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Two-Parameter Bifurcation Analysis of a Discrete-Time Glycolysis Model

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Abstract

This study involves discretizing a continuous-time glycolysis model to derive its discrete-time equivalent and investigates its dynamics using normal form theory and bifurcation analysis. The discretization employs the forward Euler's scheme, and through rigorous analysis, we delve into codimension two bifurcations, with a specific focus on the 1:2, 1:3, and 1:4 strong resonances. The 1:2 resonance unveils intricate limit-cycle patterns, the 1:3 resonance reveals intriguing periodic solutions, and the 1:4 resonance showcases co-existing periodic and chaotic regimes. Our research sheds light on the complex behaviors of the discrete glycolysis model and

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provides valuable insights into its responses under varying parametric values. Additionally, this study demonstrates the applicability of normal form theory and bifurcation analysis in understanding the dynamics of biochemical systems, enriching our comprehension of the glycolysis process and its discrete dynamics. Moreover, we present numerical simulations to substantiate and validate our theoretical investigations. These simulations offer practical evidence and reinforce the findings obtained from the analytical study.

1 Introduction

Glycolysis is the metabolic process that converts glucose into pyruvate. Typically, two molecules of pyruvate are produced for each molecule of glucose consumed during this process. The utilization of pyruvate molecules culminates in the completion of the Krebs cycle. In essence, glycolysis is a vital biochemical process that takes place in living cells, enabling them to extract energy by utilizing glucose as a fuel source, This biochemical process lends itself to modeling through a set of differential equations, which captures its intricate dynamics and behavior [1, 2]:

$$\begin{cases} \frac{dx}{dt} = ay - x + x^2 y, \\ \frac{dy}{dt} = b - ay - x^2 y. \end{cases} \tag{1}$$

Where x and y symbolize the dimensionless concentrations of adenosine diphosphate (ADP) and fructose-6-phosphate (F6P), respectively, and $a > 0$, $b > 0$ represent kinetic parameters. For particular combinations of these kinetic parameters, the biochemical process exhibits a consistent and stable oscillatory behavior, characterized by a repeating pattern over time for system (1). Furthermore, the presence of this stable limit cycle ensures that the biochemical reaction reaches its desired state or stage [3]. For more comprehensive information regarding the glycolysis process and its mathematical modeling, we recommend referring to the sources [4–7]. Furthermore, references [8–11] delve into the qualitative properties and conduct bifurcation analysis of glycolysis models. Mickens [14] developed a nonstandard difference scheme for the system (1), demonstrating the positivity of solutions and establishing the presence of limit-cycle behavior in the discrete-time model.

Notice that, the use of a discrete-time glycolysis model is significant due to its efficiency in offering valuable insights into the dynamic behavior of the glycolysis pathway.. By discretizing the continuous-time model, researchers gain the ability to analyze the system's behavior under diverse conditions, parameter values, and control strategies. This discrete-time approach facilitates the application of bifurcation analysis techniques, aiding in the identification of critical points, limit cycles, and chaos, thereby enhancing our understanding of the glycolysis process. Furthermore, the discretization process enables the exploration of stability properties and the development of control strategies to influence the system's behavior. The utility of this discrete approach extends beyond biological insights into glycolysis, finding practical applications in engineering domains such as biotechnological processes and metabolic engineering. The ability to control and optimize glycolysis is essential for various industrial and medical applications. Ultimately, a discrete-time glycolysis model serves as a valuable tool to comprehend the dynamics of this pivotal metabolic pathway and its potential in driving advancements in both biological and engineering realms [15, 16, 21].

Subsequently, employing the forward Euler's scheme yields the following discrete-time version of (1):

$$\begin{cases} x_{n+1} = x_n + h\left(ay_n - x_n + x_n^2 y_n\right), \\ y_{n+1} = y_n + h\left(b - ay_n - x_n^2 y_n\right). \end{cases} \tag{2}$$

where, h represents the step size used in Euler's scheme. For similar discretization methods, we suggest referring to sources [12–17].

In [21], Din explored the local stability, period-doubling bifurcation, and chaos control of the discretized version of system (1) using the forward Euler's scheme. The study investigated the dynamical behavior of the discrete glycolysis model and examined the impact of varying parameters on stability and bifurcations.

The novel contributions of this paper are outlined as follows:

- The study explores multi-parameter bifurcation phenomena in the context of a discrete-time glycolysis model. Investigating the 1:2, 1:3, and 1:4 strong resonance conditions reveals intricate and non-trivial dynamics, such as limit cycles and chaos, offering a deeper understanding of how the glycolysis pathway behaves under these specific resonance states.

- Understanding how discrete dynamics impact the glycolysis process is a novelty in itself. By discretizing the continuous-time model, researchers can gain insights into how discrete time steps influence the system's stability and behavior, which is crucial for comprehending real-world biochemical processes with inherent discrete nature.

- Studying the coexistence of different resonance regimes provides valuable insights into the robustness and sensitivity of the glycolysis model under varying parameter values. This knowledge is essential in understanding how the pathway responds to changes in the environment and internal conditions.

- The research bridges the gap between the fields of nonlinear dynamics, mathematical biology, and biochemical engineering. The insights gained from studying discrete-time glycolysis models have implications beyond the specific pathway, potentially benefiting other areas of research involving dynamical systems and complex networks.

The rest of this paper is structured as follows:

Stability analysis of system 2 is discussed in section 1. Codimension-two bifurcations(that is, 1:2, 1:3 and 1:4 strong resonances) are studied in Section 3 and in Section 4 numerical simulations are presented.

