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On Euler Sombor Index of Tricyclic Graphs

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Abstract

Let G be a simple graph. The Euler Sombor index of G is defined as

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y) + (d_G(x)d_G(y))},$$

where $d_G(x)$ denotes the degree of the vertex x, and the sum runs over the set of edges of G. In this paper we determine the extremal values of Euler Sombor index of tricyclic graphs.

1 Introduction

This paper considers only finite,connected and undirected graphs. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. The degree of the vertex $x \in V$ is defined as the number of vertices adjacent to x, and it is denoted by $d_G(x)$. The set of all neighbors of vertex x is $N_G(x)$. If $d_G(x) = 1$, then x is called a pendent vertex of G. If there is an edge from vertex x to vertex y, we indicate this by writing xy (or yx). For $xy \in E(G)$, denote by $G - xy$ the subgraph of G obtained from G by deleting the edge xy. For two nonadjacent vertices x and y of G, denoted by $G + xy$ the graph obtained from G by adding the edge xy. For graph-theoretical notions and terminology used in the present paper, we refer the reader to [2].

Topological indices characterize the molecular structure of a graph and are called numerical parameters used to estimate physicochemical information . The Euler Sombor index is introduced in [6, 16], where the Euler Sombor index is defined

$$EU(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y) + (d_G(x) d_G(y))}. \tag{1}$$

For other studies in the literature related to Euler Sombor index and other Sombor related indices, see [1,4,7–11,13,15,17]. Especially in recent years, a lot of work has been done on the extreme value problem of Sombor index [3, 5, 14]. A connected graph of order n and size $n + 2$ is known as a connected tricyclic graph [12]. Recently, Zhang and Zhao [18] are characterized the minimum and maximum values of graphs of the Sombor index among all tricyclic connected graphs of a given order. Inspired by the studies, we are interested in the Euler Sombor index of tricyclic graphs.

In this paper, we aim to characterize connected tricyclic graphs with minimum and maximum Euler Sombor index. For simplicity, let $G_{n,3}$ be the set of all tricyclic graphs with n vertices.

2 Tricyclic graphs with minimum Euler Sombor index

In this section, we study the tricyclic graphs with the minimum Euler Sombor index.

Lemma 1. *Let G be a connected graph and $z \in V(G)$ such that $d_G(z) \geq 3$. The two paths of G are $zx_1x_2...x_{r-1}x_r$ and $zy_1y_2...y_{s-1}y_s$ such that $d_G(x_r) = d_G(y_s) = 1$ and $d_G(x_i) = d_G(y_j) = 2$ whenever $0 < i < r$, $0 < j < s$. Let $G' = G - zy_1 + x_ry_1$. Then, $EU(G') < EU(G)$.*

Proof. Let $G_{1*} = G - \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_s\}$ and $k = d_G(z) \geq 3$. For the proof of this inequality we consider three cases.

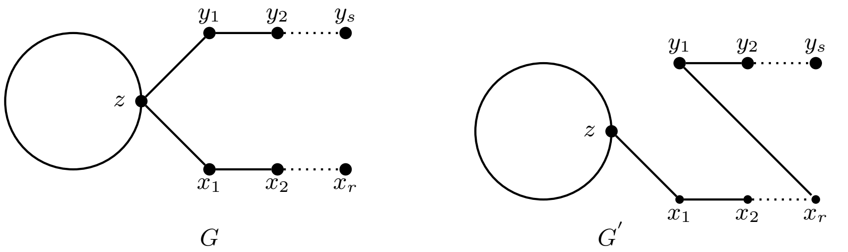

Figure 1. G and $G^{'}$ in Lemma 1.

Case 1. If $r > 1, s > 1$,we have

$$\begin{aligned}
EU(G) - EU(G^{'}) &= \sum_{y \in N_{G_{1*}}(z)} \left(\sqrt{d_G^2(y) + k^2 + d_G(y)k} \right) \\
&- \sum_{y \in N_{G_{1*}}(z)} \left(\sqrt{d_G^2(y) + (k-1)^2 + (d_G(y)(k-1))} \right) \\
&+ \left(\sqrt{k^2 + 2^2 + 2k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} \right) \\
&+ \left(\sqrt{2^2 + 1^2 + 2} - \sqrt{2^2 + 2^2 + 4} \right) \\
&+ \left(\sqrt{k^2 + 2^2 + 2k} - \sqrt{2^2 + 2^2 + 4} \right) \\
&> \sqrt{k^2 + 2^2 + 2k} + \sqrt{2^2 + 1^2 + 2} - 2\sqrt{2^2 + 2^2 + 4} \\
&\geq \sqrt{19} + \sqrt{7} - 4\sqrt{3} \approx 0.0764 > 0
\end{aligned}$$

