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Some New Results for the Hitting Time Index

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Abstract

We prove a new inequality between the HT index and the Kirchhoff index, as well as the facts that the value of the HT index for any tree is an integer, and that this new index is not monotonic under edge addition. Then we focus on the computation of the values of this index, in closed form, for several families of graphs containing one or two cutpoints.

1 Introduction

In what follows, a graph $G = (V, E)$ will be a finite simple connected undirected graph with vertex set $V = \{1, 2, \ldots, n\}$, edge set E and vertex degrees $d_1, d_2, \ldots, d_n$. For all graph theoretical details the reader is directed to reference [2].

These graphs are used in mathematical chemistry to model molecules, identifying the vertices as the atoms and the edges as the atomic bonds between the vertices. Many topological indices, or descriptors, i. e., real-valued functions on the domain of all graphs, have been defined with the

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purpose of capturing physico-chemical properties of the molecules and classifying them according to the values of their indices. Many of these indices are defined in terms of the degrees of the vertices, for example, the first Zagreb index is given by

$$M_1(G) = \sum_i d_i^2.$$

See [7] for a review of this and many other degree-based indices. Other indices are based on the distance $d(i, j)$ between the vertices i and j, defined as the length of the shortest path between i and j. An example of these is the Wiener index, defined in [14] as

$$W(G) = \sum_{i<j} d(i, j).$$

Another family of indices uses the eigenvalues of a matrix (incidence, Laplacian, normalized Laplacian, etc.) associated to the graph. The reader is directed to reference [5] for a review of these indices.

Yet another family of indices, perhaps smaller in size, relies on concepts taken from probability or electrical networks. One such index is the Kirchhoff index defined in [1] as

$$K(G) = \sum_{i<j} R_{ij}, \tag{1}$$

where R_{ij} is the effective resistance between vertices i and j when the graph is thought of as an electrical network, where all the edges have unit resistance.

The simple random walk on G is defined as the Markov chain $\{X_n, n \geq 0\}$ whose state space is V and whose transition probabilities are defined as uniform, from a vertex i to any of its d_i neighboring vertices. The hitting time T_j of the vertex j is defined as the smallest number of jumps needed by the random walk to reach the vertex j:

$$T_j = \inf\{n \geq 0 : X_n = j\},$$

and its expected value when the process is started in state i is denoted by E_iT_j. We remark that $E_iT_i = 0$, and this should not be confused with the mean return time to vertex i, $E_iT_i^+ = \frac{2|E|}{d_i}$, which involves $T_i^+ = \inf\{n \geq 1 : X_n = i\}$. For facts about hitting times of Markov chains, the reader is referred to [6].

In [10] we showed that there is a close relationship between hitting times and the Kirchhoff index, namely

$$K(G) = \frac{1}{2|E|}\sum_{i<j}(E_iT_j + E_jT_i), \tag{2}$$

so that one can use probabilistic tools and intuitions to this index, in addition to several other fruitful approaches. A good introduction to the relationship between electric networks and random walks on graphs is reference [8].

A recent probabilistic/electrical index was put forward in [3] by Camby et al., the random walk index, defined in the following way: for any pair of vertices i and j, a battery is placed between i and j so that a 1 ampere current enters i and exits j. This generates a voltage v_x^{ij} on all vertices $x \in V$, and a potential difference on any edge (x, y) given by $v_x^{ij} - v_y^{ij}$. If the polarity of the battery is inverted, then the potential drop on the edge (x, y) is $v_y^{ij} - v_x^{ij}$, and thus, in order to avoid the dependance on the polarity of the battery, the authors consider the quantity $|v_x^{ij} - v_y^{ij}|$, and they add these quantities over all edges of the graph getting

$$\hat{d}_{ij} = \sum_{(x,y)\in E} |v_x^{ij} - v_y^{ij}|.$$

The authors prove that the function $\hat{d}$ defined on the pairs of vertices ij by the value $\hat{d}_{ij}$ is a metric, and then define the random walk index as

$$RW(G) = \sum_{i<j} \hat{d}_{ij}. \tag{3}$$

We defined in [12] a new probabilistic index, the hitting time index

$HT(G)$ of a graph G, as

$$HT(G) = \sum_{i<j} D(i,j), \qquad (4)$$

where $D(i,j) = \max\{E_iT_j, E_jT_i\}$.

