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On the Volume of μ-Way $S(K_{1,3})$-Trade on $(3,6)$-Fullerene Graphs

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Abstract

A perfect star packing in a fullerene graph is a spanning subgraph whose every component is isomorphic to the star graph $K_{1,3}$. A perfect star packing in a $(3,6)$-fullerene graph is called a perfect star packing of type $T0$ if no center of a star is on a triangle of G. A μ-way G-trade consists of μ disjoint decomposition of graph H into copies of graph G. In this paper, we use the concept of packing and specify values of the number of copies of G=$S(K_{1,3})$ for which there exists a μ-way $S(K_{1,3})$-trade when the underlying graph is a non-trivial $(3,6)$-fullerene graph.

1 Introduction

A $(3,6)$-fullerene graph F on n vertices, is a 3-regular, 2-connected, or 3-connected planar graph with four triangles and $\frac{n}{2}-2$ hexagons. For given graphs, G and H, a perfect H-packing in G is a spanning subgraph of G whose all components are isomorphic to H. If H is the star graph $K_{1,3}$, it is called perfect star packing.

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T. Došlić et al. [4] categorized some fullerene graphs with perfect star packing and showed that all $(3,6)$-fullerene graphs have perfect star packing.

A decomposition of the graph G is a set of edge-disjoint subgraphs H_i such that each edge of G belongs to exactly one of the H_i and the H_i partitions the edges set of G.

A K-decomposition of G is a decomposition such that each subgraph H_i in the decomposition is isomorphic to K.

A μ-way G-trade $(\mu \geq 2)$ of volume s with underlying graph H consists of μ disjoint decompositions of H into s edge-disjoint copies of a graph G.

The first introduction to the concept of trade in the block designs was made by Hedayat [10], but trades were used back in 1917 by White, Cole, and Cummings [3]. Trades are used in various fields, including combinatorial. For example, trade is used in the Latin squares, named Latin trades [1, 11]. After that trades are introduced and used in graph theory [2,9]. Khademian et al. [12,13] determined all values s for which there exists a μ-way $K_{1,m}$-trade of volume s for underlying graph $H = K_{2n,2n}$ and $H{=}K_{2n}$ and 3-Way trades for trees with up to three Edges.

Figure 1 shows an example of 10-way $K_{1,3}$-trade of volume 2 for underlying graph $H = K_{1,6}$, where $V(H) = \{1,2,3,4,5,6,7\}$.

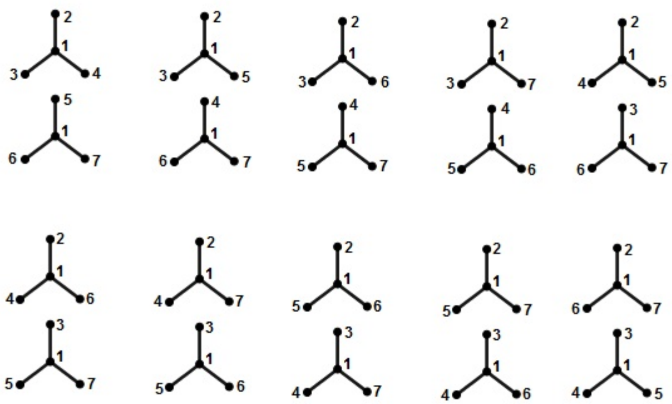

Figure 1. 10-way $K_{1,3}$-trade of volume 2.

Let G be a simple graph. A subdivided graph of a graph G, denoted by $S(G)$, is a graph obtained by placing a new vertex on each edge of G. If $G = K_{1,3}$, $S(G)$ is subdivided star graph.

Suppose G is a $(3,6)$-fullerene graph, and t, h, n, and m, are the number of triangles, hexagons, vertices, and edges of G, respectively. By Euler's formula, we have

$$m = 3h + 6, \quad n = 2h + 4, \quad t = 4$$

This shows that a $(3,6)$-fullerene graph with n vertices has four triangles and $\frac{n}{2} - 2$ hexagons, where n is an even number greater than or equal to four.

Non-trivial $(3,6)$-fullerene graphs, are fullerenes in which no two triangles have a common edge. In the rest of this paper, we will restrict our attention on non-trivial $(3,6)$-fullerene graphs.