Case 2. If $r = 1, s = 1$,we get

$$\begin{aligned}
EU(G) - EU(G^{'}) &= \sum_{y \in N_{G_{1*}}(z)} \left(\sqrt{d_G^2(y) + k^2 + d_G(y)k} \right) \\
&- \sum_{y \in N_{G_{1*}}(z)} \left(\sqrt{d_G^2(y) + (k-1)^2 + (d_G(y)(k-1))} \right) \\
&+ \left(\sqrt{k^2 + 1 + k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} \right) \\
&+ \left(\sqrt{k^2 + 1^2 + k} - \sqrt{2^2 + 1^2 + 2} \right) \\
&> \sqrt{k^2 + 1^2 + k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)}.
\end{aligned}$$

Since $\left(k^2 + 1^2 + k\right) - \left((k-1)^2 + 2^2 + 2(k-1)\right) = k - 2 \geq 1$ for $k \geq 3$, we have that $\sqrt{k^2+1^2+k}-\sqrt{(k-1)^2+2^2+2(k-1)}>0$.

Case 3. If $r=1, s>1$,we have

$$\begin{aligned} EU(G)-EU(G^{'}) &= \sum_{y\in N_{G_{1*}}(z)}\left(\sqrt{d_G^2(y)+k^2+d_G(y)k}\right) \\ &- \sum_{y\in N_{G_{1*}}(z)}\left(\sqrt{d_G^2(y)+(k-1)^2+(d_G(y)(k-1))}\right) \\ &+ \left(\sqrt{k^2+1+k}-\sqrt{(k-1)^2+2^2+2(k-1)}\right) \\ &+ \left(\sqrt{k^2+2^2+2k}-\sqrt{2^2+2^2+4}\right) \\ &> \sqrt{k^2+1^2+k}-2\sqrt{3}\geq\sqrt{13}-2\sqrt{3}\approx 0.1414>0 \end{aligned}$$

Hence we get desired result. ■

Lemma 2. *Let G be a connected graph and $yz\in E(G)$ such that $d_G(y)\geq 2$ and $d_G(z)\geq 3$. For $0<i<r$, the path of G is $zx_1x_2...x_{r-1}x_r$ such that $d_G(x_r)=1$ and $d_G(x_i)=2$. Let $G^{'}=G-zy+yx_r$. Then, $EU(G^{'})<EU(G)$.*

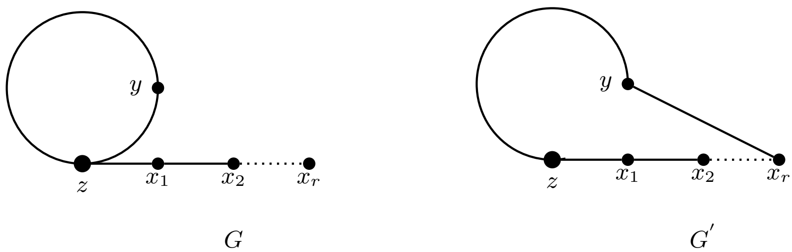

Figure 2. G and $G^{'}$ in Lemma 2.

Proof. Let $G_{2*}=G-\{y,x_1,x_2,...,x_r\}$ and $h=d_G(y)\geq 2$, $k=d_G(z)\geq 3$. We investigate the following two cases.