We showed that $D(i,j)$ is actually a distance on the set of all vertices of G, found some inequalities involving $HT(G)$, $K(G)$, $W(G)$, and $RW(G)$, and computed $HT(G)$ for some families of graphs.

A simple way to compute $HT(G)$ starts by finding the transition probability P of the random walk on G, with entries $P(i,j) = \frac{1}{d_i}$ in case i and j are neighbors, and zero otherwise. We also use the matrix W, all of whose rows are identical to the stationary distribution $\pi = \frac{1}{2|E|}[d_1, d_2, \ldots, d_n]$. Then we find the so-called fundamental matrix

$$Z = (I - P + W)^{-1},$$

where I is the $n \times n$ identity matrix. Now, the matrix E of expected hitting times is found via the matrix Z. Its entries are

$$E(i,j) = E_iT_j = (Z(j,j) - Z(i,j))/\pi_j.$$

See [6] for a discussion of the matrices Z and E. Finally, the HT index of the graph under consideration is found by adding $\binom{n-1}{2}$ terms,

$$\sum_{i<j} \max\{E(i,j), E(j,i)\}.$$

In this article, we continue with the study of $HT(G)$. First we prove three refinements for the values that this index may take, and then we focus on finding closed-from formulas for the values of HT in some families of graphs endowed with one or two cutpoints.

2 Three refinements

We showed in [12] that

$$HT(G) \leq 2|E|K(G).$$

This inequality is improved in the following

Proposition 1. *For any graph G we have*

$$HT(G) \leq 2|E|K(G) - W(G), \tag{5}$$

where the equality is attained in case $G = P_2$. For any tree T we have

$$HT(T) \leq 2|E|K(T) - W_2(T), \tag{6}$$

where $W_2(G) = \sum_{i<j} d(i,j)^2$ is the generalized Wiener index with parameter 2. The equality is attained for P_2 and P_3.

Proof. For any two real numbers x, y we have $\max\{x, y\} = x+y-\min\{x, y\}$ and therefore

$$HT(G) = 2|E|K(G) - \sum_{i<j} \min\{E_iT_j, E_jT_i\}. \tag{7}$$

The number of jumps needed for the random walk to reach j starting from i is bounded below by $d(i,j)$. Therefore, so is its expected value, i.e., $E_iT_j \geq d(i,j)$. The same holds for E_jT_i. Then $\min\{E_iT_j, E_jT_i\} \geq d(i,j)$, and inserting this inequality into (7) yields (5). For trees, we use:

$$E_iT_j = d(i,j)^2 + 2\sum_{x\in P} |E_x|d(x,j), \tag{8}$$

a formula found in [9], where P is the unique path between i and j and E_x is the connected component of $E - P$ containing x. It is obvious, then, that for i and j in a tree, we have $E_iT_j \geq d(i,j)^2$. The same occurs for E_jT_i, and a similar argument to the one used for (5), yields (6). ■

From the proof of the previous proposition we obtain the following result:

Proposition 2. *For any tree T, the value $HT(T)$ is an integer*

Proof. By the equation (8) it is clear that all expected hitting times in a tree T are integer, and therefore, so is $HT(T)$. ■

Going in a different direction now, it is worth mentioning that, due to the monotonicity principle for electric circuits, the Kirchhoff index is monotonic in the number of edges of a graph, that is, if $G^{'}$ is the graph obtained from adding a new edge to the graph G, connecting two of its vertices, then $K(G^{'}) < K(G)$. A simple proof of this fact can be read in [11]. This is not the case for the HT index, as expressed in the following

Proposition 3. *The HT index is not monotonic under edge addition.*

Proof. Consider the 4-path graph P_4. Then $HT(P_4) = 50$. If we connect the leaves of P_4 with a new edge we obtain the 4-cycle, C_4, for which $HP(C_4) = 20$. But then, if we add an extra edge to C_4 we obtain a graph G for which $HT(G) = 23.5$. ■

3 Graphs with cutpoints

A cutpoint c of a graph G is a vertex such that its removal renders the graph disconnected into two or more connected components. Graphs with cutpoints often yield easier computations of hitting times, and here is the fundamental idea in this regard: if c is a cutpoint of G, and i and j belong to two different components of the disconnected components mentioned, then if we let the random walk occur in the whole graph G, we have

$$E_iT_j = E_iT_c + E_cT_j.$$

This is intuitively obvious because in order to reach j starting from i, the random walk must necessarily reach first the cutpoint c and then, starting from c, reach the destination j. See [9] for a discussion of this theme. We will see how this idea yields closed-form formulas for the HT index of some graphs not studied in [12].