Each $(3,6)$-fullerene graph is characterized by an ordered triple (r, s, t) where the components r, s, and t are non-negative integers such that r is equal to the number of hexagonal layers, s is the number of radial edges, i.e. the edges between each layer, and t is the number of twists. If G is a non-trivial $(3,6)$-fullerene graph, then s is at least equal to 2.

In each $(3,6)$-fullerene graph, all vertices are placed on the concentric circle r. The number of vertices in each circle is twice that of radial edges. Considering that the number of radial edges is always even, the number of vertices in each circle is a multiple of four. Figure 2 shows an example of a $(3,6)$-fullerene graph with $r = 3$ and $s = 4$. For further reading about these graphs, references [6–8] are recommended.

In this paper, we obtain the values of s and μ for all μ-way $S(K_{1,3})$-trade and the underlying graph F, which is a non-trivial $(3,6)$-fullerene graph.

2 Perfect star packing of type $T0$

In this section, we introduce a special type of perfect star packing, called perfect star packing of type $T0$, for $(3,6)$-fullerene graphs and calculate the

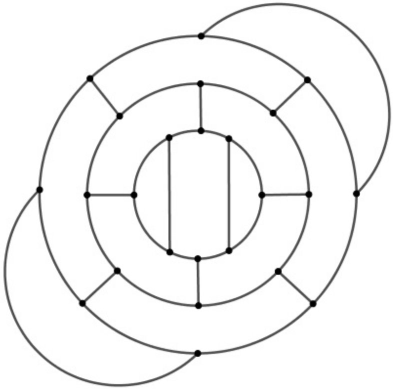

Figure 2. A $(3,6)$-fullerene with $r = 3$ and $s = 4$.

number of this packing. In the rest of this paper, we assumed all graphs are simple.

Let G and H be two graphs. A perfect H-packing in G is a spanning subgraph of G whose all components are isomorphic with H. If it impossible to cover all the vertices of G by placing disjoint copies of H in G, we say that G does not have perfect H-packing.

For further study in the field of graph packing, references [4, 5] are recommended.

It has been proven in reference [4] that all $(3,6)$-fullerene graphs have perfect star packing; that is, they have perfect H-packing, where H is the star graph $K_{1,3}$.

A perfect star packing in a $(3,6)$-fullerene graph is called a perfect star packing of type $T0$ if no center of a star is on a triangle of G. In other words, if none of the vertices of a triangle is a central vertex of a star.

Figure 3 shows an example of perfect star packing of type $T0$.

Lemma 1. *Trivial $(3,6)$-fullerene graphs do not have a perfect star packing of type $T0$.*

Proof. Let F be a trivial $(3,6)$-fullerene graph that a part of it is shown in Figure 4.

To cover the vertex v_1 in a star packing, either the vertex v_1 must be

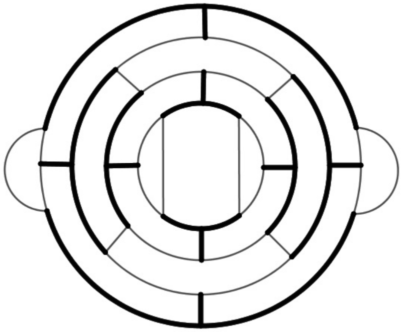

Figure 3. A perfect star packing of type $T0$.

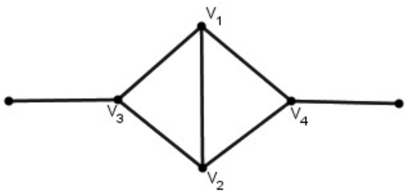

Figure 4. A part of a trivial $(3,6)$-fullerene.

the central vertex of the star, or one of the vertices v_2, v_3, v_4 should be the central vertex, and in each case, we have a central vertex on the edges of the triangle, and therefore there is no perfect star packing of type $T0$. ∎

In this part, we examine the existence of perfect star packings of type $T0$ in a $(3,6)$-fullerene graph F in different states.

First, notice that if $r = 2$, then there is a perfect star packing of type $T0$ for F. An example of this packing is shown in Figure 5.

In the following theorem, we state a condition for the existence of a perfect star packing of type $T0$ based on the parity of r.

