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Minimal Degree–Based Graph Entropy for Dense Graphs

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Abstract

Various graph entropy measures derived from classical Shannon entropy are introduced to characterize the complexity of networks, among which the degree-based graph entropy is defined by employing degrees of vertices as a graph invariant. Using graph operations to explore the properties of the extremal graphs has been demonstrated a promising method. In this paper, we give conditions for decreasing the values of the entropy by graph operations, which extend and improve some known results. As an application, we characterize graphs attaining the minimum values in some connected dense graphs with given numbers of vertices and edges.

1 Introduction

Based on Shannon's entropy [10], Rashevsky [8] introduced the concept of graph entropy to measure the structural complexity of graphs. Since then, a great deal of research has been focused on entropy measures defined by using several graph invariants, including number of vertices, number

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of edges, vertex degrees, and distance-based quantities [3, 5], which covers areas of information science, graph theory, structural chemistry, and molecular biology [6].

An (n, m)-graph is a graph with n vertices and m edges. Let G be an (n, m)-graph with vertex set $V(G)$. For any vertex $v \in V(G)$, $d_G(v)$ denotes the degree of v in G. The subscript G is omitted if it is clear in the context, namely $d(v)$.

Let f be an information functional, a function from $V(G)$ to the set of positive real numbers. According to Shannon's entropy formulas, the graph entropy, defined by Dehmer [4], is

$$I_f(G) = -\sum_{v \in V(G)} p(v) \log p(v),$$

where $p(v) = \frac{f(v)}{\sum_{u \in V(G)} f(u)}$. When $f(v) = d(v)$, we get the degree-based graph entropy [2]:

$$I_d(G) = -\sum_{v \in V(G)} \frac{d(v)}{\sum_{u \in V(G)} d(u)} \log \frac{d(v)}{\sum_{u \in V(G)} d(u)}.$$

Since $\sum_{u \in V(G)} d(u) = 2m$,

$$I_d(G) = \log(2m) - \frac{1}{2m} \sum_{v \in V(G)} d(v) \log d(v).$$

For the degree-based graph entropy, the task of determining the minimum values and the graphs attaining the minimum values is complicated as it requires an understanding of the mathematical properties of the multivariate function $\sum_{v \in V(G)} d(v) \log d(v)$. Extensive studies have been devoted to extremal results of special graph families. In [2], Cao et al. tackled the problem for trees, some cyclic graphs and chemical graphs, and afterwards, Ghalavand et al. [7] extended some extremal properties. More recently, the minimal graphs for cacti were characterized in [9]. For general graphs, the topological structure of graphs attaining the minimum values was described as a graph formed by connecting vertices of a clique and vertices of an independent set in a certain way [11]. And a conjecture to determine the extremal values for connected graphs with $m \leq 2n-3$ was proposed in the same reference. Two years later, Cambie and Mazzamurro [1] proved the conjecture. To further investigate the extremal properties, Yan and Guan [12] defined two graph operations and proved that these operations can decrease the entropy under the conditions of so-called being 'proper'. This paper extends and improves the result, and as an application, we characterize graphs attaining the minimum values in (n,m)-graphs with $m > \frac{1}{2}(n+5)(n-6)$ for $n \geq 8$, $m > 11$ for $n = 7$, and $m > 9$ for $n = 6$. It implies that the structure of the extremal graphs is related to the density of edges.

2 Preliminary definitions and results

$N_G(v)$ denotes the *neighbour set* of a vertex v in a graph G. G is called a *K_aT graph* if $V(G)$ can be partitioned into two disjoint sets S and T, such that S is a clique of a vertices and T is an independent set with $N_G(v) \subseteq N_G(u)$ for $d(u) \geq d(v)$, where $u, v \in T$. It is reasonable to assume that $d(t) \leq a-1$ for any $t \in T$, since if $d(t) = a$, we still get a K_aT graph by moving t to S. In [11], graphs attaining the minimum value of I_d are characterized as K_aT graphs, which means that the discussion of the extremal properties only needs to be carried out for such a graph family.