Specifically, let's look at the graph $G(n,m)$ which consists of two complete graphs K_n and K_m (we will call these graphs the two petals of $G(n,m)$) such that they are disjoint in their edges and their vertices except for a vertex that we will choose to be n, that is, the vertex set of K_n is $\{1,2,\ldots,n\}$, and the vertex set of K_m is $\{n, n+1, \ldots, n+m-1\}$. Then the vertex set of $G(n,m)$ is $\{1,2,\ldots,n+m-1\}$ and its edge set is the union of the edge sets of the two petals K_n and K_m. If we look at an individual petal, say, K_n, it is easy to see that

Lemma 1. *For any pair of vertices i and j in K_n we have*

(i) $R_{ij} = \frac{2}{n}$

(ii) $E_i T_j = n-1$.

Proof. For (i), it is a simple matter of working with the formulas of resistors in series and in parallel; for (ii) it is easy to see that T_j is a geometric random variable with probability of success (i.e., of hitting the vertex j) equal to $\frac{1}{n-1}$. Therefore its expectation is the inverse of the parameter, i. e., $n-1$. ■

The formulas in the following lemma will be essential for the calculations of expected hitting times below.

Lemma 2. *For any pair of vertices a and b in G we have*

$$E_a T_b + E_b T_a = 2|E| R_{ab}. \tag{9}$$

and

$$E_a T_b = \frac{1}{2} \sum_z d_z \left[R_{ab} + R_{bz} - R_{az} \right], \tag{10}$$

where d_z is the degree of vertex z.

The proof of (9) can be found in [4]. That of (10) in [13].

It is clear that the unidirectional formula (10) is stronger than (9), because if we exchange the roles of a and b in (10), and add the resulting equation to (10), we obtain

$$E_a T_b + E_b T_a = \sum_z d_z R_{ab} = R_{ab} \sum_z d_z = 2|E| R_{ab}.$$

However, most of the time we will be using (9) rather than (10).

In what follows, when we talk about a vertex $i \in G(n,m)$ which is in the K_n component, we will simply say that $i \in K_n$. Now we want to prove the following

Proposition 4. *Given $i, j \in G(n,m)$,*

(i) if $i \in K_n - \{n\}$ we have $E_iT_n = n-1$. Likewise, if $i \in K_m - \{n\}$ we have $E_iT_n = m-1$.

(ii) if $i \in K_n - \{n\}$, $j \in K_m - \{n\}$ we have

$$E_iT_j = [n(n-1) + m(m-1)]\frac{2}{m} - m + n.$$

(iii) if $i \in K_m - \{n\}$, $j \in K_n - \{n\}$ we have

$$E_iT_j = [n(n-1) + m(m-1)]\frac{2}{n} - n + m.$$

(iv) if both $i, j \in K_n - \{n\}$ then

$$E_iT_j = E_jT_i = [n(n-1) + m(m-1)]\frac{1}{n}.$$

(v) if both $i, j \in K_m - \{n\}$ then

$$E_iT_j = E_jT_i = [n(n-1) + m(m-1)]\frac{1}{m}.$$

(vi) if $i \in K_n - \{n\}$, then $E_nT_i = [n(n-1) + m(m-1)]\frac{2}{n} - n + 1$.

(vii) if $i \in K_m - \{n\}$, then $E_nT_i = [n(n-1) + m(m-1)]\frac{2}{m} - m + 1$.

Proof. (i) Since the target is n and the start is $i \in K_n$, as long as the walk has not reached n, it behaves as if the K_m did not exist. The same argument applies if the start vertex is $i \in K_m$.