Case 1. If $r>1$,we have

$$EU(G)-EU(G^{'}) = \sum_{y_0\in N_{G_{2*}}(z)}\left(\sqrt{d_G(y_0)^2+k^2+d_G(y_0)k}\right)$$

$$
\begin{aligned}
& - \sum_{y_0 \in N_{G_{2*}}(z)} \left(\sqrt{d_G^2(y_0) + (k-1)^2 + (d_G(y_0)(k-1))} \right) \\
& + \left(\sqrt{k^2 + 2^2 + 2k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} \right) \\
& + \left(\sqrt{k^2 + h^2 + kh} - \sqrt{2^2 + h^2 + 2h} \right) \\
& + \left(\sqrt{2^2 + 1^2 + 2} - \sqrt{2^2 + 2^2 + 4} \right) \\
& > \left(\sqrt{k^2 + 2^2 + 2k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} \right) \\
& + \sqrt{7} - 2\sqrt{3} \\
& \geq \sqrt{19} - 4\sqrt{3} + \sqrt{7} \approx 0.0764 > 0
\end{aligned}
$$

Case 2. If $r = 1$,we get

$$
\begin{aligned}
EU(G) - EU(G^{'}) &= \sum_{y_0 \in N_{G_{2*}}(z)} \left(\sqrt{d_G y_0^2 + k^2 + d_G(y_0)k} \right) \\
&- \sum_{y_0 \in N_{G_{2*}}(z)} \left(\sqrt{d_G^2(y_0) + (k-1)^2 + (d_G(y_0)(k-1))} \right) \\
&+ \left(\sqrt{k^2 + 1 + k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} \right) \\
&+ \left(\sqrt{k^2 + h^2 + kh} - \sqrt{2^2 + h^2 + 2h} \right) \\
&> \sqrt{k^2 + 1^2 + k} - \sqrt{(k-1)^2 + 2^2 + 2(k-1)} > 0
\end{aligned}
$$

Hence we complete the proof. ■

According to the lemmas, the minimum Euler Sombor index in $G_{n,3}$ must be a tricyclic graph with no pendent vertices. This is obtained by finding the minimum Euler Sombor index in the base tricyclic graphs with n vertices.

We know that there are precisely fifteen types of base tricyclic graphs in the set of $G_{n,3}$, which are denoted by τ_i $(i = 1, 2, ..., 15)$, respectively.

Definition 1. [18] Here $l_q \geq 1\,(q = 1,\cdots,5)$ is the length of the path connecting the cycles or the common path formed by the cycles and τ_i^n is the set of n-vertex graphs in τ_i $(i = 1, 5)$. $\tau_i^n(l_1)$ is the set of n-vertex graphs in $\tau_i(i = 2, 7, 8, 14)$. $\tau_i^n\,(l_1, l_2)$ is the set of n-vertex graphs in $\tau_i(i = 4, 6, 11)$. $\tau_i^n\,(l_1, l_2, l_3)$ is the set of n-vertex graphs in τ_i $(i = 3, 9, 12)$. $\tau_i^n(l_1, l_2, l_3, l_4)$

is the set of n-vertex graphs in $\tau_i (i = 10, 13)$ and $\tau_{15}^n (l_1, l_2, l_3, l_4, l_5)$ is the set of n-vertex graphs in τ_{15}.

When the values of the Euler Sombor indices of the graphs in these subclasses are examined, it is seen that the minimum Euler Sombor index is obtained in $\tau_{15}^n (1, 1, 1, 1, 1)$ over the set of base tricyclic graphs with n vertices. (see Figure 3.)

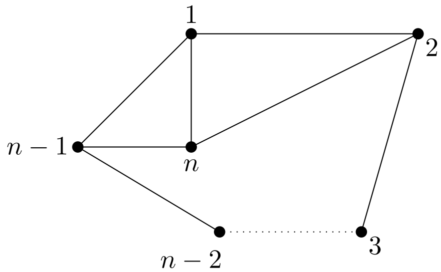

Figure 3. $\tau_{15}^n (1, 1, 1, 1, 1)$.

Theorem 1. *Let $G \in G_{n,3}$. Then, if Ω is the graph in $\tau_{15}^n (1, 1, 1, 1, 1)$, we have*

$$EU(G) \geq 2\sqrt{3}n + 5\sqrt{3} + 2\sqrt{19} = EU(\Omega).$$

Proof. From Lemma 1 and Lemma 2, we know that the minimum Euler Sombor index is obtained in $\tau_{15}^n (1, 1, 1, 1, 1)$. When $\tau_{15}^n (1, 1, 1, 1, 1)$ is examined, it is seen that there are 5 neighboring vertices with degrees 3 and 3, 2 neighboring vertices with degrees 3 and 3, and $n - 5$ neighboring vertices with degrees 2. Thus, we get

$$\begin{aligned} EU(\Omega) &= 5\sqrt{27} + 2\sqrt{19} + (n-5)\sqrt{12} \\ &= 2\sqrt{3}n + 5\sqrt{3} + 2\sqrt{19}. \end{aligned}$$

■

3 Tricyclic graphs with maximum Euler Sombor index

In this section, we study the tricyclic graphs with the maximum Euler Sombor index. The concept of contraction arises when we delete the edge $e = uv$ of a graph G and then define its ends. The resulting graph G/uv has one less edge than G.