Theorem 1. [11, Theorem 4] *Any connected graph attaining the minimum value of I_d must be isomorphic to a K_aT graph.*

Let $S = \{v_1, v_2, \ldots, v_a\}$ with $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_a)$. From the definition of K_aT graphs, we must have $N_G(t) = \{v_1, v_2, \ldots, v_{d(t)}\}$ for any vertex $t \in T$, and v_i $(1 \leq i \leq a)$ is adjacent to all the vertices whose degrees are not less than i in T. Hence, for (n,m)-graphs, if the size of the clique is given, a K_aT graph is determined uniquely by the degrees of the vertices in T up to isomorphism. Hereby, we denote a K_aT graph by $[1^{q_1}, 2^{q_2}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$, where $[1^{q_1}, 2^{q_2}, \ldots, (a-1)^{q_{a-1}}]$ is the *degree sequence* of T. That is, for any integer i with $1 \leq i \leq a-1$, if there is a vertex of degree i in T, then q_i is the number of such vertices; otherwise, $q_i = 0$. Obviously, $\sum_{i=1}^{a-1} q_i = |T|$. Note that i^{q_i} with $q_i = 0$ does not occur in the sequence for a certain graph. Now we can give the degrees of $v_1, v_2, \dots, v_a$: $a-1+\sum_{l=1}^{a-1} q_l, a-1+\sum_{l=2}^{a-1} q_l, \dots, a-1$. If $\sum_{l=2}^{a-1} q_l \le 1$, that is, there is at most one vertex of degree greater than 1 in T, then such graphs are called L^*. For given n and m, since L^* (also denoted by $L^*(n,m)$) achieves the maximum of a, L^* is determined.

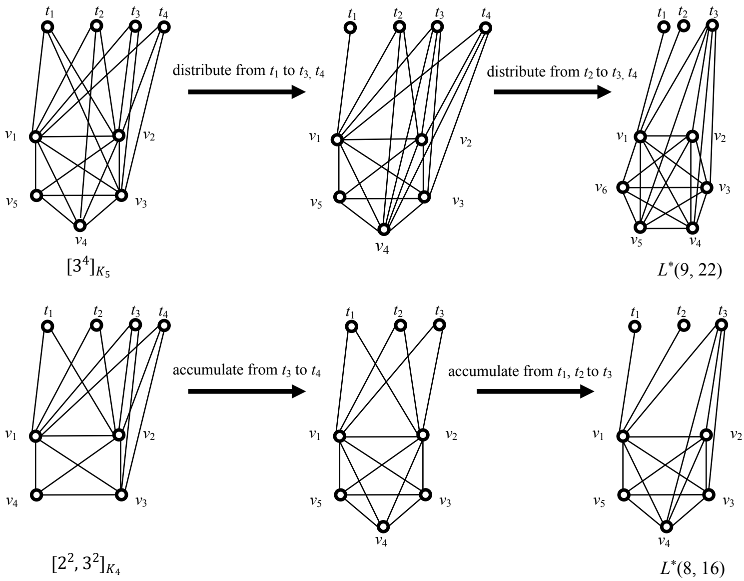

Figure 1. Transformations from K_aT graphs to L^*.

In [12], two graph operations are defined to explore extremal topological properties for (n,m)-graphs. Let $G = [1^{q_1}, 2^{q_2}, \dots, (a-1)^{q_{a-1}}]_{K_a}$ and i, j, k be three positive integers. If $k < i \le j \le a-1$, $q_i \ne 0$, and $q_j \ge k$, then there must exist a vertex t of degree i and k vertices $t_1, t_2, \dots, t_k$ of degree j. The operations of deleting the edges $tv_i, tv_{i-1}, \dots, tv_{i-k+1}$ and adding the edges $t_1v_{j+1}, t_2v_{j+1}, \dots, t_kv_{j+1}$ in G are called a *distribution* from t to $t_1, t_2, \dots, t_k$. We say that a distribution is *proper* when there is no vertex with degree greater than $i-k$ and less than $j+1$ in T of the resulting graph. If $1 < i \le j \le a-k$, $q_i \ge k$, and $q_j \ne 0$, we assume that $t_1, t_2, \dots, t_k$ are vertices of degree i and t is a vertex of degree j in T. Then the operations of deleting the edges $t_1v_i, t_2v_i, \dots, t_kv_i$ and adding the edges $tv_{j+1}, tv_{j+2}, \dots, tv_{j+k}$ in G are called an *accumulation* from $t_1, t_2, \dots, t_k$