(ii) Because n is a cutpoint, and by part (i) we have

$$E_iT_j = E_iT_n + E_nT_j = n - 1 + E_nT_j. \tag{11}$$

Now by (9)

$$E_nT_j + E_jT_n = [n(n-1) + m(m-1)]R_{jn} = [n(n-1) + m(m-1)]\frac{2}{m}.$$

Solving for E_nT_j in the previous formula, using part (i) to find $E_jT_n = m - 1$, and plugging into (11) yields the desired result.

(iii) this is immediate from (ii), exchanging the roles of n and m.

(iv) and (v) are immediate, by symmetry and (9).

(vi) By (9):

$$E_nT_i + E_iT_n = [n(n-1) + m(m-1)]R_{in} = [n(n-1) + m(m-1)]\frac{2}{n}.$$

But by part (i), $E_iT_n = n - 1$, thus solving for E_nT_i ends this part of the proof.

(vii) Similar to (vi), exchanging n with m. ■

Now we can prove the following

Proposition 5.

$$\begin{aligned} HT(G(n,m)) &= [n(n-1) + m(m-1)] \\ &\times \left\{ \frac{\binom{n-1}{2}}{n} + \frac{2(n-1)}{n} + \frac{2(m-1)}{m} + \frac{2(m-1)(n-1)}{\min\{n,m\}} + \frac{\binom{m-1}{2}}{m} \right\} \\ &- (n-1)^2 - (m-1)^2 + |m-n|(m-1)(n-1), \end{aligned} \tag{12}$$

with the particular case

$$HT(G(n,n)) = 2(n-1)^2(3n-1). \tag{13}$$

Proof. From (iv) in the previous proposition, it is clear that

$$D(i,j) = [n(n-1) + m(m-1)]\frac{1}{n}, \tag{14}$$

whenever $i, j \in K_n - \{n\}$. There are $\frac{(n-1)(n-2)}{2}$ choices for such i and j.

Also, by (v) we have

$$D(i,j) = [n(n-1) + m(m-1)]\frac{1}{m}, \tag{15}$$

whenever $i, j \in K_m - \{n\}$. There are $\frac{(m-1)(m-2)}{2}$ choices for such i and j.

Let us assume now that $m \geq n$. Then, from (ii) and (iii) it is easy to se that

$$D(i,j) = [n(n-1) + m(m-1)]\frac{2}{n} - n + m, \tag{16}$$

whenever $i \in K_n - \{n\}$, $j \in K_m - \{n\}$. There are $(n-1)(m-1)$ choices for such i and j.

From (i) and (vi) we can see that

$$D(i,n) = [n(n-1) + m(m-1)]\frac{2}{n} - n + 1, \tag{17}$$

when $i \in K_n - \{n\}$. There are $n-1$ choices for such i.

Finally, from (i) and (vii) we have that

$$D(i,n) = [n(n-1) + m(m-1)]\frac{2}{m} - m + 1, \tag{18}$$

when $i \in K_m - \{n\}$. There are $m-1$ choices for such i.

In order to get the closed form expression for (4) in this case, we add the values in (14), (15), (16), (17) and (18) multiplied by their respective number of choices. We get the expression

$$\begin{aligned} HT(G(n,m)) &= [n(n-1) + m(m-1)] \\ &\times \left\{ \frac{\binom{n-1}{2}}{n} + \frac{2(n-1)}{n} + \frac{2(m-1)}{m} + \frac{2(m-1)(n-1)}{n} + \frac{\binom{m-1}{2}}{m} \right\} \\ &-(n-1)^2 - (m-1)^2 + (m-n)(m-1)(n-1). \end{aligned} \tag{19}$$

If we assume that $n \geq m$, all previous derivations hold except (16), where we need to exchange the roles of n and m, obtaining then

$$HT(G(n,m)) = [n(n-1) + m(m-1)]$$

$$\times\left\{\frac{\binom{n-1}{2}}{n}+\frac{2(n-1)}{n}+\frac{2(m-1)}{m}+\frac{2(m-1)(n-1)}{m}+\frac{\binom{m-1}{2}}{m}\right\}$$