2 Stability analysis

It is easy to see that system Eq. 2 has unique positive fixed point $E(x_*, y_*) = (b, \frac{b}{a+b^2})$. Subsequently, we examine the local stability analysis of $E(x_*, y_*) = (b, \frac{b}{a+b^2})$ of system Eq. 2. To investigate the stability, we compute the Jacobian matrix F_J of system (2) at $E(x_*, y_*)$ as follow:

$$F_J(E) = \begin{pmatrix} \frac{h(b^2-a)}{a+b^2}+1 & h\left(a+b^2\right) \\ -\frac{2b^2h}{a+b^2} & 1-h\left(a+b^2\right) \end{pmatrix}.$$

The characteristic polynomial of F_J at $E(x_*, y_*)$ is given by:

$$\mathbb{F}(\varsigma) = \varsigma^2 - \tau_1(E)\varsigma + \tau_2(E), \tag{3}$$

where

$$\tau_1(E) = -\left(h\left(a+b^2\right)\right) + \frac{h\left(b^2-a\right)}{a+b^2} + 2,$$

and

$$\tau_2(E) = h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} + 1.$$

The following Lemma is used to explore the stability of fixed point.

Lemma 1. *Let $\mathbb{F}(\varsigma) = \varsigma^2 - \tau_1(E)\varsigma + \tau_2(E)$, and $\mathbb{F}(1) > 0$. Moreover, ς_1, $varsigma_2$ are root of 3, then:*
(i) $|\varsigma_1| < 1$ and $|\varsigma_2| < 1$ if and only if $\mathbb{F}(-1) > 0$ and $\tau_2(E) < 1$;
(ii) $|\varsigma_1| < 1$ and $|\varsigma_2| > 1$ or ($|\varsigma_1| > 1$ and $|\varsigma_2 < |1$)if and only if $\mathbb{F}(-1) < 0$;
(iii) $|\varsigma_1| > 1$ and $|\varsigma_2| > 1$ if and only if $\mathbb{F}(-1) > 0$ and $\tau_2(E) > 1$;
(iv) $\varsigma_1 = -1$ and $|\varsigma_2| \neq 1$ if and only if $\mathbb{F}(-1) = 0$ and $\tau_1(E) \neq 0, 2$;
(v) ς_1 and ς_2 are complex and $|\varsigma_1| = 1$ and $|\varsigma_2| = 1$ if and only if $\tau_1(E)^2 - 4\tau_2(E) < 0$ and $\tau_2(E) = 1$.
(vi) $\varsigma_1 = -1$ and $\varsigma_2 = -1$ if and only if $\tau_1(E) = -2$ and $\tau_2(E) = 1$;
(vii) ς_1 and ς_2 are complex and $\varsigma_{1,2} = -\frac{1}{2} \pm \iota\frac{\sqrt{3}}{2}$ if and only if $\tau_1(E) = -1$ and $\tau_2(E) = 1$;
(viii) ς_1 and ς_2 are complex and $\varsigma_{1,2} = \pm\iota$ if and only if $\tau_1(E) = 0$ and $\tau_2(E) = 1$;

As ς_1 and ς_2 are eigenvalue of (3), we have the following Topological type results. The fixed point $E(x_*, y_*)$ is known as sink if $|\varsigma_1| < 1$ and $|\varsigma_2| < 1$ thus the sink is locally asymptotic stable. The fixed point $E(x_*, y_*)$ is known as source if $|\varsigma_1| > 1$ and $|\varsigma_2| > 1$, thus source is always unstable.

The fixed point $E(x_*, y_*)$ is known as saddle point if $|\varsigma_1| < 1$ and $|\varsigma_2| > 1$ or ($|\varsigma_1| > 1$ and $|\varsigma_2| < 1$) and the fixed point $E(x_*, y_*)$ is known as non-hyperbolic fixed point either $|\varsigma_1| = 1$ and $|\varsigma_2| = 1$.

Thus, by applying Lemma 1, we study the local stability of positive equilibrium point of system (2) by stating the following proposition.

Proposition 1. *The positive equilibrium point $E(x_*, y_*)$ of system (2) satisfies the following results.*

(i) The positive fixed point $E(x_*, y_*)$ is sink if and only if:

$$h^2\left(a+b^2\right) - \frac{2h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} + 4 > 0,$$

and

$$h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} < 0.$$

(ii) The positive fixed point $E(x_*, y_*)$ is saddle point if and only if:

$$h^2\left(a+b^2\right) - \frac{2h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} + 4 < 0.$$

(iii) The positive fixed point $E(x_*, y_*)$ is source if and only if:

$$h^2\left(a+b^2\right) - \frac{2h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} + 4 > 0,$$

and

$$h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2 + a(a+1) + b^4\right)}{a+b^2} > 0.$$

(iv) The positive fixed point $E(x_*, y_*)$ is non-hyperbolic if and only if:

$$\begin{cases} h = \frac{\left(a+b^2\right)\sqrt{\frac{4a^2}{\left(a+b^2\right)^2} + 2(a-3)b^2 - \frac{4a}{a+b^2} + (a-1)^2 + b^4} + \left(a+b^2\right)^2 + a - b^2}{\left(a+b^2\right)^2}, \\ \text{and} \\ 2 - \left(h\left(a+b^2\right)\right) + \frac{h\left(b^2-a\right)}{a+b^2} \neq 0, 2. \end{cases} \tag{4}$$