to t. An accumulation is *proper* when there is no vertex with degree greater than $i-1$ and less than $j+k$ in T of the resulting graph. We see that edges are moved from vertices with smaller degree to those with lager degree after a distribution or an accumulation. Figure 1 illustrates the transforms from $[3^4]_{K_5}$ and $[2^2,3^2]_{K_4}$ to L^* by such operations, in which, except the distribution from t_1 to t_3, t_4, the other operations are proper. Theorem 2 and 3 suggest that $I_d(G)$ is decreased after a proper operation.

Theorem 2. [12, Theorem 2] *Let* $G=[1^{q_1},2^{q_2},\ldots,(a-1)^{q_{a-1}}]_{K_a}$ *and* G' *be the graph resulted from* G *by a distribution from a vertex of degree* i *to* k *vertices of degree* j*. If* $\sum_{l=i-k+1}^{j} q_l = k+1$*, then* $I_d(G) > I_d(G')$*.*

Theorem 3. [12, Theorem 3] *Let* $G=[1^{q_1},2^{q_2},\ldots,(a-1)^{q_{a-1}}]_{K_a}$ *and* G' *be the graph resulted from* G *by an accumulation from* k *vertices of degree* i *to a vertex of degree* j*. If* $\sum_{l=i}^{j+k-1} q_l = k+1$*, then* $I_d(G) > I_d(G')$*.*

3 Main results

In this section, we improve Theorem 2 and 3 by extending the conditions of being 'proper'. Recall the formula:

$$I_d(G) = \log(2m) - \frac{1}{2m}\sum_{v\in V(G)} d(v)\log d(v).$$

Let $g(x) = x\log x$. Then

$$I_d(G) = \log(2m) - \frac{1}{2m}\sum_{v\in V(G)} g(v).$$

Note that we always assume $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_a)$ for the clique $S=\{v_1,v_2,\ldots,v_a\}$ of a K_aT graph.

Theorem 4. *Let* $G=[1^{q_1},2^{q_2},\ldots,(a-1)^{q_{a-1}}]_{K_a}$ *and* G' *be the graph obtained by a distribution from a vertex of degree* i *to* k *vertices of degree* j *in* G*. If* $\sum_{l=i-k+1}^{j} q_l \leq (j-i)+(k+1)$*, then* $I_d(G) > I_d(G')$*.*

Proof. We prove the theorem by distinguishing two cases.

Case 1. $j < a-1$.

Then $G' = [1^{q_1}, \ldots, (i-k)^{q_{i-k}+1}, \ldots, i^{q_i-1}, \ldots, j^{q_j-k}, (j+1)^{q_{j+1}+k}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$, where ellipses denote the terms that are identical to those in G. It is noted that $G' = [1^{q_1}, \ldots, (i-k)^{q_{i-k}+1}, \ldots, i^{q_i-k-1}, (i+1)^{q_{i+1}+k}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$ for $i = j$, and the following discussion remains valid.