$$-(n-1)^2-(m-1)^2+(n-m)(m-1)(n-1). \qquad (20)$$

Now (19) and (20) imply (12). ■

We remark that formula (12) is symmetric in n and m, as it should, because

$$HT(G(n,m)) = HT(G(m,n)).$$

We can generalize our results in several directions. If we consider conjoining several different-sized complete graphs, the number of different parameters complicates matters rapidly. We choose to consider s equal-sized K_n's, that we will label $K_{n,1}, K_{n,2}, \ldots, K_{n,s}$ all conjoined at a single vertex, say, n. Here we have chosen for simplicity $\{1, 2, \ldots, n\}$ to be the vertex set of $K_{n,1}$, $\{n, n+1, \ldots 2n-1\}$ to be the vertex set of $K_{n,2}$, $\{n, 2n, 2n+1, \ldots 3n-2\}$ to be the vertex set of $K_{n,3}$, etc., though the labelling does not play any role in the computations below. Let us call this graph $G_s(n)$, whose edge set E satisfies $|E| = \frac{sn(n-1)}{2}$, and let us call the $K_{n,r}$, $1 \le r \le s$, its petals. Then we have

Proposition 6. *Given $G_s(n)$,*

(i) if $i \neq n$ then $E_iT_n = n-1$;

(ii) if $i \neq n$ then $E_nT_i = (2s-1)(n-1)$;

(iii) If both $i, j \in K_{n,r}$, then $E_iT_j = s(n-1)$;

(iv) if i and j are in different petals, $i \neq n$, $j, \neq n$ then $E_iT_j = 2s(n-1)$.

Proof. (i) is similar to the proof of (i) in Proposition 4;

(ii) we use (9) and (i) and solve for $E_iT_n = 2|E|R_{in} - E_iT_n = 2s(n-1) - (n-1)$;

(iiii) by symmetry and (9), $E_iT_j = E_jT_i = |E|R_{i,j} = s\frac{n(n-1)}{2}\frac{2}{n} = s(n-1)$;

(iv) by (ii) and the fact that n is a cutpoint,

$$E_iT_j = E_iT_n + E_nT_j = n-1+(2s-1)(n-1)\ .$$ ■

Since there are only four distinct expected hitting times in $G_s(n)$, the computation of its *HT* index is reasonable, as the following proposition shows.

Proposition 7.

$$HT(G_s(n)) = s(n-1)^2 \left[2s - 1 + \frac{s(n-2)}{2} + s(s-1)(n-1)\right]. \quad (21)$$

Proof. By (i) and (ii) of Proposition 5,

$$D(i,n) = E_n T_i = (2s-1)(n-1),$$

and there are $s(n-1)$ target vertices, so these choices of vertices contribute a total of

$$s(2s-1)(n-1)^2, \quad (22)$$

in the summation (4). If i and j are in the same petal

$$D(i,j) = E_i T_j = E_j T_i = s(n-1),$$

and there are $s\binom{n-1}{2}$ ways to select these two vertices, so these choices contribute

$$s^2(n-1)^2(n-2). \quad (23)$$

Finally, when i and j are in different petals, $i \neq n$, $j \neq n$, we have by (iv) that

$$D(i,j) = E_i T_j = E_j T_i = (2s-1)(n-1).$$

Since we have $\binom{s}{2}$ ways to choose the pairs of petals, and then $(n-1)^2$ ways to choose the starting vertex and the target vertex, we have that these vertices contribute a total of

$$2s(n-1)\binom{s}{2}(n-1)^2 = s^2(s-1)(n-1)^3. \quad (24)$$

Adding (22), (23), and (24) we obtain the desired result. ■

We notice that $G_2(n) = G(n,n)$ and indeed, (13) and (21) coincide when $s = 2$.

Another possible way to generalize Proposition 5 is to consider a linear chain of copies of K_n's, conjoined at one point (the two K_n's on both ends of the chain) or two points (the intermediate K_n's). This can get rapidly unmanageable, and so we will only look at the case of a 3-long chain composed of three copies of K_n, which we will call $K_{n,i}$, $1 \leq i \leq 3$, and such that $K_{n,1}$ and $K_{n,2}$ have one vertex in common, say, vertex n, and $K_{n,2}$ and $K_{n,3}$ have also one vertex in common, say, vertex $2n-1$. Here we have chosen for simplicity $\{1, 2, \ldots, n\}$ to be the vertex set of $K_{n,1}$, $\{n, n+1, \ldots 2n-1\}$ to be the vertex set of $K_{n,2}$ and $\{2n-1, 2n, \ldots 3n-2\}$ to be the vertex set of $K_{n,3}$. It is clear that the graph so defined, denoted by $K_{n,1,2,3} = (V, E)$ satisfies $|V| = 3n-2$ and $|E| = \frac{3}{2}n(n-1)$. With these conditions we can prove

