqrt{(i-\overline{d})^2+(j-\overline{d})^2}}. \end{aligned}$$

Since many quantities in the network science are often expressed in logarithmic form, we propose the logarithmic of a vertex-degree-based topological index as follows:

$$\begin{aligned} \ln DBI(G) &= \sum_{uv\in E(G)} \ln f(d_u,d_v) = \ln \prod_{uv\in E(G)} f(d_u,d_v) \\ &= \ln \prod_{(i,j)\in K} m_{i,j}(G)\varphi_{i,j}, \end{aligned}$$

where $f(d_u, d_v) > 0$ is a function of d_u and d_v. In fact, it's just a change of form on the multiplicative topological indices [15]. For a connected graph G with $n \geq 3$ vertices, we put forward the logarithmic of Sombor-type indices as follows:

$$\begin{aligned} \ln(SO(G)) &= \sum_{uv\in E(G)} \ln\sqrt{d_u^2+d_v^2} = \ln \prod_{uv\in E(G)} \sqrt{d_u^2+d_v^2}, \\ \ln(SO_{red}(G)) &= \sum_{uv\in E(G)} \ln\sqrt{(d_u-1)^2+(d_v-1)^2} \\ &= \ln \prod_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}. \end{aligned}$$

Gutman [16], Das et al. [8], Milovanović et al. [27] and Wang et al. [32] gave the many mathematical relations between the Sombor index and the first and second Zagreb indices. In order to better predict, we mix the first and second Zagreb indices and the Sombor-type indices together, and propose some new vertex-degree-based topological indices, called the

Zagreb-Sombor-type indices, as follows:

$$\begin{aligned}
ZSO_1(G) &= \sum_{uv\in E(G)} \left(\sqrt{d_u^2+d_v^2}+d_u+d_v\right) \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)\left(\sqrt{i^2+j^2}+i+j\right), \\
ZSO_2(G) &= \sum_{uv\in E(G)} \left(\sqrt{d_u^2+d_v^2}-d_u-d_v\right) \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)\left(\sqrt{i^2+j^2}-i-j\right), \\
ZSO_3(G) &= \sum_{uv\in E(G)} (d_u+d_v)\sqrt{d_u^2+d_v^2} \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)(i+j)\sqrt{i^2+j^2}, \\
ZSO_4(G) &= \sum_{uv\in E(G)} \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v} \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)\frac{\sqrt{i^2+j^2}}{i+j}, \\
MSO_1(G) &= \sum_{uv\in E(G)} \left(\sqrt{d_u^2+d_v^2}+d_ud_v\right) \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)\left(\sqrt{i^2+j^2}+ij\right), \\
MSO_2(G) &= \sum_{uv\in E(G)} \left(\sqrt{d_u^2+d_v^2}-d_ud_v\right) \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)\left(\sqrt{i^2+j^2}-ij\right), \\
MSO_3(G) &= \sum_{uv\in E(G)} d_ud_v\sqrt{d_u^2+d_v^2} \\
&= \sum_{(i,\,j)\in K} m_{i,\,j}(G)ij\sqrt{i^2+j^2}, \\
MSO_4(G) &= \sum_{uv\in E(G)} \frac{\sqrt{d_u^2+d_v^2}}{d_ud_v}
\end{aligned}$$

$$= \sum_{(i,j)\in K} m_{i,j}(G)\frac{\sqrt{i^2+j^2}}{ij},$$

where $MSO_4(G)$ is called the first Banhatti-Sombor index introduced by Kulli [21]. In [26], we established the mathematical relations between the first Banhatti-Sombor index and some other well-known vertex-degree-based topological indices.

The generalization of topological index is a trend of mathematical chemistry. In 1998, Bollobás and Erdős [4] studied the general Randić index of a graph G as follows:

$$R_p = \sum_{uv\in E(G)} (d_u d_v)^p,$$

where p is a real number. In 2005, the first general Zagreb index of a graph G was introduced by Li and Zhao [25] and is defined as

$$Z_p = \sum_{v\in V(G)} d^p = \sum_{uv\in E(G)} (d_u^{p-1} + d_v^{p-1}),$$

where p is a real number. According to Gutman, Sombor-type indices are the Euclidean metric of degrees. Meanwhile, he proposed to use other distance function [16]. Beyond the Euclidean metric, Réti et al. [30] defined the p-Sombor index as

$$SO_p(G) = \sum_{uv\in E(G)} (d_u^p + d_v^p)^{\frac{1}{p}},$$

where $p \neq 0$. Motivated by the above work, for any real $p \neq 0$, the general Zagreb-Sombor-type indices are defined as

$$\begin{aligned}
GZSO_1(G) &= \sum_{uv\in E(G)} \left((d_u^p + d_v^p)^{1/p} + d_u^{p-1} + d_v^{p-1}\right), \\
GZSO_2(G) &= \sum_{uv\in E(G)} \left((d_u^p + d_v^p)^{1/p} - d_u^{p-1} - d_v^{p-1}\right), \\
GZSO_3(G) &= \sum_{uv\in E(G)} (d_u^{p-1} + d_v^{p-1})(d_u^p + d_v^p)^{1/p},
\end{aligned}$$

$$
\begin{aligned}
GZSO_4(G) &= \sum_{uv\in E(G)} \frac{(d_u^p+d_v^p)^{1/p}}{d_u^{p-1}+d_v^{p-1}},\\
GMSO_1(G) &= \sum_{uv\in E(G)} \left((d_u^p+d_v^p)^{1/p}+d_u^{p/2}d_v^{p/2}\right),\\
GMSO_2(G) &= \sum_{uv\in E(G)} \left((d_u^p+d_v^p)^{1/p}-d_u^{p/2}d_v^{p/2}\right),\\
GMSO_3(G) &= \sum_{uv\in E(G)} d_u^{p/2}d_v^{p/2}(d_u^p+d_v^p)^{1/p},\\
GMSO_4(G) &= \sum_{uv\in E(G)} \frac{(d_u^p+d_v^p)^{1/p}}{d_u^{p/2}d_v^{p/2}}.
\end{aligned}
$$

Recently, the various topological indices of some classical networks are calculated extensively. Imran et al. [19] calculated the general Randić, first Zagreb, ABC, and GA indices for butterfly networks, Benes networks, mesh derived networks. Aslam et al. [1] computed the general Randić, first and second Zagreb, general sum connectivity, first and second multiple Zagreb, hyper Zagreb, ABC and GA indices for Optical Transpose Interconnection System (OTIS) networks. For other related results, one may refer to [18, 24] and the references therein. In this paper, we compute the Sombor-type, Estrada-Sombor-type, logarithmic-Sombor-type and Zagreb-Sombor-type indices for butterfly network, augmented butterfly networks, enhanced butterfly networks, Benes networks , mesh derived networks and the Optical Transpose Interconnection System (OTIS) networks , which provide a new kind of prediction indices for these networks. The definition, application and research value of these networks are followed in [1, 19], we omit here.

2 Results for butterfly networks

Theorem 1. *Let G be the r-dimensional butterfly network $BF(r)$. Then*
(1) $SO(G) = 2^{r+3}(\sqrt{2}r + \sqrt{5} - 2\sqrt{2})$.
(2) $SO_{red}(G) = 2^{r+1}(3\sqrt{2}r + 2\sqrt{10} - 6\sqrt{2})$.
(3) $SO_{avg}(G) = \frac{2^{r+3}}{r+1}[\sqrt{r^2 - 2r + 5} + \sqrt{2}(r - 2)]$.
(4) $e^{SO(G)} = 2^{r+1}\left[(r - 2)e^{4\sqrt{2}} + 2e^{2\sqrt{5}}\right]$.

(5) $e^{SO_{red}(G)} = 2^{r+1}\left[(r-2)e^{3\sqrt{2}} + 2e^{\sqrt{10}}\right]$.