0r

$$\begin{cases} h = \frac{a-b^2}{(a+b^2)^2} + 1, \\ and \\ \left(2 - \left(h\left(a+b^2\right)\right) + \frac{h\left(b^2-a\right)}{a+b^2}\right)^2 < 4. \end{cases} \tag{5}$$

or

$$\begin{cases} h = \frac{4}{\sqrt{4\sqrt{2}b+1}+1}, \\ a = \frac{1}{2}\left(2\left(\sqrt{2}-b\right)b + \sqrt{4\sqrt{2}b+1}+1\right). \end{cases} \tag{6}$$

or

$$\begin{cases} a = \frac{3\left(h^2+3h-3\right)}{2h^4}, \\ b = \frac{\sqrt{\frac{3}{2}}\sqrt{h^2-3h+3}}{h^2}. \end{cases} \tag{7}$$

or

$$\begin{cases} a = \frac{h^2+2h-2}{h^4}, \\ b = \frac{\sqrt{h^2-2h+2}}{h^2}. \end{cases} \tag{8}$$

Moreover, for $h = 0.915$, $a \in [0.001,\ 4]$, $b \in [0.001,\ 4]$, the topological classification for system (2) is shown in Figure 1.

3 Codimension-two bifurcations

In this section we study the codimension-two bifurcation. It is easy to see that system (2) has a unique positive equilibrium point $(x_*,\ y_*) = (b,\ \frac{b}{a+b^2})$. For an in-depth examination of local stability, co-dimension-1 bifurcation, and chaos control of (2), refer to [21]. In particular, we investigate the existence of 1:2, 1:3 and 1:4 resonances by implementing normal form theory and theory of bifurcation. The following curves identify the occurrence of these resonance points:

$$R2: \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -4,$$

$$R3: \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -3,$$

$$R4: \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -2,$$

and

$$NS: h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2+a(a+1)+b^4\right)}{a+b^2} = 0.$$

Then, it is easy to observe that $NS \cap R_2$, $NS \cap R_3$ and $NS \cap R_4$ are known as 1:2 ,1:3 and 1:4 resonance points, respectively. Moreover, for $h = 0.915$, $a \in [0.001,\ 4]$, $b \in [0.001,\ 4]$, the topological classification for system (2) is shown in Figure 1.

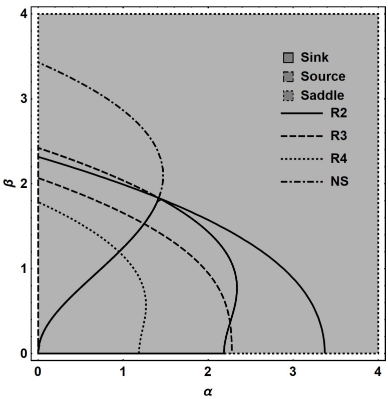


Figure 1. Topological classification for system (2).

3.1 1:2 strong resonance

This subsection delves with the investigation of 1:2 strong resonance for system (2) at its positive equilibrium point. For this, h and a are chosen to be bifurcation parameters. The Jacobian matrix of system (2) computed at positive equilibrium has eigenvalue -1 with multiplicity two if the following conditions holds true:

$$\begin{cases} Tr : \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -4 \\ Det : h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2+a(a+1)+b^4\right)}{a+b^2} = 0. \end{cases} \tag{9}$$

Solving system (9) for h and a yields the following solution (h_0, a_0):

$$h_0 = \frac{4}{\sqrt{4\sqrt{2}b+1}+1},$$

and

$$a_0 = \frac{1}{2}\left(2\left(\sqrt{2}-b\right)b + \sqrt{4\sqrt{2}b+1}+1\right).$$

Let $x_n = u_n + b$, $y_n = v_n + \frac{b}{a+b^2}$, $h = h_0 + \bar{h}$ and $a = a_0 + \bar{a}$, then the system (2) can be transformed as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} 1+\mu_{11} & -\mu_{12} \\ \mu_{21} & 1+\mu_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ f_2(u,v,) \end{pmatrix}, \tag{10}$$

where $\bar{h} << 1$ and $\bar{a} << 1$ are small perturbations,

$$\begin{aligned} f_1(u,v) &= \mu_{13}uv + \mu_{14}u^2 + O\left((|u|+|v|)^3\right), \\ f_2(u,v)) &= \mu_{23}uv + \mu_{24}u^2 + O\left((|u|+|v|)^3\right). \end{aligned}$$

$$\mu_{11} = \frac{h\left(b^2-a\right)}{a+b^2}, \quad \mu_{12} = -h\left(a+b^2\right), \quad m_{21} = -\frac{2b^2h}{a+b^2},$$

$$\mu_{22} = -h\left(a+b^2\right), \quad \mu_{13} = 2bh, \quad \mu_{23} = -2bh,$$

$$\mu_{14} = \frac{bh}{a+b^2}, \quad \mu_{24} = -\frac{bh}{a+b^2}.$$

Next, we consider the following transformation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} w \\ z \end{pmatrix}, \tag{11}$$

where T is a nonsingular matrix given by

$$T = \begin{pmatrix} \frac{\mu_{12}}{\mu_{11}+2} & \frac{\mu_{12}}{(\mu_{11}+2)^2} \\ 1 & 0 \end{pmatrix}.$$