Theorem 1. *If F is a $(3,6)$-fullerene graph, then if r is odd, F does not have a perfect star packing of type $T0$, and if r is even, then F has a perfect star packing of type $T0$.*

Proof. Figures 6, 7, and 8 show three general patterns that can be considered for a perfect star packing of type $T0$ in F, according to the type of

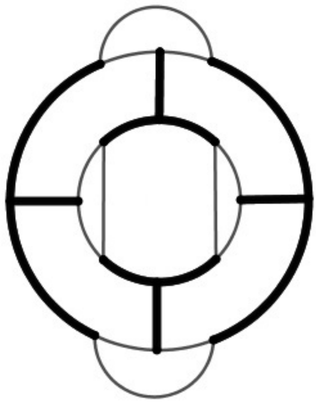

Figure 5. A perfect star packing of type $T0$ in mode $r = 2$.

packing and vertex coverage.

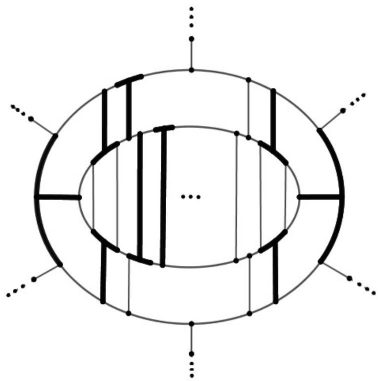

Figure 6. A type of perfect star packing of type $T0$.

If we have a packing like the one shown in Figures 6 and 7, then the number of radial edges connected to each circle that participates in the packing should be an even number. Because suppose the edge e_1 is a radial edge considered a star edge (See Figure 9). To cover the vertex u, either this vertex itself must be the central vertex, in which case the edge e_2 participates in the packing, or the vertex v must be the central vertex, and in this case, the vertex w will also be the central vertex, and as a

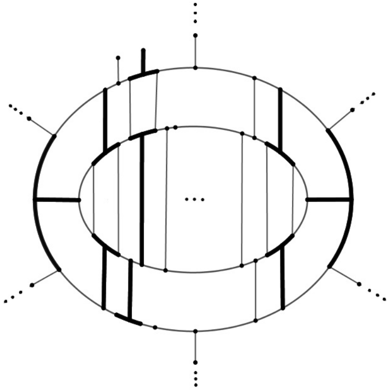

Figure 7. A type of perfect star packing of type $T0$.

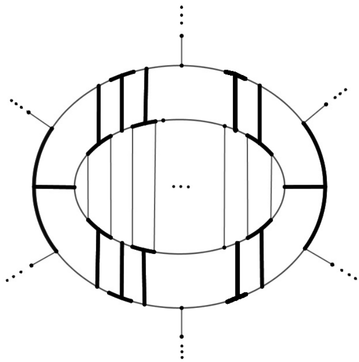

Figure 8. A type of perfect star packing of type $T0$.

result, to cover the vertex y, the vertex z must also be the central vertex of the star.

And based on this, the edge e_3 will be the radial edge participating in the star packing.

So, we will have another radial edge participating in star packing for each radial edge as a star edge. And therefore, the number of radial edges participating in the packing in each circle is an even number.

Also, if we have a packing like the one in Figures 6 and 7, in each layer, several radial edges must inevitably participate as star edges in the star packing. If we consider an arbitrary circle, there are radial edges in

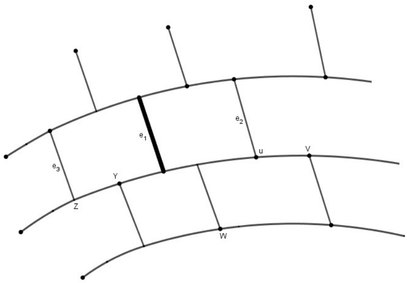

Figure 9. Radial edges in perfect star packing.

the circles before and after that circle, which are part of the perfect star packing edges. Now we claim that in these two cases, there is no perfect star packing of type $T0$. Suppose it is not true. For a perfect star packing of type $T0$, we must have the following shape.