(6) $e^{SO_{avg}(G)} = 2^{r+1}[2e^{\frac{2\sqrt{r^2-2r+5}}{r+1}} + (r-2)e^{\frac{4\sqrt{2}}{r+1}}]$.

(7) $\ln(SO(G)) = 2^r[2\ln 20 + (r-2)\ln 32]$.

(8) $\ln(SO_{red}(G)) = 2^r[2\ln 10 + (r-2)\ln 18]$.

(9) $ZSO_1(G) = 2^{r+3}[(2+\sqrt{2})r + \sqrt{5} - 2\sqrt{2} - 1]$.

(10) $ZSO_2(G) = 2^{r+3}[(\sqrt{2}-2)r + \sqrt{5} - 2\sqrt{2} + 1]$.

(11) $ZSO_3(G) = 2^{r+4}[4\sqrt{2}r + 3\sqrt{5} - 8\sqrt{2}]$.

(12) $ZSO_4(G) = 2^{r+1}\left[\frac{\sqrt{2}}{2}(r-2) + \frac{2\sqrt{5}}{3}\right]$.

(13) $MSO_1(G) = 2^{r+3}[(4+\sqrt{2})r + \sqrt{5} - 2\sqrt{2} - 4]$.

(14) $MSO_2(G) = 2^{r+3}[(\sqrt{2}-4)r + \sqrt{5} - 2\sqrt{2} + 4]$.

(15) $MSO_3(G) = 2^{r+6}[2\sqrt{2}r + \sqrt{5} - 4\sqrt{2}]$.

(16) $MSO_4(G) = 2^{r-1}(\sqrt{2}r + 2\sqrt{5} - 2\sqrt{2})$.

Proof. By the definition of the r-dimensional butterfly network, we obtain the basic information on the r-dimensional butterfly network in the following table.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{2,4}$	$m_{4,4}$
$(r+1)2^r$	$r2^{r+1}$	2^{r+2}	$(r-2)2^{r+1}$

(1) For $SO(G)$, we have

$$\begin{aligned} SO(G) &= 2^{r+2}\sqrt{2^2+4^2} + (r-2)2^{r+1}\sqrt{4^2+4^2} \\ &= 2^{r+3}(\sqrt{2}r + \sqrt{5} - 2\sqrt{2}). \end{aligned}$$

(2) For $SO_{red}(G)$, we have

$$\begin{aligned} SO_{red}(G) &= 2^{r+2}\sqrt{1^2+3^2} + (r-2)2^{r+1}\sqrt{3^2+3^2} \\ &= 2^{r+1}(3\sqrt{2}r + 2\sqrt{10} - 6\sqrt{2}). \end{aligned}$$

(3) For $SO_{avg}(G)$, we have that the average vertex degree of G is $\frac{4r}{r+1}$.

Thus

$$\begin{aligned} SO_{avg}(G) &= 2^{r+2}\sqrt{\left(2-\frac{4r}{r+1}\right)^2+\left(4-\frac{4r}{r+1}\right)^2} \\ &\quad +(r-2)2^{r+1}\sqrt{\left(4-\frac{4r}{r+1}\right)^2+\left(4-\frac{4r}{r+1}\right)^2} \\ &= \frac{2^{r+3}}{r+1}[\sqrt{r^2-2r+5}+\sqrt{2}(r-2)]. \end{aligned}$$

(4) For $e^{SO(G)}$, we have

$$\begin{aligned} e^{SO(G)} &= 2^{r+2}e^{\sqrt{2^2+4^2}}+(r-2)2^{r+1}e^{\sqrt{4^2+4^2}} \\ &= 2^{r+1}\left[(r-2)e^{4\sqrt{2}}+2e^{2\sqrt{5}}\right]. \end{aligned}$$

(5) For $e^{SO_{red}(G)}$, we have

$$\begin{aligned} e^{SO_{red}(G)} &= 2^{r+2}e^{\sqrt{1^2+3^2}}+(r-2)2^{r+1}e^{\sqrt{3^2+3^2}} \\ &= 2^{r+1}\left[(r-2)e^{3\sqrt{2}}+2e^{\sqrt{10}}\right]. \end{aligned}$$

(6) For $e^{SO_{avg}(G)}$, we have

$$\begin{aligned} e^{SO_{avg}(G)} &= 2^{r+2}e^{\sqrt{\left(2-\frac{4r}{r+1}\right)^2+\left(4-\frac{4r}{r+1}\right)^2}}+(r-2)2^{r+1}e^{\sqrt{\left(4-\frac{4r}{r+1}\right)^2+\left(4-\frac{4r}{r+1}\right)^2}} \\ &= 2^{r+1}[2e^{\frac{2\sqrt{r^2-2r+5}}{r+1}}+(r-2)e^{\frac{4\sqrt{2}}{r+1}}]. \end{aligned}$$

(7) For $\ln(SO(G))$, we have

$$\begin{aligned} \ln(SO(G)) &= 2^{r+2}\ln\sqrt{2^2+4^2}+(r-2)2^{r+1}\ln\sqrt{4^2+4^2} \\ &= 2^r[2\ln 20+(r-2)\ln 32]. \end{aligned}$$

(8) For $\ln(SO_{red}(G))$, we have

$$\begin{aligned} \ln(SO_{red}(G)) &= 2^{r+2}\ln\sqrt{1^2+3^2}+(r-2)2^{r+1}\ln\sqrt{3^2+3^2} \\ &= 2^r[2\ln 10+(r-2)\ln 18]. \end{aligned}$$

(9) For $ZSO_1(G)$, we have

$$\begin{aligned} ZSO_1(G) &= 2^{r+2}(\sqrt{2^2+4^2}+6)+(r-2)2^{r+1}(\sqrt{4^2+4^2}+8) \\ &= 2^{r+3}[(2+\sqrt{2})r+\sqrt{5}-2\sqrt{2}-1]. \end{aligned}$$

(10) For $ZSO_2(G)$, we have

$$\begin{aligned} ZSO_2(G) &= 2^{r+2}(\sqrt{2^2+4^2}-6)+(r-2)2^{r+1}(\sqrt{4^2+4^2}-8) \\ &= 2^{r+3}[(\sqrt{2}-2)r+\sqrt{5}-2\sqrt{2}+1]. \end{aligned}$$

(11) For $ZSO_3(G)$, we have

$$\begin{aligned} ZSO_3(G) &= 2^{r+2}(6\sqrt{2^2+4^2})+(r-2)2^{r+1}(8\sqrt{4^2+4^2}) \\ &= 2^{r+4}[4\sqrt{2}r+3\sqrt{5}-8\sqrt{2}]. \end{aligned}$$

(12) For $ZSO_4(G)$, we have

$$\begin{aligned} ZSO_4(G) &= 2^{r+2}\left(\frac{\sqrt{2^2+4^2}}{6}\right)+(r-2)2^{r+1}\left(\frac{\sqrt{4^2+4^2}}{8}\right) \\ &= 2^{r+1}\left[\frac{\sqrt{2}}{2}(r-2)+\frac{2\sqrt{5}}{3}\right]. \end{aligned}$$

(13) For $MSO_1(G)$, we have

$$\begin{aligned} MSO_1(G) &= 2^{r+2}(\sqrt{2^2+4^2}+8)+(r-2)2^{r+1}(\sqrt{4^2+4^2}+16) \\ &= 2^{r+3}[(4+\sqrt{2})r+\sqrt{5}-2\sqrt{2}-4]. \end{aligned}$$

(14) For $MSO_2(G)$, we have

$$\begin{aligned} MSO_2(G) &= 2^{r+2}(\sqrt{2^2+4^2}-8)+(r-2)2^{r+1}(\sqrt{4^2+4^2}-16) \\ &= 2^{r+3}[(\sqrt{2}-4)r+\sqrt{5}-2\sqrt{2}+4]. \end{aligned}$$