From (10) and (11), it follows that:

$$\begin{pmatrix} w \\ z \end{pmatrix} \to \begin{pmatrix} P_{10}-1 & P_{01}+1 \\ Q_{10} & Q_{01}-1 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} f_3(w,z,h,a) \\ f_4(w,z,h,a) \end{pmatrix}, \quad (12)$$

where

$$f_3(w,z) = P_{20}w^2 + P_{11}wz + P_{02}z^2, \quad f_4(w,z) = Q_{20}w^2 + Q_{11}wz + Q_{02}z^2,$$

$$P_{10} = \frac{\mu_{12}\mu_{21}}{\mu_{11}+2} + \mu_{22} + 2, \quad P_{01} = \frac{\mu_{12}\mu_{21}}{(\mu_{11}+2)^2} - 1,$$

$$P_{20} = \frac{\mu_{12}\left((\mu_{11}+2)\mu_{23} + \mu_{12}\mu_{24}\right)}{(\mu_{11}+2)^2},$$

$$P_{11} = \frac{\mu_{12}\left((\mu_{11}+2)\mu_{23} + 2\mu_{12}\mu_{24}\right)}{(\mu_{11}+2)^3}, \quad P_{02} = \frac{\mu_{12}^2\mu_{24}}{(\mu_{11}+2)^4},$$

$$Q_{20} = \left(\mu_{14} - \mu_{23} - \frac{\mu_{12}\mu_{24}}{\mu_{11}+2}\right)\mu_{12} + (\mu_{11}+2)\mu_{13},$$

$$Q_{11} = \frac{\mu_{13}(\mu_{11}+2)^2 + \mu_{12}\left((\mu_{11}+2)(2\mu_{14}-\mu_{23}) - 2\mu_{12}\mu_{24}\right)}{(\mu_{11}+2)^2},$$

$$Q_{02} = \frac{\mu_{12}\mu_{14}}{(\mu_{11}+2)^2}, \qquad Q_{01} = \mu_{11} - \frac{\mu_{12}\mu_{21}}{\mu_{11}+2} + 2$$

$$Q_{10} = -\mu_{12}\mu_{21} - \mu_{11}(\mu_{22}+2) - 2(\mu_{22}+2).$$

Next, we assume the following invertible linear transformation:

$$\begin{pmatrix} w \\ z \end{pmatrix} = M \begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix}, \quad (13)$$

where

$$M = \begin{pmatrix} 1 + P_{01}(h,\ a) & 0 \\ -P_{01}(h,\ a) & 1 \end{pmatrix}.$$

From (12) and (14), it follows that:

$$\begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ \omega_1 & \omega_2 - 1 \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} f_5(\bar{w}, \bar{z}, h, a) \\ f_6(\bar{w}, \bar{z}, h, a) \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}
\omega_1(h, a) &= Q_{10} + P_{01}Q_{10} - P_{10}Q_{01}, \\
\omega_2(h, a) &= P_{10} + Q_{01}, \\
f_5(\bar{w}, \bar{z}, \alpha, r) &= \bar{P}_{20}\bar{w}^2 + \bar{P}_{11}\bar{w}\bar{z} + \bar{P}_{02}\bar{z}^2, \\
f_6(\bar{w}, \bar{z}, \alpha, r) &= \bar{Q}_{20}\bar{w}^2 + \bar{Q}_{11}\bar{w}\bar{z} + \bar{Q}_{02}\bar{z}^2, \\
\bar{Q}_{20} &= \frac{(P_{01}+1)^3 Q_{20} + P_{10}(P_{01}+1)\left[P_{10}(Q_{02}-P_{11})\right.}{P_{01}+1} \\
&\quad + \frac{\left.+ (P_{01}+1)(P_{02}-Q_{11})\right] + P_{02}P_{10}^3}{P_{01}+1}, \\
\bar{P}_{20} &= \frac{P_{02}P_{10}^2}{P_{01}+1} - P_{01}P_{11} + P_{01}P_{20} + P_{20}, \\
\bar{P}_{11} &= P_{11} - \frac{2P_{02}P_{10}}{P_{01}+1}, \\
\bar{Q}_{11} &= P_{10}\left(-\frac{2P_{02}P_{10}}{P_{01}+1} + P_{11} - 2Q_{02}\right) + (P_{01}+1)\,Q_{11}, \\
\bar{P}_{02} &= \frac{P_{02}}{P_{01}+1}, \\
\bar{Q}_{02} &= \frac{P_{02}P_{10}}{P_{01}+1} + Q_{02}.
\end{aligned}$$

Taking into account ω_1 and ω_2, we define the following matrix:

$$\zeta(h_0, a_0) = \begin{pmatrix} \frac{\partial\omega_1}{\partial h}(h_0, r_0) & \frac{\partial\omega_1}{\partial a}(h_0, a_0) \\ \frac{\partial\omega_2}{\partial h}(h_0, r_0) & \frac{\partial\omega\omega_2}{\partial a}(h_0, a_0) \end{pmatrix}.$$

Then by simple calculation $det\zeta(h_0, a_0)$ is obtained as follows:

$$det\zeta(h_0, a_0) = \frac{h^2\left((2a+5)b^2 + (a-1)a + b^4\right)}{a+b^2} \neq 0. \tag{15}$$

Condition (15) is called transversality condition, and it is supposed to be true. Next, we used $\omega_1(h, a)$ and $\omega_2(h, a)$ for the following parametrization in the neighborhood of $h = h_0$ and $a = a_0$:

$$\gamma_1 = \omega_1(h, a), \quad \gamma_2 = \omega_2(h, a). \tag{16}$$

Using (16) in (14), we have the following mapping:

$$\begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 \\ \gamma_1 & -1+\gamma_2 \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} f_7(\bar{w}, \bar{z}, \gamma_1, \gamma_2) \\ f_8(\bar{w}, \bar{z}, \gamma_1, \gamma_2) \end{pmatrix}, \tag{17}$$

where

$$f_7(\bar{w}, \bar{z}, \gamma_1, \gamma_2) = g_{20}\bar{w}^2(\gamma_1, \gamma_2) + g_{11}\bar{w}\bar{z}(\gamma_1, \gamma_2) + g_{02}\bar{z}^2(\gamma_1, \gamma_2),$$