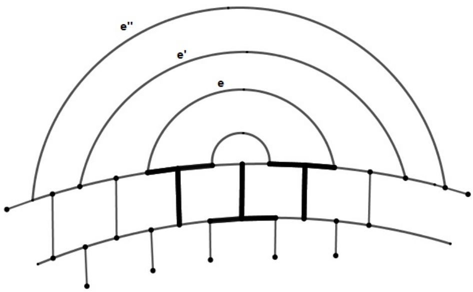

Figure 10. A part of a perfect star packing of type $T0$.

The edge e cannot be the radial edge of the packing; conversely, if none of the edges $e', e'', \ldots$ in Figure 10 are radial edges participating in the packing, then we will have a packing in Figure 8. Otherwise, suppose that at least one of these edges participates in packing; that is, one of the Figure 6 or Figure 7 will arise (Figure 11).

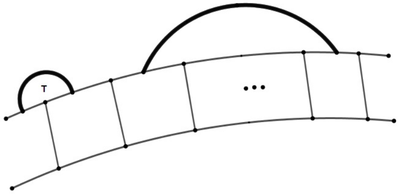

Figure 11. A part of a perfect star packing.

In this case, the triangle T cannot have a packing in the form allowed in a perfect star packing of type $T0$. Therefore, there is no perfect star packing of type $T0$ in the two cases of packing in Figures 6 and 7, regardless of whether r is even or odd.

Now suppose that the fullerene graph F has a packing of the type shown in Figure 8.

As can be seen, the packing of the first two circles in Figure 7 is clear. We inevitably have a packing in each pair of the following circles, as shown in Figure 12.

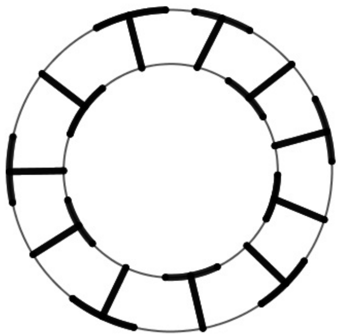

Figure 12. Perfect star packing in a pair of circles.

If r is even, the packing will be done up to the last two remaining layers in this way; that is, we continue the packing process by placing a star in each pair of circles. In the last two layers, regardless of where the triangles are placed, packing can be done similarly. In fact, in the two outer circles, we will have packing as follows.

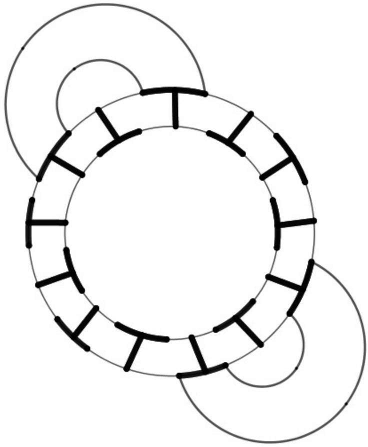

Figure 13. Perfect star packing in the caps.

We note that the structure of the caps does not affect this type of packing.

Figure 14 shows an example of a $(3, 6)$-fullerene graph with a perfect star packing of type $T0$.

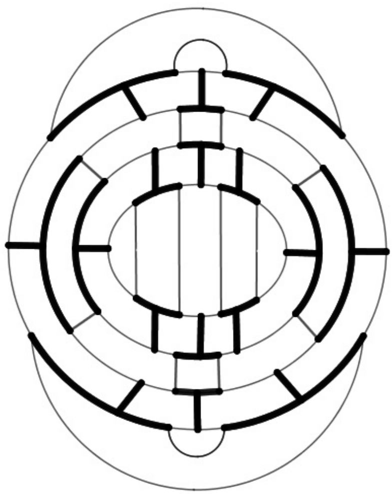

Figure 14. A perfect star packing of type $T0$ in a $(3, 6)$-fullerene graph with odd r.

If r is odd, with this packing method, one last layer remains, with several vertices that are a multiple of four. In this layer, there are two caps, as shown in Figure 15. We consider the vertex v in this cap. To cover this vertex, one of the vertices v_1 or v_2 must be the central vertex of the star, so in this case, the graph cannot have a complete star packing of type $T0$.

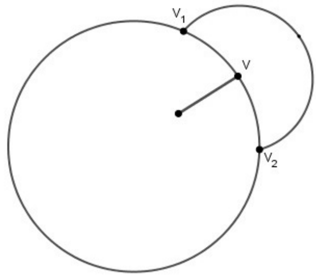

Figure 15. One cap in the last layer.