Lemma 3. *Let G be a connected graph and $xy \in E(G)$ such that $d_G(x) \geq 2$ and $d_G(y) \geq 2$. Note that $N_G(x) \setminus \{y\} \cap N_G(y) \setminus \{x\} = \varnothing$. Let $G^{'}$ be a graph obtained by contracting the edge xy to a vertex z, further adding a pendent vertex adjacent to the vertex z. Then $EU(G^{'}) > EU(G)$.*

Proof. Let $c = d_G(x) \geq 2$ and $h = d_G(y) \geq 2$. If $EU(G) - EU(G^{'})$ is represented by Δ_1

$$\begin{aligned}
\Delta_1 &= \sum_{y_0 \in N_G(x)\setminus\{y\}} \left(\sqrt{d_G^2(y_0) + c^2 + (d_G(y_0)c)}\right) \\
&- \sum_{y_0 \in N_G(x)\setminus\{y\}} \left(\sqrt{d_G^2(y_0) + (c+h-1)^2 + (d_G(y_0)(c+h-1))}\right) \\
&+ \sum_{y_1 \in N_G(y)\setminus\{x\}} \left(\sqrt{d_G^2(y_1) + h^2 + (d_G(y_1)h)}\right) \\
&- \sum_{y_1 \in N_G(y)\setminus\{x\}} \left(\sqrt{d_G^2(y_1) + (c+h-1)^2 + (d_G(y_1)(c+h-1))}\right) \\
&+ \left(\sqrt{c^2 + h^2 + ch} - \sqrt{(c+h-1)^2 + 1^2 + (c+h-1)}\right) \\
&< \sqrt{c^2 + h^2 + ch} - \sqrt{(c+h-1)^2 + 1^2 + (c+h-1)}
\end{aligned}$$

Since $c = d_G(x) \geq 2$ and $h = d_G(y) \geq 2$, we obtain $EU(G^{'}) > EU(G)$.

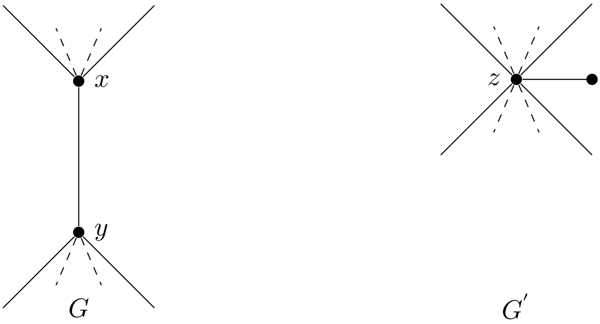

Figure 4. G and G' in Lemma 3.

■

Lemma 4. *Let* $\psi(x,y) = \sqrt{x^2+y^2+xy} - \sqrt{(x+1)^2+y^2+(x+1)y}$, *where* $x \geq 1$ *and* $y \geq 1$. *Then for any value of* $y \geq 1$, ψ *is decreasing as a function of* x *and for any value of* $x \geq 1$, ψ *is increasing as a function of* y.

Proof. The partial derivative of the function ψ about x is

$$\begin{aligned}\frac{\partial\psi(x,y)}{\partial x} &= \frac{(2x+y)\sqrt{(x+1)^2+y^2+(x+1)y}}{\sqrt{x^2+y^2+xy}\sqrt{(x+1)^2+y^2+(x+1)y}} \\ &- \frac{(2x+y+2)\sqrt{x^2+y^2+xy}}{\sqrt{x^2+y^2+xy}\sqrt{(x+1)^2+y^2+(x+1)y}}\end{aligned}$$

Since $(2x+y)^2\left((x+1)^2+y^2+(x+1)y\right)-(2x+y+2)^2\left(x^2+y^2+xy\right)$ $= -3y^2(2x+y+1) < 0$ for $x \geq 1$ and $y \geq 1$, we get

$$(2x+y)\sqrt{(x+1)^2+y^2+(x+1)y} - (2x+y+2)\sqrt{x^2+y^2+xy} < 0.$$

Hence we obtain $\frac{\partial\psi(x,y)}{\partial x} < 0$ for $x \geq 1$ and $y \geq 1$.