Then $d_{G'}(v_p) = d_G(v_p) - 1 = a - 2 + \sum_{l=p}^{a-1} q_l$ for $i-k+1 \leq p \leq i$; $d_{G'}(v_{j+1}) = d_G(v_{j+1}) + k = a - 1 + k + \sum_{l=j+1}^{a-1} q_l$; and the degree of any other vertex in S stays the same. Then we have

$$I_d(G) - I_d(G') = -\frac{1}{2m}\left[\sum_{v\in V(G)} g(v) - \sum_{v\in V(G')} g(v)\right]$$
$$= -\frac{1}{2m}\left[(A-B)-(C-D)\right],$$

where

$$A = \sum_{p=i-k+1}^{i}\left[g\left(a-1+\sum_{l=p}^{a-1} q_l\right) - g\left(a-2+\sum_{l=p}^{a-1} q_l\right)\right],$$

$$B = g\left(a-1+k+\sum_{l=j+1}^{a-1} q_l\right) - g\left(a-1+\sum_{l=j+1}^{a-1} q_l\right),$$

$$C = k\left[g(j+1)-g(j)\right],$$

$$D = g(i) - g(i-k).$$

Since $g(x) - g(x-1)$ is an increasing function of x,

$$A \leq k\left[g\left(a-1+\sum_{l=i-k+1}^{a-1} q_l\right) - g\left(a-2+\sum_{l=i-k+1}^{a-1} q_l\right)\right]. \tag{1}$$

Since $\sum_{l=i-k+1}^{j} q_l \leq j-i+k+1$ by the assumption,

$$A \leq k\left[g\left(a+j-i+k+\sum_{l=j+1}^{a-1} q_l\right) - g\left(a+j-i+k-1+\sum_{l=j+1}^{a-1} q_l\right)\right].$$

Let $X_1 = a+j-i+k+\sum_{l=j+1}^{a-1} q_l$, $X_2 = j+1$, and $h = j-i+1$. Then

$$A - B \leq k\left[g(X_1) - g(X_1-1)\right] - \left[g(X_1-h) - g(X_1-h-k)\right], \tag{2}$$

$$C - D = k\left[g(X_2) - g(X_2-1)\right] - \left[g(X_2-h) - g(X_2-h-k)\right]. \tag{3}$$

Since the derivative of $k\left[g(x)-g(x-1)\right]-\left[g(x-h)-g(x-h-k)\right]$ with respect to x equals $k[\log x - \log(x-1)] - [\log(x-h) - \log(x-h-k)] = k(\frac{1}{\xi_1} - \frac{1}{\xi_2})$, where $\xi_1 \in (x-1, x)$ and $\xi_2 \in (x-h-k, x-h)$, this function is strictly decreasing with x by $k(\frac{1}{\xi_1} - \frac{1}{\xi_2}) < 0$. And since $X_1 > X_2$ for $i \leq a-1$ and $k \geq 1$, $A-B < C-D$, and thus $I_d(G) > I_d(G')$.

Case 2. $j = a-1$.

We obtain $G' = [1^{q_1}, \ldots, (i-k)^{q_{i-k}+1}, \ldots, i^{q_i-1}, \ldots, (a-1)^{q_{a-1}-k}, a^k]_{K_a}$. Thus, $d_{G'}(v_a) = d_G(v_a) + k = a-1+k$, and the other vertices in S have the same degrees as they are in G' of Case 1. We have

$$I_d(G) - I_d(G') = -\frac{1}{2m}\left[(A-B) - (C-D)\right],$$

where A and D are the same here as in Case 1, $B = g\left(a-1+k\right) - g\left(a-1\right)$, and $C = k[g(a) - g(a-1)]$. It follows from the assumption that $\sum_{l=i-k+1}^{a-1} q_l \leq a-i+k$. Hence by inequality (1), we have

$$A \leq k\left[g\left(2a-i+k-1\right) - g\left(2a-i+k-2\right)\right].$$

Now let $X_1 = 2a-i+k-1$, $X_2 = a$, and $h = a-i$, which makes inequalities (2) and (3) hold as well. Since $X_1 > X_2$, $I_d(G) > I_d(G')$.