(15) For $MSO_3(G)$, we have

$$MSO_3(G) = 2^{r+2}(8\sqrt{2^2+4^2})+(r-2)2^{r+1}(16\sqrt{4^2+4^2})$$

$$= \quad 2^{r+6}[2\sqrt{2}r + \sqrt{5} - 4\sqrt{2}].$$

(16) For $MSO_4(G)$, we have

$$\begin{aligned} MSO_4(G) &= 2^{r+2}\left(\frac{\sqrt{2^2+4^2}}{8}\right) + (r-2)2^{r+1}\left(\frac{\sqrt{4^2+4^2}}{16}\right) \\ &= 2^{r-1}(\sqrt{2}r + 2\sqrt{5} - 2\sqrt{2}). \end{aligned}$$

This completes the proof. ■

3 Results for augmented butterfly networks

Place an new edge on the antipodal vertices in a cycle in $BF(r)$. The resulting graph is called an augmented butterfly networks $ABF(r)$. The normal and diamond representations of 2-dimensional and 3-dimensional augmented butterfly network are given in Fig. 3.1, respectively.

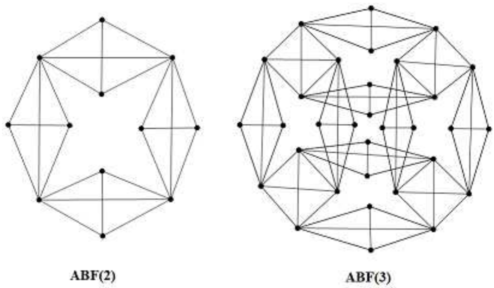

Fig. 3.1 Graphs $ABF(2)$ and $ABF(3)$.

By the definition of the r-dimensional augmented butterfly network, we obtain the basic information on the r-dimensional augmented butterfly network in the following table. By a similar reasoning as the proofs of Theorem 1, we have Theorem 2.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{3,3}$	$m_{3,6}$	$m_{6,6}$
$(r+1)2^r$	$3r2^r$	2^r	2^{r+2}	$(3r-5)2^r$

Theorem 2. *Let G be the r-dimensional augmented butterfly network $ABF(r)$. Then*
(1) $SO(G) = 2^r(18\sqrt{2}r + 12\sqrt{5} - 27\sqrt{2})$.
(2) $SO_{red}(G) = 2^r(15\sqrt{2}r + 4\sqrt{29} - 23\sqrt{2})$.
(3) $SO_{avg}(G) = \frac{2^r}{r+1}(12\sqrt{r^2 - 2r + 5} + 21\sqrt{2}r - 33\sqrt{2})$.
(4) $e^{SO(G)} = 2^r\left[(3r-5)e^{6\sqrt{2}} + 4e^{3\sqrt{5}} + e^{3\sqrt{2}}\right]$.
(5) $e^{SO_{red}(G)} = 2^r\left[(3r-5)e^{5\sqrt{2}} + 4e^{\sqrt{29}} + e^{2\sqrt{2}}\right]$.
(6) $e^{SO_{avg}(G)} = 2^r[e^{\frac{3\sqrt{2}(r-1)}{r+1}} + 4e^{\frac{3\sqrt{r^2-2r+5}}{r+1}} + (3r-5)e^{\frac{6\sqrt{2}}{r+1}}]$.
(7) $\ln(SO(G)) = 2^{r-1}[\ln 18 + 4\ln 45 + (3r-5)\ln 72]$.
(8) $\ln(SO_{red}(G)) = 2^{r-1}[\ln 8 + 4\ln 29 + (3r-5)\ln 50]$.
(9) $ZSO_1(G) = 2^r[18(2+\sqrt{2})r + 12\sqrt{5} - 27\sqrt{2} - 18]$.
(10) $ZSO_2(G) = 2^r[18(\sqrt{2}-2)r + 12\sqrt{5} - 27\sqrt{2} + 18]$.
(11) $ZSO_3(G) = 2^r[216\sqrt{2}r + 108\sqrt{5} - 342\sqrt{2}]$.
(12) $ZSO_4(G) = 2^r\left[\frac{3\sqrt{2}}{2}r + \frac{4\sqrt{5}}{3} - 2\sqrt{2}\right]$.
(13) $MSO_1(G) = 2^r[18(6+\sqrt{2})r + 12\sqrt{5} - 27\sqrt{2} - 99]$.
(14) $MSO_2(G) = 2^r[18(\sqrt{2}-6)r + 12\sqrt{5} - 27\sqrt{2} + 99]$.
(15) $MSO_3(G) = 2^r[648\sqrt{2}r + 216\sqrt{5} - 1053\sqrt{2}]$.
(16) $MSO_4(G) = 2^r\left[\frac{\sqrt{2}}{2}r + \frac{2\sqrt{5}}{3} - \frac{\sqrt{2}}{2}\right]$.

4 Results for enhanced butterfly networks

Place a new vertex in each 4-cycle of $BF(r)$ and join this vertex to the four vertices of the 4-cycle. The resulting graph is called an enhanced butterfly networks $EBF(r)$. The normal and diamond representations of 2-dimensional and 3-dimensional enhanced butterfly network are given in Fig. 4.1, respectively.

By the definition of the r-dimensional enhanced butterfly network, we obtain the basic information on the r-dimensional enhanced butterfly network in the following table. By a similar reasoning as the proofs of Theorem 1, we have Theorem 3.

Theorem 3. *Let G be the r-dimensional enhanced butterfly network $EBF(r)$. Then*

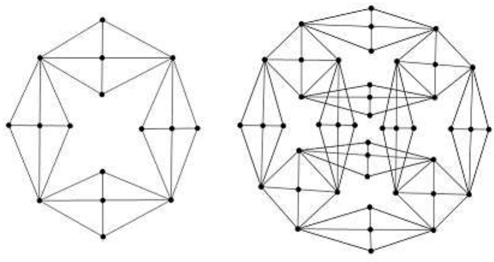

Fig. 4.1 Graphs $EBF(2)$ and $EBF(3)$.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{3,4}$	$m_{3,6}$	$m_{4,6}$	$m_{6,6}$
$(3r+2)2^{r-1}$	$r2^{r+2}$	2^{r+1}	2^{r+2}	$(r-1)2^{r+1}$	$(r-2)2^{r+1}$