The partial derivative of the function ψ about y is

$$\frac{\partial\psi(x,y)}{\partial y} = \frac{(2y+x)\sqrt{(x+1)^2+y^2+(x+1)y}}{\sqrt{x^2+y^2+xy}\sqrt{(x+1)^2+y^2+(x+1)y}}$$

$$- \quad \frac{(2y+x+1)\sqrt{x^2+y^2+xy}}{\sqrt{x^2+y^2+xy}\sqrt{(x+1)^2+y^2+(x+1)y}}.$$

Since $(2y+x)^2\left((x+1)^2+y^2+(x+1)y\right)-(2y+x+1)^2\left(x^2+y^2+xy\right)$ $=3\left(2xy^2+y^2+x^2y+xy\right)$ is greater than zero for $x \geq 1$ and $y \geq 1$, we obtain $\frac{\partial\psi(x,y)}{\partial y} > 0$. ∎

Lemma 5. *Let $d_G(x_1) \geq 2$, $d_G(x_2) \geq 2$, $x_1 \geq x_2 > s > 0$ and $c > 0$. Then,*

$$\begin{aligned}\sqrt{(x_1+s)^2+(x_1+s)c+c^2} &+ \sqrt{(x_2-s)^2+(x_2-s)c+c^2}\\ &> \sqrt{x_1{}^2+x_1c+c^2}\\ &+ \sqrt{(x_2^2+x_2c+c^2}.\end{aligned}$$

Proof. To prove the Lemma, it is sufficient to compare the increase in x_1 with the increase in x_2 in the inequality. In other words, since

$$\left|s^2+2sx_1+sc\right| > \left|s^2-2sx_2-sc\right|,$$

the desired result is obtained. ∎

Corollary. *Let $d_G(x) \geq 2$, $d_G(y) \geq 2$, and $z > 0$. Then,*

$$\begin{aligned}\sqrt{(x+y-1)^2+(x+y-1)z+z^2}+\sqrt{1+z+z^2} &> \sqrt{x^2+xz+z^2}\\ &+ \sqrt{(y^2+yz+z^2}.\end{aligned}$$

Proof. In Lemma 5., let $x_1 = x, x_2 = y, s = y-1$ and $c = z$, we get desired result. ∎

It is possible to identify non-adjacent vertices u and v of a graph G by replacing these vertices with a single vertex w that is incident to all edges in G that are incident to u or v.

Lemma 6. *Let G be a connected graph and $x, y, t \in V(G)$ such that $xy \notin E(G)$ such that $d_G(x) \geq 2$ and $d_G(y) \geq 2$. Note that $N_G(x) \cap N_G(y) = \{t\}$. Let $G^{'}$ be a graph obtained by identifying the vertices x, y and deleting one*

Figure 5. G and $G^{'}$ in Lemma 6.

edge of tx and ty, further adding a pendent vertex adjacent to the vertex t. Then $EU(G^{'}) > EU(G)$.

Proof. Let $c = d_G(x) \geq 2$, $h = d_G(y) \geq 2$ and $l = d_G(t)$

$$\begin{aligned} EU(G) \quad - \quad EU(G^{'}) = & \sum_{y_0 \in N_G(x)\setminus\{t\}} \left(\sqrt{d_G^2(y_0) + c^2 + (d_G(y_0)c)} \right) \\ - & \sum_{y_0 \in N_G(x)\setminus\{t\}} \left(\sqrt{d_G^2(y_0) + (c+h-1)^2 + (d_G(y_0)(c+h-1))} \right) \\ + & \sum_{y_1 \in N_G(y)\setminus\{t\}} \left(\sqrt{d_G^2(y_1) + h^2 + (d_G(y_1)h)} \right) \\ - & \sum_{y_1 \in N_G(y)\setminus\{t\}} \left(\sqrt{d_G^2(y_1) + (c+h-1)^2 + (d_G(y_1)(c+h-1))} \right) \\ + & \left(\sqrt{c^2 + l^2 + cl} + \sqrt{h^2 + l^2 + hl} \right) \\ - & \left(\sqrt{(c+h-1)^2 + l^2 + (c+h-1)l} + \sqrt{1^2 + l^2 + l} \right) \\ < & \left(\sqrt{c^2 + l^2 + cl} + \sqrt{h^2 + l^2 + hl} \right) \\ - & \left(\sqrt{(c+h-1)^2 + l^2 + (c+h-1)l} + \sqrt{1^2 + l^2 + l} \right). \end{aligned}$$