$$f_8(\bar{w}, \bar{z}, \gamma_1, \gamma_2) = h_{20}\bar{w}^2(\gamma_1, \gamma_2) + h_{11}\bar{w}\bar{z}(\gamma_1, \gamma_2) + h_{02}\bar{z}(\gamma_1, \gamma_2),$$

$$\begin{aligned} g_{20}(\gamma, \gamma_2) &= \bar{P}_{20}(\gamma, \gamma_2), \quad g_{11}(\gamma, \gamma_2) = \bar{P}_{11}(\gamma, \gamma_2), \quad g_{11} = \bar{P}_{11}, \\ h_{20}(\gamma, \gamma_2) &= \bar{Q}_{20}(\gamma, \gamma_2), \quad h_{11}(\gamma, \gamma_2) = \bar{Q}_{11}(\gamma, \gamma_2), \quad h_{02} = \bar{Q}_{02}. \end{aligned}$$

Then according to Lemma 9.9 [[22], p. 437], there exists a near identity map such that system (14) can be transformed as follows:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 \\ \gamma_1 & -1+\gamma_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ Cz_1^3 + Dz_1z_2 \end{pmatrix} \\ + O(|z_1 + z_2|^4), \end{aligned} \tag{18}$$

where

$$C(\gamma_1,\gamma_2) = g_{20}(\gamma_1,\gamma_2)h_{20}(\gamma_1,\gamma_2) + \frac{1}{2}h_{20}^2(\gamma_1,\gamma_2) + \frac{1}{2}h_{20}(\gamma_1,\gamma_2)h_{11}(\gamma_1,\gamma_2),$$

$$\begin{aligned} D(\gamma_1,\gamma_2) &= \frac{1}{2}g_{20}(\gamma_1,\gamma_2)h_{11}(\gamma_1,\gamma_2) + \frac{5}{4}h_{20}(\gamma_1,\gamma_2)h_{11}(\gamma_1,\gamma_2) \\ &+ h_{20}^2(\gamma_1,\gamma_2) + \frac{1}{2}h_{11}^2(\gamma_1,\gamma_2) + h_{20}(\gamma_1,\gamma_2)h_{02}(\gamma_1,\gamma_2) \\ &+ 3g_{20}^2(\gamma_1,\gamma_2) + \frac{5}{2}g_{20}(\gamma_1,\gamma_2)h_{20}(\gamma_1,\gamma_2) \\ &+ \frac{5}{2}g_{11}(\gamma_1,\gamma_2)h_{20}(\gamma_1,\gamma_2). \end{aligned}$$

Taking into account theoretical results cited in [22] and the above computations, we have the following result.

Theorem 2. *Assume that $C(0,0) \neq 0$, $D(0,0) + 3C(0,0) \neq 0$, and $det\zeta(h_0, a_0) \neq 0$ then system (2) experiences 1:2 strong resonance at its positive equilibrium point whenever h and a vary in small neighborhoods of h_0 and a_0, respectively.*

3.2 1:3 strong resonance

In this subsection we study codimension-two bifurcation associated with 1:3 strong resonance. For this, assume that a and b are bifurcation parameters. Then characteristic equation of variational matrix of system (2) at $(x_*,\ y_*)$ has eigenvalues $-\frac{1}{2} \pm \iota\frac{\sqrt{3}}{2}$ if the following condition holds true:

$$\begin{cases} Tr : \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -3 \\ Det : h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2+a(a+1)+b^4\right)}{a+b^2} = 0. \end{cases} \tag{19}$$

We have the following solution of system (19) for a and b:

$$a_1 = \frac{3(h(h+3)-3)}{2h^4},$$

$$b_1 = \frac{\sqrt{\frac{3}{2}}\sqrt{(h-3)h+3}}{h^2}.$$

Next, assume that $u_n = x_n - b$, $v_n = y_n - \frac{b}{a+b^2}$ and $a = a_1$ and $b = b_1$, then equilibrium point (x_*, y_*) of (2) is shifted at $(0,0)$. In this case (2) transformed into the following map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ f_2(u,v,) \end{pmatrix}, \tag{20}$$

$$\begin{aligned} \xi_{11} &= \frac{h\left(b^2 - a\right)}{a+b^2} + 1, \quad \xi_{12} = h\left(a+b^2\right), \\ \xi_{21} &+ -\frac{2b^2 h}{a+b^2}, \quad \xi_{22} = 1 - h\left(a+b^2\right), \end{aligned}$$

$$\begin{aligned} f_1(u,v) &= r_{11}uv + r_{02}u^2 + O\left((|u|+|v|)^3\right), \\ f_2(u,v)) &= q_{11}uv + q_{02}u^2 + O\left((|u|+|v|)^3\right). \end{aligned}$$

$$r_{02} = \frac{bh}{a+b^2}, \quad r_{11} = 2bh,$$

$$q_{02} = -\frac{bh}{a+b^2}, \quad q_{11} = -2bh.$$

The eigenvalues of characteristics equation of Jacobian matrix of system (20) are $\frac{-1}{2} \pm \frac{\sqrt{3}}{2}\iota$, let $\rho_1(a_1, b_1)$ and $\varrho_1(a_1, b_1)$ are eigenvector associated with Jacobian matrix of (20) and its transpose, respectively and satisfying $\langle \rho_1(a_1, b_1), \varrho_1(a_1, b_1)\rangle = 1$. Then, by simple computation one has;

$$\rho_1(a_1, b_1) = \begin{pmatrix} \frac{6i}{\left(3i+\sqrt{3}\right)h-6i} \\ 1 \end{pmatrix},$$

and

$$\varrho_1(a_1, b_1) = \begin{pmatrix} 1 + \frac{1}{6}i\left(3i+\sqrt{3}\right)h \\ 1 \end{pmatrix}.$$