Proposition 8. *In $K_{n,1,2,3}$ there are eight different values for the expected hitting times with the form $k(n-1)$, for $k = 1, 2, 3, 4, 5, 7, 8, 9$, according to the following choices:*

(i) $E_iT_n = n-1$, *for* $i \in K_{n,1} - \{n\}$; $E_iT_{2n-1} = n-1$, *for* $i \in K_{n,3} - \{2n-1\}$.

(ii) $E_nT_i = 5(n-1)$, *for* $i \in K_{n,1} - \{n\}$, $E_{2n-1}T_i = 5(n-1)$, *for* $i \in K_{n,3} - \{2n-1\}$.

(iii) $E_nT_{2n-1} = E_{2n-1}T_n = 3(n-1)$.

(iv) $E_iT_j = 3(n-1)$ *for* $i, j \in K_{n,2} - \{n, 2n-1\}$. *This also holds if both* $i, j \in K_{n,1} - \{n\}$ *or if both* $i, j \in K_{n,3} - \{2n-1\}$.

(v) $E_iT_n = 4(n-1)$ *for* $i \in K_{n,3} - \{2n-1\}$, $E_iT_{2n-1} = 4(n-1)$ *for* $i \in K_{n,1} - \{n\}$.

(vi) $E_iT_j = 9(n-1)$, *for* $i \in K_{n,1} - \{n\}$, $j \in K_{n,3} - \{2n-1\}$. *This also holds when exchanging the roles of* i *and* j.

(vii) $E_iT_n = 2(n-1)$, *for* $i \in K_{n,2} - \{n, 2n-1\}$. $E_iT_{2n-1} = 2(n-1)$, *for* $i \in K_{n,2} - \{n, 2n-1\}$.

(viii) $E_iT_j = 7(n-1)$, *for* $i \in K_{n,2} - \{n, 2n-1\}$, $j \in K_{n,1} - \{n\}$. *This also holds if* $j \in K_{n,3} - \{2n-1\}$.

(ix) $E_nT_i = 8(n-1)$ *for* $i \in K_{n,3} - \{2n-1\}$, $E_{2n-1}T_i = 8(n-1)$, $i \in K_{n,1} - \{n\}$.

(x) $E_nT_i = 4(n-1)$, *for* $i \in K_{n,2} - \{n, 2n-1\}$. *This also holds when replacing the starting vertex* n *with* $2n-1$.

(xi) $E_iT_j = 5(n-1)$ *for* $i \in K_{n,1} - \{n\}$, $j \in K_{n,2} - \{n, 2n-1\}$. *This also holds if* $i \in K_{n,3} - \{2n-1\}$.

Proof. (i) The argument is similar to that of (i) in Proposition 4.

For (ii) we have by (9):

$$E_iT_n + E_nT_i = 2|E|R_{in} = 3n(n-1)\frac{2}{n} = 6(n-1).$$

Therefore, by (i) we have

$$E_nT_i = 6(n-1) - (n-1) = 5(n-1),$$

for $i \in K_{n,1}$, $i \neq n$. The argument for $E_{2n-1}T_i$ is similar.

For (iii), by symmetry and (9) we get

$$E_nT_{2n-1} = E_{2n-1}T_n = |E|R_{n,2n-1} = \frac{3}{2}n(n-1)\frac{2}{n} = 3(n-1).$$

The argument for (iv) is the same as that for (iii). The same argument holds if $i, j \in K_{n,1}$ with $i, j \neq n$ and if $i, j \in K_{n,3}$, $i, j \neq 2n-1$.

For (iv), if $i \in K_{n,3}$, $i \neq 2n-1$, since $2n-1$ is a cutpoint we have

$$E_iT_n = E_iT_{2n-1} + E_{2n-1}T_n = n - 1 + 3(n-1) = 4(n-1).$$

The argument for E_iT_{2n-1}, with $i \in K_{n,1}$, $i \neq n$ is similar.