(1) $SO(BF(r)) = 2^{r+1}[2\sqrt{13}(r-1) + 6\sqrt{2}(r-2) + 6\sqrt{5} + 5]$.
(2) $SO_{red}(G) = 2^{r+1}[\sqrt{34}(r-1) + 5\sqrt{2}(r-2) + \sqrt{13} + 2\sqrt{29}]$.
(3) $SO_{avg}(G) = \frac{2^{r+1}}{3r+2}[\sqrt{65r^2 - 148r + 100} + 2\sqrt{53r^2 - 36r + 180} + (r-1)\sqrt{20r^2 - 16r + 208} + 2\sqrt{2}(r-2)(r+6)]$.
(4) $e^{SO(G)} = 2^{r+1}\left[e^5 + 2e^{3\sqrt{5}} + (r-1)e^{2\sqrt{13}} + (r-2)e^{6\sqrt{2}}\right]$.
(5) $e^{SO_{red}(G)} = 2^{r+1}\left[e^{\sqrt{13}} + 2e^{\sqrt{29}} + (r-1)e^{\sqrt{34}} + (r-2)e^{5\sqrt{2}}\right]$.
(6) $e^{SO_{avg}(G)} = 2^{r+1}[e^{\frac{\sqrt{65r^2-148r+100}}{3r+2}} + 2e^{\frac{\sqrt{53r^2-36r+180}}{3r+2}} + (r-1)e^{\frac{\sqrt{20r^2-16r+208}}{3r+2}} + (r-2)e^{\frac{2\sqrt{2}(r+6)}{3r+2}}]$.
(7) $\ln(SO(G)) = 2^r[2\ln 5 + 2\ln 45 + (r-1)\ln 52 + (r-2)\ln 72]$.
(8) $\ln(SO_{red}(G)) = 2^r[\ln 13 + 2\ln 29 + (r-1)\ln 34 + (r-2)\ln 50]$.
(9) $ZSO_1(G) = 2^{r+1}[2\sqrt{13}(r-1) + 6\sqrt{2}(r-2) + 22r + 6\sqrt{5} - 4]$.
(10) $ZSO_2(G) = 2^{r+1}[2\sqrt{13}(r-1) + 6\sqrt{2}(r-2) - 22r + 6\sqrt{5} + 14]$.
(11) $ZSO_3(G) = 2^{r+1}[20\sqrt{13}(r-1) + 72\sqrt{2}(r-2) + 54\sqrt{5} + 35]$.
(12) $ZSO_4(G) = 2^{r+1}\left[\frac{\sqrt{13}}{5}(r-1) + \frac{\sqrt{2}}{2}(r-2) + \frac{2\sqrt{5}}{3} + \frac{5}{7}\right]$.
(13) $MSO_1(G) = 2^{r+1}[2\sqrt{13}(r-1) + 6\sqrt{2}(r-2) + 60r + 6\sqrt{5} - 43]$.
(14) $MSO_2(G) = 2^{r+1}[2\sqrt{13}(r-1) + 6\sqrt{2}(r-2) - 60r + 6\sqrt{5} + 89]$.
(15) $MSO_3(G) = 2^{r+1}[48\sqrt{13}(r-1) + 216\sqrt{2}(r-2) + 108\sqrt{5} + 60]$.
(16) $MSO_4(G) = 2^{r+1}\left[\frac{\sqrt{13}}{12}(r-1) + \frac{\sqrt{2}}{6}(r-2) + \frac{\sqrt{5}}{3} + \frac{5}{12}\right]$.

5 Results for Benes networks

By the definition of the r-dimensional Benes network, we obtain the basic information on the r-dimensional Benes network in the following table.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{2,4}$	$m_{4,4}$
$(2r+1)2^r$	$r2^{r+2}$	2^{r+2}	$(r-1)2^{r+2}$

Theorem 4. *Let G be the r-dimensional Benes network $B(r)$. Then*
(1) $SO(G) = 2^{r+3}(2\sqrt{2}r + \sqrt{5} - 2\sqrt{2})$.
(2) $SO_{red}(G) = 2^{r+2}(3\sqrt{2}r + \sqrt{10} - 3\sqrt{2})$.
(3) $SO_{avg}(G) = \frac{2^{r+3}}{2r+1}(\sqrt{4r^2-4r+5} + 2\sqrt{2}r - 2\sqrt{2})$.
(4) $e^{SO(G)} = 2^{r+2}[e^{2\sqrt{5}} + (r-1)e^{4\sqrt{2}}]$.
(5) $e^{SO_{red}(G)} = 2^{r+2}[e^{\sqrt{10}} + (r-1)e^{3\sqrt{2}}]$.
(6) $e^{SO_{avg}(G)} = 2^{r+2}[e^{\frac{2\sqrt{4r^2-4r+5}}{2r+1}} + (r-1)e^{\frac{4\sqrt{2}}{2r+1}}]$.
(7) $\ln(SO(G)) = 2^{r+1}[\ln 20 + (r-1)\ln 32]$.
(8) $\ln(SO_{red}(G)) = 2^{r+1}[\ln 10 + (r-1)\ln 18]$.
(9) $ZSO_1(G) = 2^{r+3}[2(\sqrt{2}+2)r - 2\sqrt{2} + \sqrt{5} - 1]$.
(10) $ZSO_2(G) = 2^{r+3}[2(\sqrt{2}-2)r - 2\sqrt{2} + \sqrt{5} + 1]$.
(11) $ZSO_3(G) = 2^{r+4}(8\sqrt{2}r + 3\sqrt{5} - 8\sqrt{2})$.
(12) $ZSO_4(G) = 2^{r+2}[\frac{\sqrt{5}}{3} + \frac{\sqrt{2}(r-1)}{2}]$.
(13) $MSO_1(G) = 2^{r+3}[2(\sqrt{2}+4)r - 2\sqrt{2} + \sqrt{5} - 4]$.
(14) $MSO_2(G) = 2^{r+3}[2(\sqrt{2}-4)r - 2\sqrt{2} + \sqrt{5} + 4]$.
(15) $MSO_3(G) = 2^{r+6}(4\sqrt{2}r + \sqrt{5} - 4\sqrt{2})$.
(16) $MSO_4(G) = 2^r(\sqrt{2}r + \sqrt{5} - \sqrt{2})$.

6 Results for mesh derived networks

By the definition of the $MDN1(m, n)$ network, we obtain the basic information on the $MDN1(m, n)$ network in the following table.

Theorem 5. *Let G be the mesh derived network $MDN1(m, n)$. Then*
(1) $SO(G) = (8\sqrt{13} + 24\sqrt{2})mn + (20 + 6\sqrt{5} - 8\sqrt{13} - 48\sqrt{2})(m+n)$
$+ 112\sqrt{2} - 8\sqrt{5} - 80$.

$\lvert V(G)\rvert$	$3mn-m-n$
$\lvert E(G)\rvert$	$8mn-6(m+n)+4$
$m_{2,4}$	8
$m_{3,4}$	$4(m+n-4)$
$m_{3,6}$	$2(m+n-4)$
$m_{4,6}$	$4(mn-m-n)$
$m_{4,4}$	4
$m_{6,6}$	$4(mn-2m-2n+4)$

(2) $SO_{red}(G) = (4\sqrt{34}+20\sqrt{2})mn + (4\sqrt{13}+2\sqrt{29}-4\sqrt{34}-40\sqrt{2})(m+n) + 8\sqrt{10} - 16\sqrt{13} - 8\sqrt{29} + 92\sqrt{2}$.

(3) $SO_{avg}(G) = 8\sqrt{(10+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2} + 4(m+n-4)\sqrt{(9+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2} + 2(m+n-4)\sqrt{(10+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2} + 4(mn-m-n)\sqrt{(8+\frac{8-20mn}{3mn-m-n})^2+(6+\frac{8-20mn}{3mn-m-n})^2} + 4\sqrt{2}(8+\frac{8-20mn}{3mn-m-n}) + 4\sqrt{2}(mn-2m-2n+4)(8+\frac{8-20mn}{3mn-m-n})$.

(4) $e^{SO(G)} = 8e^{2\sqrt{5}} + 4(m+n-4)e^{5} + 2(m+n-4)e^{3\sqrt{5}} + 4(mn-m-n)e^{2\sqrt{13}} + 8e^{4\sqrt{2}} + 4(mn-2m-2n+4)e^{6\sqrt{2}}$.

(5) $e^{SO_{red}(G)} = 8e^{\sqrt{10}} + 4(m+n-4)e^{\sqrt{13}} + 2(m+n-4)e^{\sqrt{29}} + 4(mn-m-n)e^{\sqrt{34}} + 8e^{3\sqrt{2}} + 4(mn-2m-2n+4)e^{5\sqrt{2}}$.