The proof is complete. ■

Theorem 5. *Let $G = [1^{q_1}, 2^{q_2}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$ and G' be the graph obtained by an accumulation from k vertices of degree i to a vertex of degree j in G. If $\sum_{l=i}^{j+k-1} q_l \leq (j-i)+(k+1)$, then $I_d(G) > I_d(G')$.*

Proof. There are two cases to be considered.

Case 1. $j+k \leq a-1$.

We have $G' = [1^{q_1}, \ldots, (i-1)^{q_{i-1}+k}, i^{q_i-k}, \ldots, j^{q_j-1}, \ldots, (j+k)^{q_{j+k}+1}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$. Noted that $G' = [1^{q_1}, \ldots, (i-1)^{q_{i-1}+k}, i^{q_i-k-1}, \ldots, (i+k)^{q_{i+k}+1}, \ldots, (a-1)^{q_{a-1}}]_{K_a}$ when $i = j$, and it makes no difference to the subsequent discussion. For any vertex v_p in the independent set S of G, if $j+1 \leq p \leq j+k$, $d_{G'}(v_p) = d_G(v_p) + 1 = a + \sum_{l=p}^{a-1} q_l$; $d_{G'}(v_i) = d_G(v_i) - k = a - 1 - k + \sum_{l=i}^{a-1} q_l$; $d_{G'}(v_p) = d_G(v_p)$ for the rest vertices. Let

$$A = g\left(a - 1 + \sum_{l=i}^{a-1} q_l\right) - g\left(a - 1 - k + \sum_{l=i}^{a-1} q_l\right),$$

$$B = \sum_{p=j+1}^{j+k}\left[g\left(a + \sum_{l=p}^{a-1} q_l\right) - g\left(a - 1 + \sum_{l=p}^{a-1} q_l\right)\right],$$

$$C = g(j+k) - g(j),$$

$$D = k[g(i) - g(i-1)].$$

Then

$$I_d(G) - I_d(G') = -\frac{1}{2m}[(A - B) - (C - D)].$$

Since $\sum_{l=i}^{j+k-1} q_l \leq j - i + k + 1$,

$$A \leq g\left(a + j - i + k + \sum_{l=j+k}^{a-1} q_l\right) - g\left(a + j - i + \sum_{l=j+k}^{a-1} q_l\right).$$

On the other hand, since $g(x) - g(x-1)$ is strictly increasing,

$$B \geq k\left[g\left(a + \sum_{l=j+k}^{a-1} q_l\right) - g\left(a - 1 + \sum_{l=j+k}^{a-1} q_l\right)\right].$$

Let $X_1 = a + j - i + \sum_{l=j+k}^{a-1} q_l$, $X_2 = j$, and $h = j - i$. Together, these two inequalities yield

$$A - B \leq [g(X_1 + k) - g(X_1)] - k[g(X_1 - h) - g(X_1 - h - 1)]. \qquad (4)$$

In addition,

$$C - D = [g(X_2 + k) - g(X_2)] - k[g(X_2 - h) - g(X_2 - h - 1)]. \qquad (5)$$

Taking the derivative of $[g(x+k) - g(x)] - k[g(x-h) - g(x-h-1)]$ as we have done in Theorem 4, we can show that $[g(x+k) - g(x)] - k[g(x-h) - g(x-h-1)]$ is strictly decreasing with x. Since $X_1 > X_2$, $A - B < C - D$, and hence, $I_d(G) > I_d(G')$.

Case 2. $j + k = a$.

Then $G' = [1^{q_1}, \ldots, (i-1)^{q_{i-1}+k}, i^{q_i - k}, \ldots, (a-k)^{q_{a-k}-1}, \ldots, (a-1)^{q_{a-1}}, a]_{K_a}$. We treat the case $k \geq 2$ as the case $k = 1$ is similar. Since the only difference is that there is no point v_p with $a - k + 1 \leq p \leq a - 1$ when $k = 1$, $B = 0$ and A, C, D stay the same, but this does not affect the result. For $v_p \in S$, $d_{G'}(v_p) = d_G(v_p) + 1 = a + \sum_{l=p}^{a-1} q_l$ when $a - k + 1 \leq p \leq a - 1$, $d_{G'}(v_a) = d_G(v_a) + 1 = a$, and the rest vertices are the same as they are in G' of Case 1. Let