Bu using Corollary, we have $EU(G) - EU(G^{'}) < 0$. ■

Remark. By Lemma 3 and Lemma 6, it is seen that a graph with maximum Sombor index in $G_{n,3}$ is of the form $T_{14}(n, p_1, p_2, q_1, q_2, q_3)$ or $T_{15}(n, q_1, q_2, q_3, q_4)$ (shown in [18]) where $p_1, p_2, q_1, q_2, q_3, q_4 \geq 0$ are pendent vertices such that $p_1+p_2+q_1+q_2+q_3 = n-5$ and $q_1+q_2+q_3+q_4 = n-4$ respectively.

Lets consider that $\psi(x,y) = g(x,y) - g(x+1,y)$,where $g(x,y) = \sqrt{x^2+y^2+xy}$ and $x,y \geq 1$.

Lemma 7. *Let $n \geq 5$. For any values of $p_1,p_2,q_1,q_2,q_3 \geq 0$ and $p_1+p_2+q_1+q_2+q_3 = n-5$. We obtain $EU(T_{14}(n,p_1,p_2,q_1,q_2,q_3)) < EU\left(T_{14}(n,n-5,0,0,0,0)\right)$.*

Proof. We distinguish the following four cases.

Case 1. Let $p_1 \geq p_2 \geq 1$. If

$$EU(T_{14}(n,p_1,p_2,q_1,q_2,q_3)) - EU(T_{14}(n,p_1+1,p_2-1,q_1,q_2,q_3))$$

represented by δ_1,

$$\begin{aligned}
\delta_1 &= p_1[g(p_1+4,1)-g(p_1+5,1)] - p_2[g(p_2+3,1)-g(p_2+4,1)] \\
&+ [g(p_2+3,1)-g(p_1+5,1)] + [g(p_1+4,p_2+4)-g(p_1+5,p_2+3)] \\
&+ [g(p_1+4,q_1+2)-g(p_1+5,q_1+2)] \\
&- [g(p_2+3,q_1+2)-g(p_2+4,q_1+2)] \\
&+ [g(p_1+4,q_2+2)-g(p_1+5,q_2+2)] \\
&- [g(p_2+3,q_2+2)-g(p_2+4,q_2+2)] \\
&+ [g(p_1+4,q_3+2)-g(p_1+5,q_3+2)] \\
&- [g(p_2+3,q_3+2)-g(p_2+4,q_3+2)] \\
&= [p_1\psi(p_1+4,1)-p_2\psi(p_2+3,1)] \\
&+ [\psi(p_1+4,q_1+2)-\psi(p_2+3,q_1+2)] \\
&+ [\psi(p_1+4,q_2+2)-\psi(p_2+3,q_2+2)] \\
&+ [\psi(p_1+4,q_3+2)-\psi(p_2+3,q_3+2)] \\
&+ [g(p_2+3,1)-g(p_1+5,1)] + [g(p_1+4,p_2+4)-g(p_1+5,p_2+3)] \\
&< [g(p_1+4,p_2+4)-g(p_1+5,p_2+3)]
\end{aligned}$$

By using Lemma 4, we get $[g(p_1+4,p_2+4)-g(p_1+5,p_2+3)] < 0$. Hence we get $EU(T_{14}(n,p_1,p_2,q_1,q_2,q_3)) < EU(T_{14}(n,p_1+1,p_2-1,q_1,q_2,q_3))$.

Case 2. Let $q_1 \geq q_2 \geq 1$.