(6) $e^{SO_{avg}(G)} = 8e^{\sqrt{(10+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2}} + 4(m+n-4)e^{\sqrt{(9+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2}} + 2(m+n-4)e^{\sqrt{(10+\frac{8-20mn}{3mn-m-n})^2+(8+\frac{8-20mn}{3mn-m-n})^2}} + 4(mn-m-n)e^{\sqrt{(8+\frac{8-20mn}{3mn-m-n})^2+(6+\frac{8-20mn}{3mn-m-n})^2}} + 4e^{\sqrt{2}(8+\frac{8-20mn}{3mn-m-n})} + 4(mn-2m-2n+4)e^{\sqrt{2}(8+\frac{8-20mn}{3mn-m-n})}$.

(7) $\ln(SO(G)) = 4\ln 20 + 4(m+n-4)\ln 5 + (m+n-4)\ln 45 + 2(mn-m-n)\ln 52 + 4\ln 32 + 2(mn-2m-2n+4)\ln 72$.

(8) $\ln(SO_{red}(G)) = 4\ln 10 + 2(m+n-4)\ln 13 + (m+n-4)\ln 29 + 2(mn-m-n)\ln 34 + 4\ln 18 + 2(mn-2m-2n+4)\ln 50$.

(9) $ZSO_1(G) = (8\sqrt{13}+24\sqrt{2}+136)mn + (6\sqrt{5}-8\sqrt{13}-48\sqrt{2}-166)(m+n) + 96\sqrt{2} - 8\sqrt{5} - 184$.

(10) $ZSO_2(G) = (8\sqrt{13}+24\sqrt{2}-88)mn+(6\sqrt{5}-8\sqrt{13}-48\sqrt{2}+110)(m+n) + 112\sqrt{2} - 8\sqrt{5} - 168.$

(11) $ZSO_3(G) = (80\sqrt{13}+288\sqrt{2})mn+(54\sqrt{5}+140-80\sqrt{13}-576\sqrt{2})(m + n) + 1280\sqrt{2} - 120\sqrt{5} - 560.$

(12) $ZSO_4(G) = (\frac{4\sqrt{13}}{5} + 2\sqrt{2})mn + (\frac{2\sqrt{5}}{3} + \frac{20}{7} - \frac{4\sqrt{13}}{5} - 4\sqrt{2})(m + n) + 10\sqrt{2} - \frac{80}{7}.$

(13) $MSO_1(G) = (8\sqrt{13} + 24\sqrt{2} + 288)mn + (6\sqrt{5} - 8\sqrt{13} - 48\sqrt{2} - 224)(m + n) + 112\sqrt{2} - 8\sqrt{5} + 398.$

(14) $MSO_2(G) = (8\sqrt{13} + 24\sqrt{2} - 240)mn + (6\sqrt{5} - 8\sqrt{13} - 48\sqrt{2} + 320)(m + n) + 112\sqrt{2} - 8\sqrt{5} - 448.$

(15) $MSO_3(G) = (192\sqrt{13} + 864\sqrt{2})mn + (108\sqrt{5} + 240 - 192\sqrt{13} - 1728\sqrt{2})(m + n) + 3712\sqrt{2} - 304\sqrt{5} - 960.$

(16) $MSO_4(G) = (\frac{\sqrt{13}}{3} + \frac{2\sqrt{2}}{3})mn + (\frac{\sqrt{5}}{3} + \frac{5}{3} - \frac{\sqrt{13}}{3} - \frac{4\sqrt{2}}{3})(m + n) + \frac{11\sqrt{2}}{3} + \frac{2\sqrt{5}}{3} - \frac{20}{3}.$

By the definition of the $MDN2(m, n)$ network, we obtain the basic information on the $MDN2(m, n)$ network in the following table.

$\lvert V(G)\rvert$	$2mn - m - n + 1$
$\lvert E(G)\rvert$	$8(mn - m - n + 1)$
$m_{3,6}$	4
$m_{3,5}$	8
$m_{5,6}$	8
$m_{5,5}$	$2(m + n - 6)$
$m_{6,8}$	4
$m_{5,8}$	$2(m + n - 4)$
$m_{5,7}$	$4(m + n - 6)$
$m_{7,7}$	$2(m + n - 8)$
$m_{6,7}$	8
$m_{7,8}$	$6(m + n - 6)$
$m_{8,8}$	$8mn - 24(m + n) + 72$

Theorem 6. *Let G be the mesh derived network $MDN2(m, n)$. Then*
(1) $SO(G) = 64\sqrt{2}mn + (6\sqrt{113} + 2\sqrt{89} + 4\sqrt{74} - 168\sqrt{2})(m + n) + 12\sqrt{5} + 8(\sqrt{34} + \sqrt{61} + \sqrt{85}) + 524\sqrt{2} + 40 - 8\sqrt{89} - 24\sqrt{74} - 36\sqrt{113}.$

(2) $SO_{red}(G)=56\sqrt{2}mn+(2\sqrt{65}+8\sqrt{13}+6\sqrt{85}-148\sqrt{2})(m+n)$
$+4\sqrt{29}+16\sqrt{5}+8\sqrt{41}+4\sqrt{74}+8\sqrt{61}+456\sqrt{2}$
$-8\sqrt{65}-48\sqrt{13}-36\sqrt{85}$.

(3) $SO_{avg}(G)=4\sqrt{(13-\frac{16mn}{2mn-m-n+1})^2+(10-\frac{16mn}{2mn-m-n+1})^2}$
$+8\sqrt{(13-\frac{16mn}{2mn-m-n+1})^2+(11-\frac{16mn}{2mn-m-n+1})^2}$
$+8\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(10-\frac{16mn}{2mn-m-n+1})^2}$
$+2\sqrt{2}(m+n-6)(11-\frac{16mn}{2mn-m-n+1})$
$+4\sqrt{(10-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}$
$+2(m+n-4)\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}$
$+4(m+n-6)\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(9-\frac{16mn}{2mn-m-n+1})^2}$
$+2\sqrt{2}(m+n-8)(9-\frac{16mn}{2mn-m-n+1})$
$+8\sqrt{(10-\frac{16mn}{2mn-m-n+1})^2+(9-\frac{16mn}{2mn-m-n+1})^2}$
$+6(m+n-6)\sqrt{(9-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}$
$+\sqrt{2}[8mn-24(m+n)+72](8-\frac{16mn}{2mn-m-n+1})$.

(4) $e^{SO(G)}=4e^{3\sqrt{5}}+8e^{\sqrt{34}}+8e^{\sqrt{61}}+2(m+n-6)e^{5\sqrt{2}}+4e^{10}+2(m$
$+n-8)e^{\sqrt{89}}+4(m+n-6)e^{\sqrt{74}}+2(m+n-16)e^{7\sqrt{2}}+8e^{\sqrt{85}}$
$+6(m+n-6)e^{\sqrt{113}}+[8mn-24(m+n)+72]e^{8\sqrt{2}}$.

(5) $e^{SO_{red}(G)}=4e^{\sqrt{29}}+8e^{2\sqrt{5}}+8e^{\sqrt{41}}+2(m+n-6)e^{4\sqrt{2}}+4e^{\sqrt{74}}$
$+2(m+n-4)e^{\sqrt{65}}+4(m+n-6)e^{2\sqrt{13}}+2(m+n-8)e^{6\sqrt{2}}$
$+8e^{\sqrt{61}}+6(m+n-6)e^{\sqrt{85}}+[8mn-24(m+n)+72]e^{7\sqrt{2}}$.