Further, any $Y \in \mathbb{R}^2$ can be uniquely described as follows:

$$Y = w\rho_1(a_1, b_1) + \bar{w}\bar{\rho_1}(a_1, b_1), \quad w \in \mathcal{C}.$$

Therefore, the complex form for the map (20) can be written as follows:

$$w \longrightarrow \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\iota\right) w + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} G_{jk} w^j \bar{w}^k, \tag{21}$$

where

$$G_{20} = \frac{2bh\left(2\sqrt{3}h - 3\left(\sqrt{3}+i\right)\right)\left(\left(\sqrt{3}+3i\right)h\left(a+b^2\right) - 3i\left(2a+2b^2-1\right)\right)}{\left(\left(\sqrt{3}+3i\right)h - 6i\right)^2\left(a+b^2\right)},$$

$$G_{11} = -\frac{ibh\left(2\sqrt{3}h - 3\left(\sqrt{3}+i\right)\right)\left(a(h-2) + b^2(h-2) + 1\right)}{\left((h-3)h+3\right)\left(a+b^2\right)},$$

$$G_{02} = -\frac{2bh\left(2\sqrt{3}h - 3\left(\sqrt{3}+i\right)\right)\left(\left(\sqrt{3}-3i\right)h\left(a+b^2\right) + 3i\left(2a+2b^2-1\right)\right)}{\left(\left(\sqrt{3}-3i\right)h + 6i\right)^2\left(a+b^2\right)},$$

$$\text{and } G_{30} = G_{03} = G_{12} = G_{21} = 0.$$

Next, according to Lemma 9.12 [[22], p. 448], there exists a smoothly parameter dependent change of variable such that the map (21) can be converted into the following form:

$$z \longrightarrow \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\iota\right) z + F(a_1, b_1)\bar{z} + K(a_1, b_1)z|z|^2 + \left(|z|^4\right), \tag{22}$$

where

$$F(a_1, b_1) = \frac{1}{2}G_{02},$$

and

$$K(a_1, b_1) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\iota\right) G_{02}G_{11} + \left(\frac{1}{2} + \frac{-1}{2\sqrt{3}}\iota\right)|G_{11}|.$$

Next, we consider the following quantities:

$$F_1(a_1, b_1) = \left(\frac{-3}{2} + \frac{3\sqrt{3}}{2}\iota\right) F(a_1, b_1)$$

$$K_1(a_1, b_1) = -3\,|F(a_1, b_1)|^2 - \frac{3}{2}(1 + \sqrt{3}\iota)K(a_1, b_1).$$

Arguing as in Lemma 9.13 [[22], p. 450], we have the following result.

Theorem 3. *Assume that $a = a_1$, $b = b_1$, $ReK_1(a_1, b_1) \neq 0$ and $F(a_1, b_1)$ $\neq 0$ then the system (2) undergoes a 1:3 resonance about its positive fixed point, $ReK_1(a_1, b_1) \neq 0$ determines the stability nature for the bifurcating closed invariant curve.*

3.3 1:4 strong resonance

In this subsection we study codimension-two bifurcation associated with 1:4 strong resonance. For this, assume that a and b are bifurcation parameters. Then characteristic equation of variational matrix of system (2) at $(x_*,\ y_*)$ has eigenvalues $\pm\iota$ if the following condition holds true:

$$\begin{cases} Tr : \frac{h\left(b^2-a\right)}{a+b^2} - h\left(a+b^2\right) = -2 \\ Det : h^2\left(a+b^2\right) - \frac{h\left((2a-1)b^2+a(a+1)+b^4\right)}{a+b^2} = 0. \end{cases} \tag{23}$$

We have the following solution of system (23) for a and b:

$$a_2 = \frac{h(h+2)-2}{h^4},$$

$$B_2 = \frac{\sqrt{(h-2)h+2}}{h^2}.$$

Next, assume that $u_n = x_n - b$, $v_n = y_n - \frac{b}{a+b^2}$ and $a = a_2$ and $b = b_2$, then equilibrium point (x_*, y_*) of (2) is shifted at $(0, 0)$. In this case (2) transformed into the following map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_3(u, v) \\ f_4(u, v,) \end{pmatrix}, \tag{24}$$

$$
\begin{aligned}
\theta_{11} &= \frac{h\left(b^2-a\right)}{a+b^2}+1, \quad \theta_{12}=h\left(a+b^2\right), \\
\theta_{21} &+ -\frac{2b^2h}{a+b^2}, \quad \theta_{22}=1-h\left(a+b^2\right),
\end{aligned}
$$

$$
\begin{aligned}
f_3(u,v) &= \chi_{11}uv+\chi_{02}u^2+O\left((|u|+|v|)^3\right), \\
f_4(u,v)) &= \varsigma_{11}uv+\varsigma_{02}u^2+O\left((|u|+|v|)^3\right).
\end{aligned}
$$

$$
\chi_{02}=\frac{bh}{a+b^2}, \quad \chi_{11}=2bh,
$$

$$
\varsigma_{02}=-\frac{bh}{a+b^2}, \quad \varsigma_{11}=-2bh.
$$

The eigenvalues of Jacobian matrix of system (24) are $\pm\iota$, let $p(a_2,b_2)$ and $q(a_2,b_2)$ are eigenvector associated with Jacobian matrix of (24) and its transpose, respectively and satisfying $\langle p(a_2,b_2), q(a_2,b_2)\rangle = 1$. Then, by simple computation one has;

$$
p(a_2,b_2)=\begin{pmatrix} \frac{1+i}{h+(-1-i)} \\ 1 \end{pmatrix},
$$

and

$$
q(a_2,b_2)=\begin{pmatrix} 1-\left(\frac{1}{2}-\frac{i}{2}\right)h \\ 1 \end{pmatrix}.
$$