■

Theorem 2. *The necessary condition for a* $(3,6)$*-fullerene graph* F *to have a perfect star packing of type* $T0$ *is that it has* $8k$ $(k \in \mathbb{N})$ *vertices in each circle.*

Proof. According to the proof of Theorem 1, the $(3,6)$-Fullerene graph, which has a perfect star packing of type $T0$, appears as shown in Figure 16.

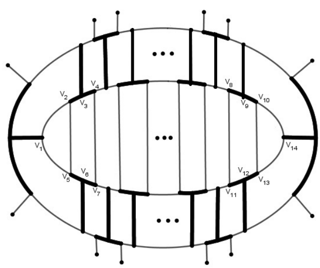

Figure 16. A part of a $(3,6)-$fullerene graph with perfect star packing of type $T0$.

In the innermost circle, fourteen vertices v_1, v_2, ..., v_{14} are covered by the initial packing of triangles. To complete the packing, other vertices of this circle from vertex v_4 to vertex v_8 must participate in the packing with the following pattern.

$$1 \quad 3 \quad 1 \quad 3 \cdots 3 \quad 1$$

In other words, first one vertex and then three vertices of the circle are covered by stars, and this pattern continues. In this pattern, the number 3 is repeated $t(t \in \mathbb{N})$ times, and the number 1 is repeated $t+1$. Therefore, the sum of these vertices is equal to

$$3t + (t+1) = 4t + 1$$

In the same way, there are $4t+1$ vertices in the path from vertex v_7 to vertex v_{11}, and therefore the sum of these vertices is equal to $8t+2$. These vertices plus the initial 14 vertices covered by triangles will be the number of $8 + 8t + 8 = 8s$ vertices $(s \in \mathbb{N})$ in the circle. ■

The results of Theorems 1 and 2 are summarized in the following Theorem

Theorem 3. *A* $(3,6)$*-fullerene graph* F *has complete star packings of type* $T0$ *if and only if* r *is even and the number of vertices of each circle is a multiple of 8.*

If we denote the number of perfect star packings of type $T0$ of the $(3,6)$-fullerene graph F by $T0(F)$, then we have the following theorem.

Theorem 4. *Let* F *be a* $(3,6)$*-fullerene graph that applies in the conditions of the previous corollary then*

$$T0(F) = \begin{cases} 1 & r = 2 \\ 2^{\frac{r-4}{2}} & r > 2 \end{cases}$$

Proof. First, we consider the case where the number of vertices on the circle is equals 8. In this case, an example of a perfect star packing of type $T0$ for F is shown in Figure 17.

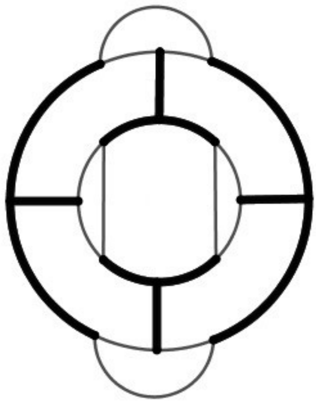

Figure 17. A perfect star packing of type $T0$.

This packing is the only perfect star packing of type $T0$ for F; therefore, in this case, $T0(F) = 1$. According to the argument in the proof of Theorem 1, only in the case of Figure 8, the graph F has a complete star packing of type $T0$. Now suppose $r = 2$. There is only one state to cover the first two circles; therefore, $T0(F) = 1$. Now suppose $r > 2$. In this case, we cover the first two circles, as shown in Figure 8, and for the last two circles, we have the same packing as shown in Figure 14. Therefore, the number of $r - 4$ circles remains, which we cover according to the explanation of the proof of Theorem 1 as shown in Figure 12, and so in this case, we will have

$$T0\,(F) = 2^{\frac{r-4}{2}} \qquad \blacksquare$$

3 μ-way $S(K_{1,3})$-trade for $(3,6)$-fullerene graphs

In this section, according to the contents of the previous topics, our goal is to obtain the values of s and μ for all μ-way $S(K_{1,3})$-trade for the underlying graph F, which is a non-trivial $(3,6)$-fullerene graph. Before that, we state and prove the following lemma.