(6) $e^{SO_{avg}(G)}=4e^{\sqrt{(13-\frac{16mn}{2mn-m-n+1})^2+(10-\frac{16mn}{2mn-m-n+1})^2}}$
$+8e^{\sqrt{(13-\frac{16mn}{2mn-m-n+1})^2+(11-\frac{16mn}{2mn-m-n+1})^2}}$
$+8e^{\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(10-\frac{16mn}{2mn-m-n+1})^2}}$
$+2(m+n-6)e^{\sqrt{2}(11-\frac{16mn}{2mn-m-n+1})}$
$+4e^{\sqrt{(10-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}}$
$+2(m+n-4)e^{\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}}$
$+4(m+n-6)e^{\sqrt{(11-\frac{16mn}{2mn-m-n+1})^2+(9-\frac{16mn}{2mn-m-n+1})^2}}$
$+2(m+n-8)e^{\sqrt{2}(9-\frac{16mn}{2mn-m-n+1})}$
$+8e^{\sqrt{(10-\frac{16mn}{2mn-m-n+1})^2+(9-\frac{16mn}{2mn-m-n+1})^2}}$
$+6(m+n-6)e^{\sqrt{(9-\frac{16mn}{2mn-m-n+1})^2+(8-\frac{16mn}{2mn-m-n+1})^2}}$
$+[8mn-24(m+n)+72]e^{\sqrt{2}(8-\frac{16mn}{2mn-m-n+1})}$.

(7) $\ln(SO(G))=2\ln(45)+4\ln 34+4\ln 61+(m+n-6)\ln 50+4\ln 10$

$+ (m+n-4)\ln 89 + 2(m+n-6)\ln 74 + (m+n$
$-8)\ln 98 + 4\ln 85 + 3(m+n-6)\ln 113 + [4mn - 12(m$
$+n) + 36]\ln 128.$

(8) $\ln(SO_{red}(G)) = 2\ln 29 + 4\ln 20 + 4\ln 41 + (m+n-6)\ln 32 + 2\ln 74$
$+ (m+n-4)\ln 65 + 2(m+n-6)\ln 52 + (m+n$
$-8)\ln 72 + 4\ln 61 + 3(m+n-6)\ln 85$
$+ [4mn - 12(m+n) + 36]\ln 98.$

(9) $ZSO_1(G) = (64\sqrt{2} + 128)mn + (6\sqrt{113} + 2\sqrt{89} + 4\sqrt{74} - 168\sqrt{2}$
$-290)(m+n) + 12\sqrt{5} + 8\sqrt{34} + 8\sqrt{61} + 8\sqrt{85} + 524\sqrt{2}$
$+ 464 - 8\sqrt{89} - 24\sqrt{74} - 36\sqrt{113}.$

(10) $ZSO_2(G) = (64\sqrt{2} - 128)mn + (6\sqrt{113} + 2\sqrt{89} + 4\sqrt{74} - 168\sqrt{2}$
$+168)(m+n) + 12\sqrt{5} + 8\sqrt{34} + 8\sqrt{61} + 8\sqrt{85} + 404\sqrt{2} - 160$
$-8\sqrt{89} - 24\sqrt{74} - 36\sqrt{113}.$

(11) $ZSO_3(G) = 8192\sqrt{2}mn + (26\sqrt{89} + 48\sqrt{74} + 90\sqrt{113} - 24280\sqrt{2})(m$
$+n) + 108\sqrt{5} + 64\sqrt{34} + 88\sqrt{61} + 104\sqrt{85} + 560$
$+ 71000\sqrt{2} - 104\sqrt{89} - 288\sqrt{74} - 540\sqrt{113}.$

(12) $ZSO_4(G) = 4\sqrt{2}mn + (\frac{2\sqrt{89}}{13} + \frac{\sqrt{74}}{3} + \frac{2\sqrt{113}}{5} - 10\sqrt{2})(m+n) + \frac{4\sqrt{5}}{3}$
$+ \sqrt{34} + \frac{8\sqrt{61}}{11} + \frac{8\sqrt{85}}{13} + \frac{20}{7} + 34\sqrt{2} - \frac{8\sqrt{89}}{13} - 2\sqrt{74} - \frac{12\sqrt{113}}{5}.$

(13) $MSO_1(G) = (64\sqrt{2} + 512)mn + (6\sqrt{113} + 2\sqrt{89} + 4\sqrt{74} - 168\sqrt{2}$
$-1168)(m+n) + 12\sqrt{5} + 8\sqrt{34} + 8\sqrt{61} + 8\sqrt{85} + 404\sqrt{2}$
$+ 1602 - 8\sqrt{89} - 24\sqrt{74} - 36\sqrt{113}.$

(14) $MSO_2(G) = (64\sqrt{2} - 512)mn + (6\sqrt{113} + 2\sqrt{89} + 4\sqrt{74} - 168\sqrt{2}$
$+832)(m+n) + 12\sqrt{5} + 8\sqrt{34} + 8\sqrt{61} + 8\sqrt{85} + 404\sqrt{2}$
$-1082 - 8\sqrt{89} - 24\sqrt{74} - 36\sqrt{113}.$

(15) $MSO_3(G) = 32768\sqrt{2}mn + (80\sqrt{89} + 140\sqrt{74} + 336\sqrt{113} - 97368\sqrt{2})(m$
$+n) + 216\sqrt{5} + 120\sqrt{34} + 240\sqrt{61} + 336\sqrt{85} + 1920$
$+ 289924\sqrt{2} - 320\sqrt{89} - 840\sqrt{74} - 2016\sqrt{113}.$

(16) $MSO_4(G) = \sqrt{2}mn + (\frac{\sqrt{89}}{20} + \frac{4\sqrt{74}}{35} + \frac{3\sqrt{113}}{28} - \frac{91\sqrt{2}}{35})(m+n) + \frac{2\sqrt{5}}{3}$
$+ \frac{8\sqrt{34}}{15} + \frac{4\sqrt{61}}{15} + \frac{4\sqrt{85}}{21} + \frac{5}{6} + \frac{369\sqrt{2}}{35} - \frac{\sqrt{89}}{5} - \frac{24\sqrt{74}}{35} - \frac{9\sqrt{113}}{14}.$

7 Results for OTIS networks

By the definition of the OTIS swapped network O_{P_n}, we obtain the basic information on the OTIS swapped network O_{P_n} in the following table.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{1,3}$	$m_{2,2}$	$m_{2,3}$	$m_{3,3}$
n^2	$\frac{3n(n-1)}{2}$	2	3	$6n-14$	$\frac{3(n-2)(n-3)}{2}$