If $EU(T_{14}(n,p_1,0,q_1,q_2,q_3)) - EU(T_{14}(n,p_1,0,q_1+1,q_2-1,q_3))$ represented by δ_2,

$$\begin{aligned}\delta_2 &= g(p_1+4,q_1+2)) - g(p_1+4,q_1+3) + g(p_1+4,q_2+2))\\ &- [g(p_1+4,q_2+1) + g(4,q_1+2) - g(4,q_1+3)\\ &+ g(4,q_2+2) - g(4,q_2+1)\end{aligned}$$

By using Lemma 4, we get

$$EU(T_{14}(EU(T_{14}(n,p_1,0,q_1,q_2,q_3)) < EU(T_{14}(n,p_1,0,q_1+1,q_2-1,q_3)).$$

Case 3. Let $p_1 \geq q_1 \geq 1$. If $EU(T_{14}(n,p_1,0,q_1,0,0)) - EU(T_{14}(n,p_1+1,0,q_1-1,0,0))$ represented by δ_3,

$$\begin{aligned}\delta_3 &= p_1[g(p_1+4,1) - g(p_1+5,1)] + q_1[g(q_1+2,1) - g(q_1+1,1)]\\ &+ g(p_1+4,q_1+2) - g(p_1+5,q_1+1) + g(q_1+1,1) - g(p_1+5,1)\\ &+ 2[g(p_1+4,2) - g(p_1+5,2)] + g(p_1+4,4) - g(p_1+5,4)\\ &+ g(4,q_1+2) - g(4,q_1+1)\\ &< g(p_1+4,q_1+2) - g(p_1+5,q_1+1)\end{aligned}$$

By using Lemma 4, we obtain

$$EU(T_{14}(n,p_1,0,q_1,0,0)) < EU(T_{14}(n,p_1+1,0,q_1-1,0,0)).$$

Case 4. Let $q_1 \geq p_1+1$. If $EU(T_{14}(n,p_1,0,q_1,0,0)) - EU(T_{14}(n,p_1-1,0,q_1-1,0,0))$ represented by δ_4,

$$\begin{aligned}\delta_4 &= q_1[g(q_1+2,1) - g(q_1+3,1)] - p_1[g(p_1+3,1) - g(p_1+4,1)]\\ &+ [g(p_1+3,1) - g(q_1+3,1)] + 2[g(p_1+4,2) - g(p_1+3,2)]\\ &+ [g(q_1+2,4) - g(p_1+3,4)] - [g(p_1+3,4) - g(p_1+4,4)]\\ &+ [g(p_1+4,q_1+2) - g(p_1+3,q_1+3)]\\ &= [q_1\psi(q_1+2,1) - p_1\psi(p_1+3,1)] + [\psi(q_1+2,4) - \psi(p_1+3,4)]\\ &+ [g(p_1+3,1) - g(q_1+3,1)] + 2[g(p_1+4,2) - g(p_1+3,2)]\\ &+ [g(p_1+4,q_1+2) - g(p_1+3,q_1+3)]\end{aligned}$$

$$< \quad [g(p_1+4,p_2+4)-g(p_1+5,p_2+3)]$$

By using Lemma 4 and $q_1 \geq p_1+1$,we have

$$\begin{aligned} q_1\psi(q_1+2,1)-p_1\psi(p_1+3,1) &\leq (p_1+1)\psi(p_1+3,1)-p_1\psi(p_1+3,1) \\ &= \psi(p_1+3,1) \\ &= g(p_1+3,1)-g(p_1+4,1), \end{aligned}$$

$$\psi(q_1+2,4)-\psi(p_1+3,4) \leq 0,$$

and

$$2[g(p_1+4,2)-g(p_1+3,2)] < 2[g(p_1+4,1)-g(p_1+3,1)].$$

Hence we get

$$\begin{aligned} \delta_4 &< [g(p_1+3,1)-g(p_1+4,1)]+2[g(p_1+4,1)-g(p_1+3,1)] \\ &+ [g(p_1+4,q_1+2)-g(p_1+3,q_1+3)] \\ &< [g(p_1+4,q_1+2)-g(p_1+3,q_1+3)] < 0. \end{aligned}$$

■

Thus, we see that the graph of the form $T_{14}(n,p_1,p_2,q_1,q_2,q_3)$ with maximum value of the Euler Sombor index is $T_{14}(n,n-5,0,0,0,0)$ or $T_{14}(n,0,0,n-5,0,0)$.(see Figure 6)