Lemma 2. *The number of edges of a* $(3,6)$*-fullerene graph that participate* *in a perfect star packing is precisely equal to the number of edges that do not.*

Proof. Let F be a $(3,6)$-fullerene graph with n vertices and m edges, with perfect star packing. The number of stars in this packing is equal to $\frac{n}{4}$, and therefore the number of edges of stars in the star packing is $\frac{3n}{4}$. On the other hand, since F is a 3-regular graph, therefore $m = \frac{3n}{2}$. Thus, the number of remaining edges other than star edges equals $\frac{3n}{2} - \frac{3n}{4} = \frac{3n}{4}$. ■

It is necessary to answer the following question in order to achieve the stated goal.

Does choosing G=$S(K_{1,3})$ cover all edges of F?

To answer this question, we note that firstly, because F has a perfect star packing, then all the vertices of F are covered by stars. Now, according to Lemma 2, for each edge of the star, there is one edge of graph F that is not covered by the stars so that edge can be connected to one of the edges of the star. Since each edge of the star has two choices for connecting the edge, such a pattern can be continued. We will further examine this subject. First, we examine the forbidden states of decomposition of graph F by $S(K_{1,3})$.

Consider Figure 18, which contains a hexagon with a star packing.

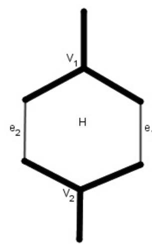

Figure 18. Packing of a hexagon.

Two vertices, v_1 and v_2, are the central vertices of the star in the hexagon H in a star packing of graph F. Suppose we have two copies of graph $S(K_{1,3})$ with central vertices v_1 and v_2. If both edges e_1 and e_2 are outside the hexagon in the decomposition of the graph into versions $S(K_{1,3})$, then these two edges will not be present in the decomposition of

F, so at least one of e_1 or e_2 must be the edge of $S(K_{1,3})$ in the graph decomposition. So, there is one choice per edge for a star centered on a hexagon.

Now we consider packing a hexagon in the form of Figure 19. If none of the edges e_1 and e_2 are edges of $S(K_{1,3})$ in the star with center v_1, then these edges must be the edges of $S(K_{1,3})$ in stars with centers v_2 and v_3. Therefore the edges e_3 and e_4 are not covered in the decomposition of F. So, in this case, at least one of the edges of $S(K_{1,3})$, with the center v_1 (which is on the hexagon), must be the edges e_1 and e_2, and there is no second choice.

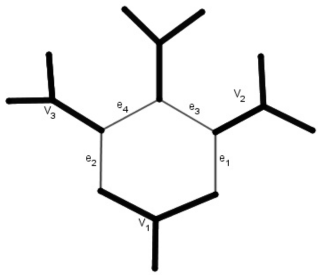

Figure 19. Packing of a hexagon.

From the above discussion, it is concluded that in every star whose center is on an arbitrary hexagon H, at least one edge $S\,(K_{1,3})$ must be placed on the hexagon. In other words, in this situation, two edges cannot be outside the hexagon. (That is, there should not be any hexagon where two edges of a graph $S\,(K_{1,3})$ are outside that hexagon). Therefore, to count the virtual states in which the graph $S\,(K_{1,3})$ can cover all the edges of F, i.e. $S\,(K_{1,3})$ is a decomposition for F, we need the states in which two edges of a $K_{1,3}$ are outside of the hexagons subtracted from the sum of the states.

Suppose e_1,$e_2$and e_3 are three edges of a star. (Figure 20)

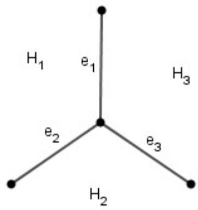

Figure 20. Three edges of a star in a hexagon.

Considering that every edge connected to a star edge can be a hexagonal edge or not, the different states of the edges connected to these three edges, compared to the corresponding hexagons, are given in Table 1.