Theorem 7. *Let G be the OTIS swapped network O_{P_n}. Then*
(1) $SO(G) = \frac{9\sqrt{2}}{2}n^2 + (6\sqrt{13} - \frac{45}{2}\sqrt{2})n - 14\sqrt{13} + 33\sqrt{2} + 2\sqrt{10}$.
(2) $SO_{red}(G) = 3\sqrt{2}n^2 - (15\sqrt{2} - 6\sqrt{5})n - 14\sqrt{5} + 21\sqrt{2} + 4$.
(3) $SO_{avg}(G) = \frac{9\sqrt{2}}{2}n^2 - \frac{45\sqrt{2}}{2}n - \frac{9\sqrt{2}}{n} - 30\sqrt{2} + \frac{2}{n}\sqrt{4n^2 - 12n + 18}$
$+ \frac{6n-14}{n}\sqrt{n^2 - 6n + 18}$.
(4) $e^{SO(G)} = 2e^{\sqrt{10}} + 3e^{2\sqrt{2}} + (6n-14)e^{\sqrt{13}} + \frac{3(n-2)(n-3)}{2}e^{3\sqrt{2}}$.
(5) $e^{SO_{red}(G)} = 2e^2 + 3e^{\sqrt{2}} + (6n-14)e^{\sqrt{5}} + \frac{3(n-2)(n-3)}{2}e^{2\sqrt{2}}$.
(6) $e^{SO_{avg}(G)} = 2e^{\frac{\sqrt{4n^2-12n+18}}{n}} + 3e^{\sqrt{2}(1-\frac{3}{n})} + (6n-14)e^{\frac{\sqrt{n^2-6n+18}}{n}}$
$+ \frac{3(n-2)(n-3)}{2}e^{\frac{3\sqrt{2}}{n}}$.
(7) $\ln(SO(G)) = 2\ln\sqrt{10} + 3\ln(2\sqrt{2}) + (6n-14)\ln\sqrt{13} + \frac{3(n-2)(n-3)}{2}\ln(3\sqrt{2})$.
(8) $\ln(SO_{red}(G)) = 2\ln 2 + 3\ln(\sqrt{2}) + (6n-14)\ln\sqrt{5} + \frac{3(n-2)(n-3)}{2}\ln(2\sqrt{2})$.
(9) $ZSO_1(G) = \frac{9\sqrt{2}+54}{2}n^2 + (6\sqrt{13} - \frac{45}{2}\sqrt{2} - 57)n - 14\sqrt{13} + 33\sqrt{2}$
$+ 2\sqrt{10} + 24$.
(10) $ZSO_2(G) = \frac{9\sqrt{2}-18}{2}n^2 + (6\sqrt{13} - \frac{45}{2}\sqrt{2} + 15)n - 14\sqrt{13}$
$+ 33\sqrt{2} + 2\sqrt{10} - 4$.
(11) $ZSO_3(G) = 27\sqrt{2}n^2 + (30\sqrt{13} - 135\sqrt{2})n + 186\sqrt{2} + 8\sqrt{10} - 70\sqrt{13}$.
(12) $ZSO_4(G) = \frac{\sqrt{2}}{4}n^2 + (\frac{6\sqrt{13}}{5} - \frac{5\sqrt{2}}{4})n + \frac{\sqrt{10}}{2} + 3\sqrt{2} - \frac{14\sqrt{13}}{5}$.
(13) $MSO_1(G) = \frac{9\sqrt{2}+27}{2}n^2 + (6\sqrt{13} - \frac{45}{2}\sqrt{2} - \frac{63}{2})n - 14\sqrt{13} + 33\sqrt{2}$
$+ 2\sqrt{10} + 9$.
(14) $MSO_2(G) = \frac{9\sqrt{2}-27}{2}n^2 + (6\sqrt{13} - \frac{45}{2}\sqrt{2} + \frac{63}{2})n - 14\sqrt{13} + 33\sqrt{2}$
$+ 2\sqrt{10} - 15$.
(15) $MSO_3(G) = \frac{81\sqrt{2}}{2}n^2 + (36\sqrt{13} - \frac{405\sqrt{2}}{2})n + 267\sqrt{2} + 6\sqrt{10} - 84\sqrt{13}$.
(16) $MSO_4(G) = \frac{\sqrt{2}}{2}n^2 + (\sqrt{13} - \frac{5\sqrt{2}}{2})n + \frac{2\sqrt{10}}{3} + \frac{9\sqrt{2}}{2} - \frac{7\sqrt{13}}{3}$.

By the definition of the OTIS swapped network O_{R_k}, we obtain the basic information on the OTIS swapped network O_{R_k} in the following table.

Theorem 8. *Let G be the OTIS swapped network O_{R_k}. Then*
(1) $SO(G) = \frac{(k+1)^2\sqrt{2}}{2}n^2 + [k\sqrt{2k^2+2k+1} - \frac{\sqrt{2}(1+2k)(k+1)}{2}]n$.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{k,\,k+1}$	$m_{k+1,\,k+1}$
n^2	$\frac{n^2(k+1)-n}{2}$	kn	$\frac{n^2(k+1)-n(1+2k)}{2}$

(2) $SO_{red}(G) = \frac{k(k+1)\sqrt{2}}{2}n^2 + [k\sqrt{2k^2-2k+1} - \frac{\sqrt{2}k(1+2k)}{2}]n.$
(3) $SO_{avg}(G) = k[\sqrt{n^2-2kn+2k^2} + \sqrt{2}\frac{n(k+1)-(1+2k)}{2}].$
(4) $e^{SO(G)} = nke^{\sqrt{2k^2+2k+1}} + \frac{n^2(k+1)-n(1+2k)}{2}e^{\sqrt{2}(k+1)}.$
(5) $e^{SO_{red}(G)} = nke^{\sqrt{2k^2-2k+1}} + \frac{n^2(k+1)-n(1+2k)}{2}e^{\sqrt{2}k}.$
(6) $e^{SO_{avg}(G)} = nke^{\frac{\sqrt{n^2-2kn+2k^2}}{n}} + k\frac{n^2(k+1)-n(1+2k)}{2}e^{\frac{\sqrt{2}}{n}}.$
(7) $\ln(SO(G)) = nk\ln(\sqrt{2k^2+2k+1}) + \frac{n^2(k+1)-n(1+2k)}{2}\ln(\sqrt{2}(k+1)).$
(8) $\ln(SO_{red}(G)) = nk\ln(\sqrt{2k^2-2k+1}) + \frac{n^2(k+1)-n(1+2k)}{2}\ln(\sqrt{2}k).$
(9) $ZSO_1(G) = \frac{(k+1)^2(\sqrt{2}+2)}{2}n^2 + [k\sqrt{2k^2+2k+1} - \frac{\sqrt{2}(1+2k)(k+1)+2k(1+2k)}{2}]n.$
(10) $ZSO_2(G) = \frac{(k+1)^2(\sqrt{2}-2)}{2}n^2 + [k\sqrt{2k^2+2k+1} - \frac{(1+2k)(k+1)\sqrt{2}-2(1+2k)}{2}]n.$
(11) $ZSO_3(G) = \sqrt{2}(k+1)^3n^2 + n[k(2k+1)\sqrt{2k^2+2k+1}$
$-\sqrt{2}(k+1)^2(1+2k)].$
(12) $ZSO_4(G) = \frac{\sqrt{2}(k+1)}{4}n^2 + [\frac{k\sqrt{2k^2+2k+1}}{2k+1} - \frac{\sqrt{2}(1+2k)}{4}]n.$
(13) $MSO_1(G) = \frac{(k+1)^2(\sqrt{2}+k+1)}{2}n^2 + n[k\sqrt{2k^2+2k+1}$
$-\frac{\sqrt{2}(1+2k)(k+1)+(1+k)(3k+1)}{2}].$
(14) $MSO_2(G) = \frac{(k+1)^2(\sqrt{2}-k-1)}{2}n^2 + [k\sqrt{2k^2+2k+1}$
$-\frac{\sqrt{2}(1+2k)(k+1)-(1+k)(3k+1)}{2}]n.$
(15) $MSO_3(G) = \sqrt{2}(k+1)^3n^2 + n[k(2k+1)\sqrt{2k^2+2k+1}$
$-\sqrt{2}(k+1)^2(1+2k)].$
(16) $MSO_4(G) = \frac{\sqrt{2}}{2}n^2 + [\frac{\sqrt{2k^2+2k+1}}{k+1} - \frac{\sqrt{2}(1+2k)}{2(k+1)}]n.$

By the definition of the Biswapped network $Bsw(P_n)$, we obtain the basic information on the Biswapped network $Bsw(P_n)$ in the following table.