$$B = \sum_{p=a-k+1}^{a-1} \left[g\left(a + \sum_{l=p}^{a-1} q_l\right) - g\left(a - 1 + \sum_{l=p}^{a-1} q_l\right)\right] + [g(a) - g(a-1)],$$

$$C = g(a) - g(a-k),$$

and A, D still denote the same expressions as in Case 1. Then

$$I_d(G) - I_d(G') = -\frac{1}{2m}[(A - B) - (C - D)].$$

We infer that

$$\begin{aligned} B &\geq (k-1)[g(a + q_{a-1}) - g(a - 1 + q_{a-1})] + [g(a) - g(a-1)] \\ &\geq k[g(a) - g(a-1)] \end{aligned}$$

In addition, from the assumption of $\sum_{l=i}^{a-1} q_l \leq a-i+1$, it follows that

$$A \leq g(2a-i) - g(2a-i-k).$$

Let $X_1 = 2a-i-k$, $X_2 = a-k$, and $h = a-k-i$. It is easy to check that equalities (4) and (5) still hold. Since $X_1 > X_2$, $A-B < C-D$, and so $I_d(G) > I_d(G')$.

The proof is complete. ■

4 An application

In this section, we characterize the minimal graphs for some dense graphs as an application of Theorem 4 and 5. To avoid the interference of 1-degree vertices, let $T_{\geq 2} = \{t \in T | d(t) \geq 2\}$.

Theorem 6. *For any connected (n,m)-graph G with $m > \frac{1}{2}(n+5)(n-6)$ for $n \geq 8$, $m > 11$ for $n = 7$, and $m > 9$ for $n = 6$, we have*

$$I_d(G) \geq I_d(L^*(n,m)),$$

where the equality holds if and only if G is isomorphic to $L^*(n,m)$.

Proof. By Theorem 1, it suffices to prove this for K_aT graphs. We first prove two preliminary claims.

Claim 1. If $m > \frac{1}{2}(n+5)(n-6)$ for a K_aT graph, then $|T| \leq 4$.

Suppose, to the contrary, $|T| \geq 5$, namely $a \leq n-5$. Since a K_aT graph with n vertices attains the maximum number of edges only if each vertex in T is adjacent to $a-1$ vertices in K_a, we have $m \leq \frac{1}{2}a(a-1)+(n-a)(a-1) = (-a+1)(\frac{a}{2}-n)$. Since $(-a+1)(\frac{a}{2}-n)$ is an increasing function of a on $[1,n]$ and $a \leq n-5$, $m \leq \frac{1}{2}(n+5)(n-6)$, a contradiction.

Claim 2. $a \geq 4$ for any K_aT graph satisfying the hypotheses.

Otherwise, there is a K_aT graph with $a \leq 3$. Since $|T| \leq 4$ by Claim 1, $n \leq 7$. But then $m \leq 11$ when $n = 7$, and $m \leq 9$ when $n = 6$. It is a contradiction.

By Claim 1, we have $|T_{\geq 2}| \leq 4$. If $|T_{\geq 2}| = 0$ or 1, G is just L^*; if $|T_{\geq 2}| = 2$, we can get L^* after performing a distribution from one vertex to the other, and $I_d(G) > I_d(L^*)$ by Theorem 4.