Moreover, any $Y\in\mathbb{R}^2$ can be described uniquely as follows:

$$
Y=wp(a_2,b_2)+\bar{w}\bar{p}(a_2,b_2), \quad w\in\mathcal{C}.
$$

Consequently, the complex form for the map (24) can be written as follows:

$$
w \longrightarrow (\iota)\,w+\sum_{2\le j+k\le 3}\frac{1}{j!k!}\bar{G}_{jk}w^j\bar{w}^k, \tag{25}
$$

where

$$\bar{G}_{20} = \frac{\sqrt{(h-2)h+2}(-4+h(h+(2-2i)))}{2(h+(-1-i))h},$$

$$\bar{G}_{11} = -\frac{i\sqrt{(h-2)h+2}(h(h+2)-4)}{(h+(-1+i))h},$$

$$\bar{G}_{02} = -\frac{(h+(-1-i))\sqrt{(h-2)h+2}(-4+h(h+(2+2i)))}{2(h+(-1+i))^2 h},$$

and $\bar{G}_{30} = \bar{G}_{03} = \bar{G}_{12} = \bar{G}_{21} = 0$.

Next, according to Lemma 9.13 [[22], p. 448], there exists a smoothly parameter–dependent change of variable such that the map (25) can be converted into the following form:

$$z_1 \longrightarrow (\iota)\, z_1 + F_2(a_2, b_2) z_1 |z_1|^2 + K_2(a_2, b_2) z_1^3 + \left(|z_1|^4\right), \tag{26}$$

where

$$F_2 = \iota \bar{G}_{11} - \frac{1}{2}\bar{G}_{11}\bar{\bar{G}}_{20}(1+\iota) + \bar{\bar{G}}_{11}\bar{G}_{20} + \bar{G}_{02}\bar{G}_{11}(\iota - 1) - \frac{1}{2}\bar{G}_{11}\bar{G}_{20}(1-2\iota),$$

and

$$K_2(a_2, b_2) = \frac{\iota - 1}{4}\bar{G}_{11}\bar{G}_{02} - \frac{\iota + 1}{4}\bar{G}_{11}\bar{G}_{20}.$$

Next, we consider the following quantities:

$$F_3(a_2, b_2) = -4\iota F_2(a_2, b_2)$$

$$K_3(a_2, b_2) = -4\iota K_2(a_2, b_2),$$

whenever $K_3(a_2, b_2) \neq 0$, thus we can write Jacobian matrix $J(a_2, b_2) = \frac{F_3(a_2,b_2)}{|K_3(a_2,b_2)|}$. Arguing as in Lemma 9.15 [[22], p. 450], we have the following result.

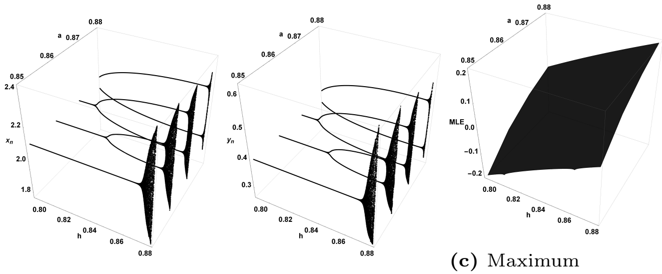
Theorem 4. *Assume that $a = a_2$, $b = b_2$, $ReJ(a_2, b_2) \neq 0$ and $ImJ(a_2, b_2)$ $\neq 0$ then the system (2) undergoes a 1:4 resonance about its positive fixed* *point, $ReJ(a_2, b_2) \neq 0$ determines the stability nature for the bifurcating closed invariant curve.*

4 Numerical simulation

Let $(a,\ b,\ h) = (0.85424,\ 2.1,\ 0.87169)$, then $(x_*,\ y_*) = (2.1,\ 0.39891)$. In this case eigenvalue of Jacobian Matrix at $(x_*,\ y_*)$ is -1 with multiplicity two. Moreover, $det\left(\zeta(h_0, a_0)\right) = \frac{h^2\left((2a+5)b^2+(a-1)a+b^4\right)}{a+b^2} = 7.05941$, $C(0,\ 0) = -8.61177$ and $D(0,\ 0) + 3C(0,\ 0) = 79.6377$, which shows the correctness of Theorem 2. Hence, system (2) undergoes codimension-two bifurcation associated with 1:2 strong resonance whenever $h \in [0.76,\ 0.879]$ and $a \in [0.85,\ 0.88]$. Alternatively, the bifurcation diagram in $(h,\ a,\ x_n)$, $(h,\ a,\ y_n)$ spaces and MLE are depicted in Figure 2a, 2b and 2c, respectively.