$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$m_{2,\,2}$	$m_{2,\,3}$	$m_{3,\,3}$
$2n^2$	$3n^2-2n$	4	$8(n-1)$	$3n^2-10n+4$

Theorem 9. *Let G be the Biswapped network $Bsw(P_n)$. Then*
(1) $SO(G) = 9\sqrt{2}n^2 + (8\sqrt{13} - 30\sqrt{2})n + 20\sqrt{2} - 8\sqrt{13}$.
(2) $SO_{red}(G) = 6\sqrt{2}n^2 + (8\sqrt{5} - 20\sqrt{2})n + 12\sqrt{2} - 8\sqrt{5}$.
(3) $SO_{avg}(G) = 6\sqrt{2}n + 8(n-1)\frac{\sqrt{n^2-4n+8}}{n} - 16\sqrt{2}$.
(4) $e^{SO(G)} = 4e^{2\sqrt{2}} + 8(n-1)e^{\sqrt{13}} + (3n^2 - 10n + 4)e^{3\sqrt{2}}$.
(5) $e^{SO_{red}(G)} = 4e^{\sqrt{2}} + 8(n-1)e^{\sqrt{5}} + (3n^2 - 10n + 4)e^{2\sqrt{2}}$.
(6) $e^{SO_{avg}(G)} = 4e^{\frac{\sqrt{2}(n-2)}{n}} + 8(n-1)e^{\frac{\sqrt{n^2-4n+8}}{n}} + (3n^2 - 10n + 4)e^{\frac{2\sqrt{2}}{n}}$.
(7) $\ln(SO(G)) = 4\ln(2\sqrt{2}) + 8(n-1)\ln\sqrt{13} + (3n^2 - 10n + 4)\ln(3\sqrt{2})$.
(8) $\ln(SO_{red}(G)) = 4\ln\sqrt{2} + 8(n-1)\ln\sqrt{5} + (3n^2 - 10n + 4)\ln(2\sqrt{2})$.
(9) $ZSO_1(G) = (9\sqrt{2} + 18)n^2 + (8\sqrt{13} - 30\sqrt{2} - 20)n + 20\sqrt{2} - 8\sqrt{13}$.
(10) $ZSO_2(G) = (9\sqrt{2} - 18)n^2 + (8\sqrt{13} - 30\sqrt{2} + 20)n + 20\sqrt{2} - 8\sqrt{13}$.
(11) $ZSO_3(G) = 54\sqrt{2}n^2 + (40\sqrt{13} - 180\sqrt{2})n + 104\sqrt{2} - 40\sqrt{13}$.
(12) $ZSO_4(G) = \frac{3\sqrt{2}}{2}n^2 + (\frac{8\sqrt{13}}{5} - \frac{10\sqrt{2}}{2})n + 2\sqrt{2} - \frac{8\sqrt{13}}{5}$.
(13) $MSO_1(G) = (9\sqrt{2} + 27)n^2 + (8\sqrt{13} - 30\sqrt{2} - 42)n + 20\sqrt{2} - 8\sqrt{13} + 4$.
(14) $MSO_2(G) = (9\sqrt{2} - 27)n^2 + (8\sqrt{13} - 30\sqrt{2} + 42)n + 20\sqrt{2} - 8\sqrt{13} - 4$.
(15) $MSO_3(G) = 81\sqrt{2}n^2 + (48\sqrt{13} - 270\sqrt{2})n + 104\sqrt{2} - 48\sqrt{13}$.
(16) $MSO_4(G) = \sqrt{2}n^2 + (\frac{4\sqrt{13}}{3} - \frac{10\sqrt{2}}{3})n + \frac{10\sqrt{2} - 4\sqrt{13}}{3}$.

8 Conclusion

Based on the Sombor-type indices, we propose the Estrada-Sombor-type, logarithmic-Sombor-type, Zagreb-Sombor-type and general Zagreb-Sombor-type indices, and calculate these indices for butterfly networks, augmented butterfly networks, enhanced butterfly networks, Benes networks, mesh derived networks and the Optical Transpose Interconnection System networks, which provide a new kind of prediction indices for these networks. In addition, the bound and the extremal value problem of these new indices would be interesting.

***Acknowledgment*:** The author would like to thank Professor Ivan Gutman for his valuable suggestions and comments.

References

[1] A. Aslam, S. Ahmad, M. A. Binyamin, W. Gao, Calculating topological indices of certain OTIS interconnection networks, *Open Chem.* **17** (2019) 220–228.

[2] S. Alikhani, N. Ghanbari, Sombor index of polymers, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 715–728.

[3] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, Bounds for Zagreb Indices, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 17–100.

[4] B. Bollobás, P. Erdős, Graphs of extremal weights, *Ars Comb.* **50** (1998) 225–233.

[5] A. T. Balaban, I. Motoc, D. Bonchev, O. Mekenyan, Topological indices for structure-activity correlations, *Top. Curr. Chem.* **114** (1983) 21–55.

[6] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, *Appl. Math. Comput.* **399** (2021) #126018.

[7] R. Cruz, J. Rada, Extremal values of the Sombor index in unicyclic and bicyclic graphs, *J. Math. Chem.* **59** (2021) 1098–1116.

[8] K. C. Das, A. S. Çevik, I. N. Cangul, Y. Shang, On Sombor index, *Symmetry* **13** (2021) #140.

[9] M. Dehmer, M. Grabner, The discrimination power of molecular identification numbers revisited, *MATCH Commun. Math. Comput. Chem.* **69** (2013) 785–794.

[10] M. Dehmer, M. Grabner, K. Varmuza, Information indices with high discriminative power for graphs, *PLoS ONE* **7** (2012) #e31214.

[11] T. Došlić, T. Réti, A. Ali, On the structure of graphs with integer Sombor indices, *Discrete Math. Lett.* **7** (2021) 1–4.

[12] H. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, *Int. J. Quantum Chem.* **121** (2021) #e26622.

[13] X. Fang, L. You, H. Liu, The expected values of Sombor indices in random hexagonal chains, phenylene chains and Sombor indices of some chemical graphs, *Int. J. Quantum Chem.* **121** (2021) #e26740.

[14] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 11–16.

[15] I. Gutman, Multiplicative Zagreb indices of trees, *Bull. Soc. Math. Banja Luka* **18** (2011) 17–23.

[16] I. Gutman, Some basic properties of Sombor indices, *Open J. Discr. Appl. Math.* **4** (2021) 1–3.

[17] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total π electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[18] S. Hayat, M. Imran, Computation of topological indices of certain networks, *Appl. Math. Comput.* **240** (2014) 213–228.

[19] M. Imran, S. Haya, M. Y. H. Mailk, On topological indices of certain interconnection networks, *Appl. Math. Comput.* **244** (2014) 936–951.

[20] E. V. Konstantinova, On some applications of information indices in chemical graph theory, in: R. Ahlswede, L. Bäumer, N. Cai, H. Aydinian, V. Blinovsky, C. Deppe, H. Mashurian (Eds.), *General Theory of Information Transfer and Combinatorics*, Springer, 2006, pp. 831–852.

[21] V. R. Kulli, On Banhatti-Sombor indices, *Int. J. Appl. Chem.* **8** (2021) 21–25.

[22] H. Liu, Ordering chemical graphs by their Sombor indices, *arXiv:2103.05995*.

[23] Z. Lin, T. Zhou, L. Miao, On the spectral radius, energy and Estrada index of the Sombor matrix of graphs, *Trans. Comb.* **12** (2023) 191-205.

[24] J. Liu, S. Wang, C. Wang, S. Hayat, Further results on computation of topological indices of certain networks, *IET Control Theory Appl.* **11** (2017) 2065–2071.

[25] X. Li, H. Zhao, Trees with the first smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 57–62.

[26] Z. Lin, T. Zhou, V. R. Kulli, L. Miao, On the first Banhatti-Sombor index, *J. Int. Math. Virtual Inst.* **11** (2021) 53–68.

[27] I. Milovanović, E. Milovanović, M. Matejić, On some mathematical properties of sombor indices, *Bull. Int. Math. Virtual Inst.* **11** (2021) 341–353.

[28] I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.* **86** (2021) 445–457.

[29] J. Rada, Exponential vertex-degree-based topological indices and discrimination, *MATCH Commun. Math. Comput. Chem.* **82** (2019) 29–41.

[30] T. Réti, T. Došlić, A. Ali, On the Sombor index of graphs, *Contrib. Math.* **3** (2021) 11–18.

[31] R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.

[32] Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, *J. Appl. Math. Comput.* **68** (2022) 1-17.