For (v), using that n is a cutpoint, we have

$$E_iT_n = E_iT_{2n-1} + E_{2n-1}T_n = n - 1 + 3(n-1) = 4(n-1).$$

The argument for E_iT_{2n-1}, $i \in K_{n,1}$, $i \neq n$ is similar.

For (vi), using that both n and $2n-1$ are cutpoints, (i), (ii) and (iii) we get

$$E_iT_j = E_iT_n + E_nT_{2n-1} + E_{2n-1}T_j = n-1+3(n-1)+5(n-1) = 9(n-1).$$

For (vii), this is the only expected hitting time that cannot be calculated in closed form using only (9). Thus we will use (10), taking $a = i$ and $b = n$.

We notice that the term in the brackets in (10) becomes 0 when $z = n$ (because we get $\left[\frac{2}{n} + 0 - \frac{2}{n}\right]$) or when $z \in K_{n,1}$, $z \neq n$, (because we get $\left[\frac{2}{n} + \frac{2}{n} - \frac{4}{n}\right]$).

If we take $z \in K_{n,2}$, with $z \notin \{n, i, 2n-1\}$ then the summand becomes

$$(n-1)\left[\frac{2}{n} + \frac{2}{n} - \frac{2}{n}\right] = \frac{2(n-1)}{n}, \tag{25}$$

and there are $n-3$ such summands. If $z = i$ we get the additional summand

$$(n-1)\left[\frac{2}{n} + \frac{2}{n} - 0\right] = \frac{4(n-1)}{n}. \tag{26}$$

If we take $z \in K_{n,3}$, with $z \neq 2n-1$ we get

$$(n-1)\left[\frac{2}{n} + \frac{4}{n} - \frac{4}{n}\right] = \frac{2(n-1)}{n}, \tag{27}$$

and there are $n-1$ such summands; finally, if $z = 2n-1$ we obtain the additional summand

$$2(n-1)\left[\frac{2}{n} + \frac{2}{n} - \frac{2}{n}\right] = \frac{4(n-1)}{n}. \tag{28}$$

Putting together (25), (26), (27) and (31) we get

$$E_iT_n = \frac{1}{2}\left[\frac{2(n-1)(n-3)}{n} + \frac{4(n-1)}{n} + \frac{2(n-1)^2}{n} + \frac{4(n-1)}{n}\right] = 2(n-1). \tag{29}$$

This implies by symmetry that also $E_iT_{2n-1} = 2(n-1)$, for $i \in K_{n,2}$ with $i \neq 2n-1$.

For (viii), because of the cutpoint, and using (vii) and (ii) we get

$$E_iT_j = E_iT_n + E_nT_j = 2(n-1) + 5(n-1) = 7(n-1),$$

for $i \in K_{n,2}$, $i \notin \{n, 2n-1\}$, $j \in K_{n,1}$, $j \neq n$.

The same result holds by symmetry if the target vertex j satisfies $j \in$

$K_{n,3}$, $j \neq 2n-1$.

For (ix), because $2n-1$ is a cutpoint, and using (ii) and (iii) we have

$$E_nT_i = E_nT_{2n-1} + E_{2n-1}T_i = 3(n-1) + 5(n-1) = 8(n-1),$$

for $i \in K_{n,3}$, $i \neq 2n-1$. The same result applies when exchanging the roles of n with $2n-1$ and $K_{n,3}$ with $K_{n,1}$

For (x), we have $E_nT_i + E_iT_n = E_nT_i + 2(n-1) = 2|E|R_{ni} = 3n(n-1)\frac{2}{n} = 6(n-1)$. Solving for E_nT_i shows the result. The other case is similar.

For (xi), when $i \in K_{n,1}$ $i \neq n$ and $j \in K_{n,2}$, $j \notin \{n, 2n-1\}$, write $E_iT_j = E_iT_n + E_nT_j$ and use (i) and (x). The other case is similar. ■

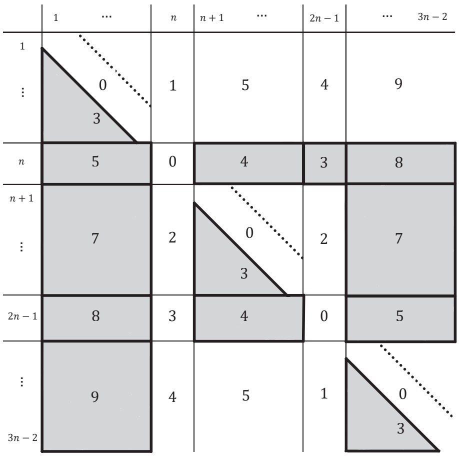

Figure 1. Shaded regions used in computing $HT(K_{n,1,2,3})$.