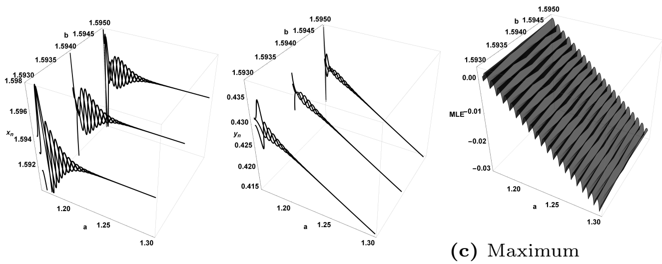
Next, suppose that $(a,\ b,\ h) = (1.16598,\ 1.59302,\ 0.9)$, then $(x_*,\ y_*) = (1.59302, 0.43011)$. In this case eigenvalues of Jacobian Matrix at $(x_*,\ y_*)$ are $-\frac{1}{2} \pm \iota\frac{\sqrt{3}}{2}$. Moreover, $ReK_1(a_1, b_1) = -19.5329 \neq 0$ and $F(a_1, b_1) = 1.10825 - 0.321837\iota \neq 0$, which shows the correctness of Theorem 3. Hence, system (2) undergoes codimension-two bifurcation associated with 1:3 strong resonance whenever $a \in [1.162,\ 1.3]$ and $b \in [1.5930,\ 1.596]$. Alternatively, the bifurcation diagram in $(a,\ b,\ x_n)$, $(a,\ b,\ y_n)$ spaces and MLE are depicted in Figure 3a, 3b and 3c, respectively.

Finally, assume that $(a,\ b,\ h) = (0.809377,\ 11.399567,\ 0.85)$, then $(x_*,\ y_*) = (1.399567, 0.505593)$. In this case eigenvalues of Jacobian Matrix at $(x_*,\ y_*)$ are $\pm\iota$. Moreover, $K_3(a_2, b_2) = 1.98415 + 6.46503i$, $Re\left(J(a_2, b_2)\right) = 2.29359$ and $Im\left(J(a_2, b_2)\right) = 0.0750144$, which shows the correctness of Theorem 4. Hence, system (2) undergoes codimension-two bifurcation associated with 1:4 strong resonance whenever $a \in [0.8,\ 0.84]$ and $b \in [1.3,\ 1.5]$. Alternatively, the bifurcation diagram in $(a,\ b,\ x_n)$, $(a,\ b,\ y_n)$ spaces and MLE are depicted in Figure 4a, 4b and 4c, respectively.



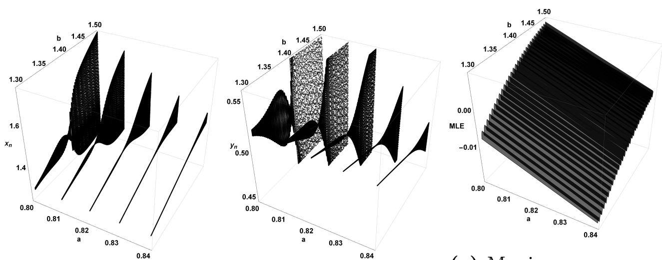
(a) Bifurcation diagram for x_n. **(b)** Bifurcation diagram for y_n. **(c)** Maximum Lyaponov Exponent

Figure 2. Plots of the system (2) for $b = 2.1$, $h \in [0.79,\ 0.879]$ and $a \in [0.85,\ 0.88]$ with initial conditions $x_0 = 2.1$ and $y_0 = 0.398918$.



(a) Bifurcation diagram for x_n. **(b)** Bifurcation diagram for y_n. **(c)** Maximum Lyaponov Exponent

Figure 3. Plots of the system (2) for $h = 0.9$, $a \in [1.162,\ 1.3]$ and $b \in [1.5930,\ 1.596]$ with initial conditions $x_0 = 1.594$ and $y_0 = 0.430$.



(a) Bifurcation diagram for x_n. **(b)** Bifurcation diagram for y_n. **(c)** Maximum Lyaponov Exponent

Figure 4. Plots of the system (2) for $h = 0.85$, $a \in [0.8,\ 0.84]$ and $b \in [1.3,\ 1.5]$ with initial conditions $x_0 = 1.399$ and $y_0 = 0.505$.

5 Conclusion

A glycolysis model is considered for its discretization and qualitative analysis. By applying the forward Euler scheme the discrete-time glycolysis model is obtained. By implementing normal form method and bifurcation theory, it is proved that system (2) undergoes codimension-two bifurcation associated with 1:2, 1:3 and 1:4 strong resonances at its positive fixed point. In the case of 1:2 resonance, the system (2) displays the emergence of intricate limit-cycle patterns. This implies that the glycolysis pathway exhibits oscillatory behavior, where the concentrations of key metabolites undergo periodic fluctuations. The resonant interaction between the system's parameters leads to stable limit cycles, indicating persistent and predictable oscillations in glycolytic activity. In the case of 1:3 resonance, the system (2) reveals intriguing periodic solutions. This resonance condition causes the system to exhibit more complex oscillatory patterns compared to the 1:2 resonance. The system now undergoes three periods of oscillation before returning to its initial state, highlighting the intricate interplay of parameters that drive this behavior. Under 1:4 resonance, the system (2) showcases co-existing periodic and chaotic regimes. This resonance condition leads to the coexistence of both stable periodic solutions and chaotic behavior within the system. The presence of chaos implies that the glycolysis pathway becomes highly sensitive to initial conditions, resulting in unpredictable and erratic dynamics.

This study investigates the complex and multi-parameter bifurcation phenomena observed in a discrete-time glycolysis model. Specifically, it explores the dynamics of the system under resonances mentioned above. Through rigorous analysis, the research reveals intricate behaviors, such as limit cycles and chaos, offering valuable insights into how the glycolysis pathway behaves under these specific resonance states. Understanding the coexistence of different resonance regimes and their sensitivity to parameter variations provides crucial information for comprehending the robustness and control of the glycolysis process. By employing advanced mathematical tools, including normal form theory and bifurcation analysis, this study contributes to the field of mathematical biology and discrete dynamical systems, enhancing our understanding of complex biochemical networks and their responses in both biological and engineering contexts.

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