Table 1. Different positions of the edges connected to the edges of the star (Inside or Outside).

e_2 and e_1 (Hexagon H_1)		e_3 and e_2 (Hexagon H_2)		e_3 and e_1 (Hexagon H_3)	
I	I	I	I	I	I
I	O	I	O	I	O
O	I	O	I	O	I
O	O	O	O	O	O

Therefore, among the 12 possible states that have edges connected to three-star edges, three states are forbidden states. So every star between three hexagons can have nine allowed states to decompose the subdivided star graph $S\left(K_{1,3}\right)$.

In the last part of this discussion, we will examine the different states of triangles. First, we consider the case that F is a trivial $(3,6)$-fullerene graph. In this case, we have the next star packings for two triangles.

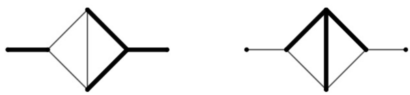

Figure 21. Different packings of two stars.

In both cases of Figure 21, the graph cannot be decomposed into copies of the subdivided star graph $S\left(K_{1,3}\right)$, so for this purpose, F must be a non-trivial $(3,6)$-fullerene graph. In the non-trivial case, we have two packing, as shown below for a star.

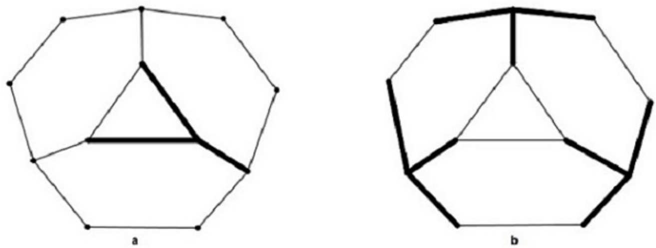

Figure 22. Different packings of a triangle.

In case **a**, it is impossible to determine the decomposition in the form of $S\left(K_{1,3}\right)$ for the graph. Still, in the case of **b**, considering the previous limitations, we can have such a decomposition.

The above discussions lead to the following final theorem.

Theorem 5. *Let H be a $(3,6)$-fullerene graph on n vertices. If there is μ-way $S(K_{1,3})$-trade of volume s with underlying graph H, we have*

$$s = \frac{n}{4} \qquad\qquad \mu = 9\ T0(H)$$

Where $T0(H)$ is the number of perfect star packing of type $T0$ in graph H.

Proof. According to the stated content, the underlying graph H can be decomposed into versions of $S(K_{1,3})$, and all edges of H can be covered by $S(K_{1,3})$. On the other hand, since H has a perfect star packing, and according to the type of formation of $S(K_{1,3})$, the number of these copies is equal to the number of stars in the perfect star packing of H that is, it is equal to $\frac{n}{4}$, so $s{=}\frac{n}{4}$. On the other hand, there are $9.T0(H)$ of this type of decomposition for graph H because each star can be transformed into an $S(K_{1,3})$ in any perfect star packing of type $T0$. ■

Figure 23 shows an example of a decomposition of a $(3,6)$-fullerene graph into subdivided star graph.

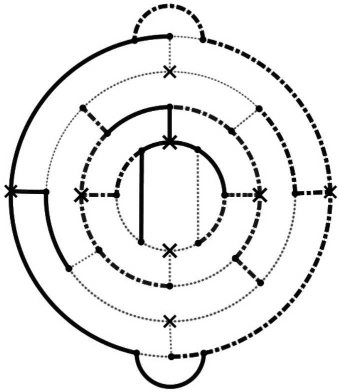

Figure 23. A decomposition of a $(3,6)$-fullerene graph.

4 Open problems

There are many open problems in the field of packing of fullerene graphs and the values of s and μ for all μ-way G-trade for the underlying graph F, which is a General fullerene graph. In the final part of this article, we list some of these open problems. $(4,6)$-fullerene graphs are cubic planar graphs with only square and hexagonal faces.

In [4], the authors categorized some $(4,6)$ and $(5,6)$-fullerene graphs with a perfect star packing.

Problem 1. *Is it possible to count all these perfect star packings?*

Problem 2. *What are the values of s and μ for all μ-way G-trade for the underlying graph F, which is a* $(4,6)$ *or* $(5,6)$*-fullerene graph?*

Problem 3. *Can we obtain different decompositions for fullerene graphs like decomposition $S(K_{1,3})$ for non-trivial* $(3,6)$*-fullerene graphs?*

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