Proposition 9. *For all n we have*

$$HT(K_{n,1,2,3}) = (n-1)\left(\frac{55n^2}{2} - \frac{85n}{2} + 13\right) \tag{30}$$

Proof. The results of the previous proposition can be summarized in the matrix E of expected hitting times shown in figure 1, where all the integer values are multiplied by $n-1$. The maxima $D(i,j) = \max\{E_iT_j, E_jT_i\}$ are found in the shaded regions. For example, in the upper left triangle we add $1+2+\cdots+n-2 = \frac{(n-1)(n-2)}{2}$ times the constant 3, etc. When we add all those maxima we get

$$\frac{HT(K_{n,1,2,3})}{n-1} = \frac{3(n-1)(n-2)}{2} + \frac{3(n-3)(n-2)}{2} + \frac{3(n-1)(n-2)}{2}$$
$$+ 5(n-1) + 4(n-2) + 7(n-1)(n-2) + 7(n-1)(n-2) + 8(n-1)$$
$$+ 4(n-2) + 5(n-1) + 9(n-1)(n-1) + 8(n-1) + 3$$

$$= \frac{55n^2}{2} - \frac{85n}{2} + 13,$$

and solving for $HT(K_{n,1,2,3})$ we get the expression:

$$HT(K_{n,1,2,3}) = (n-1)\left(\frac{55n^2}{2} - \frac{85n}{2} + 13\right). \qquad \blacksquare$$

When $n=2$ the formula turns out $HT(K_{2,1,2,3}) = 38$, which coincides with $HT(P_4)$ found with the formula for $HT(P_n)$ given in [12].

3.1 Final note

We have seen in this section how to use cutpoints to our advantage when computing hitting times because when c is a cutpoint and i and j belong to different connected components of $G - \{c\}$, then $E_iT_j = E_iT_c + E_cT_j$. One may ask if, under these conditions, it is also true that

$$D(i,j) = D(i,c) + D(c,j). \tag{31}$$

In other words, when c is a cutpoint, does the triangular inequality for the distance D become an equality? The answer in general is no, as we can attest by looking at all the graphs covered in the previous propositions. However, adding another hypothesis, which will occur under mild symmetry conditions, we can get the equality as in the following

Proposition 10. *If c is a cutpoint, and $E_iT_c = E_cT_i$ or $E_cT_j = E_jT_c$, then (31) holds.*

Proof. Suppose without loss of generality that $E_cT_j = E_jT_c$ then $D(c,j) = E_cT_j$. Suppose, also without loss of generality, that $D(i,j) = E_iT_j = E_iT_c + E_cT_j \geq E_jT_i = E_jT_c + E_cT_i$. This implies that $E_iT_c \geq E_cT_i$, so that $D(i,c) = E_iT_c$ and then $D(i,j) = E_iT_j = E_iT_c + E_cT_j = D(i,c) + D(c,j)$. ■

For example, consider the path graph P_{2n} on $2n$ vertices. By symmetry and (9), we have

$$D(n, n+1) = E_nT_{n+1} = E_{n+1}T_n = |E| = 2n - 1. \tag{32}$$

Also, it is easy to see that $E_1T_{n+1} = n^2$ and $E_1T_n = (n-1)^2$. Using (9), we get $E_{n+1}T_1 = 3n^2 - 2n$ and $E_nT_1 = 3n^2 - 4n + 1$. Therefore

$$D(1, n+1) = 3n^2 - 2n \tag{33}$$

and

$$D(1, n) = 3n^2 - 4n + 1. \tag{34}$$

From (32), (33) and (34) it is plain to see that

$$D(1, n+1) = D(1, n) + D(n, n+1).$$

We must remark that even in the graph P_{2n}, not all choices of i, j and c lead to (31). It would be interesting to know if the conditions of Proposition 10 are also necessary for (31) to hold.

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