Next we suppose $|T_{\geq 2}| = 3$. If there exist two vertices u and v of G with $d(u) < d(v)$, distribute from u to v repeatedly until the degree of u decreases to 1, or the degree of v increases to a. Since the inequality $\sum_{l=d(u)}^{d(v)} q_l \leq d(v) - d(u) + 2$ always holds in the process, we have $I_d(G) > I_d(G')$ for the resulting graph G' by Theorem 4. In addition, $|T_{\geq 2}| \leq 2$ in G' and so, as we have proved, $I_d(G') \geq I_d(L^*)$. Thus, $I_d(G) > I_d(L^*)$. Now let all three vertices in $T_{\geq 2}$ have the same degree d. Let G' be the graph obtained by a distribution from one vertex to the other two as $d \geq 3$, or an accumulation from two vertices to the other one as $d = 2$. By Theorem 4 and 5, $I_d(G) > I_d(G')$. On the other hand, we see that either there exist vertices with different degrees in $T_{\geq 2}$ or $|T_{\geq 2}| \leq 2$ for G', and so $I_d(G') \geq I_d(L^*)$. Hence, $I_d(G) > I_d(L^*)$.

Now let $|T_{\geq 2}| = 4$. We start with a useful claim.

Claim 3. If there are two vertices u and v in $T_{\geq 2}$ of G with $d(u) - d(v) \geq 2$, then $I_d(G) > I_d(L^*)$.

Let G' be the graph obtained by distributing from u to v repeatedly in G as we did above. Since $\sum_{l=d(u)}^{d(v)} q_l \leq 4 \leq d(v) - d(u) + 2$ for each operation, $I_d(G) > I_d(G')$ by Theorem 4. So $I_d(G) > I_d(G') \geq I_d(L^*)$ for $|T_{\geq 2}| \leq 3$ in G'.

Hence by Claim 3, we only need to deal with graphs in the form of $G = [i^{q_i}, (i+1)^{q_{i+1}}]_{K_a}$, where $i \geq 2$ and $q_i + q_{i+1} = 4$. There are three distinct cases.

Case 1. $q_i = 2$.

Perform a distribution between two vertices with the same degree. By Theorem 4, the operation decreases $I_d(G)$. And since there are vertices whose degrees differ by at least two in $T_{\geq 2}$ or $|T_{\geq 2}| \leq 3$ in the resulting graph, we have $I_d(G) > I_d(L^*)$.

Case 2. $q_i = 1$ or $q_i = 3$.

Then $G = [i, (i+1)^3]_{K_a}$ or $[i^3, i+1]_{K_a}$. If $i \geq 3$, then distribute from a vertex of degree i to another two vertices with the same degree. The resulting graph $G' = [i-2, i+1, (i+2)^2]_{K_a}$ or $[i-2, (i+1)^3]_{K_a}$. Since the distribution satisfies the condition of Theorem 4, $I_d(G) > I_d(G')$. For similar reasons as in Case 1, $I_d(G') \geq I_d(L^*)$, and thus $I_d(G) > I_d(L^*)$.

Otherwise, $i = 2$. We first consider the case of $a = 4$. Then $G = [2, 3^3]_{K_4}$ or $[2^3, 3]_{K_4}$. By a simple calculation, we have $I_d(G) > I_d(L^*)$. Now assume that $a \geq 5$. We accumulate from two vertices with the same degree to another vertex of degree 3 and get $G' = [2^3, 5]_{K_a}$ or $[1^2, 2, 5]_{K_a}$. Since the accumulation satisfies the condition of Theorem 5, $I_d(G) > I_d(G')$. Furthermore, $I_d(G') > I_d(L^*)$ from Claim 3 so $I_d(G) \geq I_d(L^*)$.

Case 3. $q_i = 4$.

If $i \geq 4$, then distribute from one vertex to the other three vertices. We have $I_d(G) > I_d(L^*)$.

Otherwise, $i = 2$ or 3, that is, $G = [2^4]_{K_a}$ or $[3^4]_{K_a}$. If $a \geq 5$ for $[2^4]_{K_a}$ and $a \geq 6$ for $[3^4]_{K_a}$, accumulate from three vertices to the other one. We have $I_d(G) > I_d(L^*)$. And it is easy to see that the inequality also holds for $[2^4]_{K_4}, [3^4]_{K_4}$, and $[3^4]_{K_5}$.

The proof is complete. ∎

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