Using the graph structures, we can see that

$$\begin{aligned} EU(T_{14}(n,n-5,0,0,0,0)) &= (n-5)\sqrt{(n-1)^2+1^2+n-1} \\ &+ 3\sqrt{(n-1)^2+2^2+2(n-1)} \\ &+ \sqrt{(n-1)^2+4^2+4(n-1)}+6\sqrt{7}, \end{aligned}$$

$$\begin{aligned} EU(T_{14}(n,0,0,n-5,0,0)) &= (n-5)\sqrt{(n-3)^2+1^2+n-3} \\ &+ 2\sqrt{(n-3)^2+4^2+4(n-3)} \\ &+ \sqrt{4^2+4^2+16}+8\sqrt{7} \end{aligned}$$

and

$$EU(T_{14}(n,0,0,n-5,0,0)) < EU(T_{14}(n,n-5,0,0,0,0)).$$

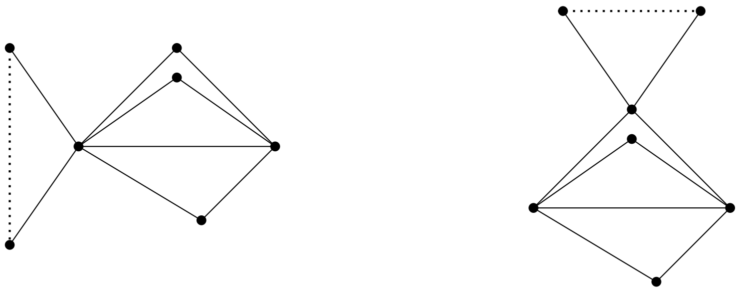

Figure 6. $T_{14}(n,n-5,0,0,0,0)$ and $T_{14}(n,0,0,n-5,0,0)$.

Similarly, it is seen that $EU(T_{15}(n,q_1,q_2,q_3,q_4)) < EU(T_{15}(n,n-4,0,0,0,0))$ is valid for $n \geq 4$, $q_1,q_2,q_3,q_4 \geq 0$ and $q_1+q_2+q_3+q_4 = n-4$.

Lemma 8. *Let $n \geq 5$. Then, we have*

$$EU\left(T_{15}(n,n-4,0,0,0)\right) < EU(T_{14}(n,n-5,0,0,0,0).$$

Proof. If $EU(T_{14}(n,n-5,0,0,0,0)) - EU\left(T_{15}(n,n-4,0,0,0)\right)$ represented by Δ, we get

$$\begin{aligned} \Delta &= 3\sqrt{(n-1)^2+2^2+2(n-1)} \\ &+ \sqrt{(n-1)^2+4^2+4(n-1)} + 6\sqrt{7} \\ &- 3\sqrt{(n-1)^2+3^2+3(n-1)} \\ &- \sqrt{(n-1)^2+1^2+(n-1)} - 9\sqrt{3} \end{aligned}$$

If $g(n) = 3\left(\sqrt{(n-1)^2+2^2+2(n-1)} - \sqrt{(n-1)^2+3^2+3(n-1)}\right)$, $g(n)$ is increasing for $n \geq 5$ by Lemma 4. Thus we have

$$g(n) \geq g(5) = 6\sqrt{7} - 3\sqrt{37}.$$

Then,

$$\Delta \quad > \quad 12\sqrt{7} - 3\sqrt{37} - 5\sqrt{3} - \sqrt{21} > 0.$$

Hence we get the desired result. ■

Theorem 2. *Let $G \in G_{n,3}$ and $n \geq 5$. Then we have*

$$\begin{aligned} EU(G) \quad &\leq \quad (n-5)\sqrt{(n-1)^2 + 1^2 + n - 1} \\ &+ \quad 3\sqrt{(n-1)^2 + 2^2 + 2(n-1)} \\ &+ \quad \sqrt{(n-1)^2 + 4^2 + 4(n-1)} + 6\sqrt{7}. \end{aligned}$$

The equality holds if and only if G is isomorphic to the graph $EU(T_{14}(n, n-5, 0, 0, 0, 0))$.

Proof. By using Lemma 3,6,7 and 8, it is easy to see the proof. ■

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