MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **94** (2025) 229–246
ISSN: 0340–6253
doi: 10.46793/match.94-1.229L

On Sombor Index for Uniform Hypergraphs

Zhuanzhuan Li[a], Liyuan Shi[d], Lei Zhang[a,b,c,*], Haizhen Ren[a,c], Fan Li[a]

[a] *Department of Mathematics and Statistics, Qinghai Normal University, Xining, China*

[b] *Department of Mathematics, Hunan Normal University, Changsha, China*

[c] *The State Key Laboratory of Tibetan Information Processing and Application, Xining, China*

[d] *Qinghai Vocational and Technical University, Xining, China*

lzz200228@163.com, 946111610@qq.com, shuxuezhanglei@163.com, haizhenr@126.com, f18392266629@163.com

(Received November 12, 2024)

Abstract

The Sombor index for graphs is given by Gutman in 2021. Since Hypergraphs can more accurately describe certain chemical scenarios. Then it has been proposed to generalize the Sombor index from graphs to hypergraphs. Recently, Shetty and Bhat defined the Sombor index $SO(H)$ of a hypergraph H as $SO(H) = \sum_{e_i \in E(H)} \sqrt{\sum_{u \in e_i} d_H(u)^2}$, where $d_H(u)$ is the degree of the vertex u of H. In this paper, we study the Sombor indices of uniform hypergraphs by hypergraph operations. The extremal hypergraph with minimum Sombor index is obtained among uniform hypertrees with maximum degree $\Delta \geq 3$, and the corresponding value of minimum Sombor index is also obtained. Furthermore, we consider the Sombor index for uniform unicyclic hypergraphs. The extremal hypergraph with maximum(minimum) Sombor index for uniform unicyclic hypergraphs is given, and the corresponding values for maximum(minimum) Sombor index are also given.

*Corresponding author.

1 Introduction

Let H=(V,E) be a connected hypergraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and hyperedge set $E = \{e_1, e_2, \cdots, e_m\}$. A *walk* in a hypergraph is a sequence of vertices and hyperedges, $(v_1, e_1, v_2, e_2, v_3, \cdots, e_{t-1}, v_t)$ with $v_{i-1}, v_i \in e_i$ and $v_{i-1} \neq v_i$. A walk w in a hypergraph H is called a *path* in H if all e_i's and all v_i's are distinct in w, and its length is $t-1$. A walk w in a hypergraph H is called a *cycle* in H if all e_i's and v_i's are distinct except $v_1 = v_t$, and its length is t. The length of the shortest cycle in H is called the girth. Two vertices in a hypergraph are *adjacent* if there is a hyperedge which contains both vertices. A hypergraph is *connected* if for any pair of vertices, there is a path which connects these vertices. A hypergraph is k*-uniform* if it has k vertices in every hyperedge. If $k = 2$, the hypergraph is a graph. A hypergraph is *linear* if its any two hyperedges have at most one vertex in common. The *degree* of a vertex u in H is the number of hyperedges that contain u, denoted by $d_H(u)$. The largest of degrees of all vertices in H is called the *maximum degree*, denoted by Δ. In a k-uniform hypergraph, denote by $H - e$ a subgraph of H obtained from H by deleting the hyperedge $e \in E$. For two nonadjacent vertices u and v of H, denote by $H + e$ the hypergraph obtained from H by adding the hyperedge e containing u and v and adding another $k - 2$ vertices to e. For a hyperedge e containing vertex u, denote by $e\backslash u$ the remaining vertices removing u from e. For the hyperedges containing u, denote by $\{e_i | u \in e_i\}\backslash e_1$ the remaining hyperedges removing e_1 from $\{e_i | i = 1, 2, \cdots, d(u), u \in e_i\}$.

Hu, Qi and Shao [6] defined the power hypergraph as Definition 1.

Definition 1. [6] *Let $G = (V, E)$ be a 2-uniform graph. For any $k \geq 3$, the kth power of G, $G^k = (V^k, E^k)$ is defined as the power hypergraph with the set of edges $E^k = \{e \bigcup \{i_{e,1}, \cdots, i_{e,k-2}\} \mid e \in E\}$, and the set of vertices $V^k = V \cup \{i_{e,1}, \cdots, i_{e,k-2}, e \in E\}$.*

From Definition 1. we can obtain the definition of hypertrees as following.

Definition 2. *Let $G = (V, E)$ be a 2-uniform tree. For any $k \geq 3$, the kth power of G, $G^k = (V^k, E^k)$ is defined as the hypertree with the set* *of edges* $E^k = \{e\bigcup\{i_{e,1}, \cdots, i_{e,k-2}\} \mid e \in E\}$*, and the set of vertices* $V^k = V \cup \{i_{e,1}, \cdots, i_{e,k-2}, e \in E\}$*.*

A hypertree is said to be *star-like* if it has exactly one vertex of degree greater than two. A hyperedge e is said to be a *pendent hyperedge* if it has a vertex of degree greater than or equal two and degree of every other vertex in e is one. A vertex x is said to be the neighbor of a vertex u in H if u and x are in the same hyperedge in H. A hyperedge e in H becomes a vertex of e, called contraction of e in H.

Based on an alternative interpretation of vertex-degree-based topological indices, and consider their chemical applications. Gutman [4] defined a new vertex-degree-based graph invariant, named "Sombor index" of a graph G, denoted by $SO(G)$, i.e.

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Since then, Sombor index of graphs have been widely studied. Horoldabva and Xu [5] gave the sharp lower and upper bounds on $SO(G)$ of connected graphs, and characterized the corresponding extremal graph. Das and Gutman [3] obtained bounds on $SO(G)$ of trees in terms of order, independence number, and number of pendent vertices, and characterized the corresponding extremal graph. Alidadi, Parsian and Arianpoor [1] obtained the minimum Sombor index for unicyclic graphs with fixed diameter. Li, Wang and Zhang [7] characterized the extremal graph with respect to Sombor index among all the n-vertex trees with given diameter. Chen, Li and Wang [2] obtained the extremal values of the Sombor index of trees with some given parameters, including matching number, pendant vertices, bipartition, diameter, segment number and branching number. Sepehr and Rad [8] obtained r-degree connected graphs with integer Sombor index for $r \in \{5, 6, 7\}$.

Since hypergraphs offer a more accurate depiction of certain chemical scenarios, such as transition states in reactions, which involve multiple atoms simultaneously changing their bonding conffgurations. Recently, Shetty and Bhat [9] consider to generalize the idea of vertex degree-based topological indices from graphs to hypergraphs, and defined the Sombor index SO of a hypergraph H as

$$SO(H) = \sum_{e_i \in E} \sqrt{\sum_{u \in e_i} d_H(u)^2},$$

where $d_H(u)$ is the degree of the vertex u of H. They gave the bounds for the Sombor index of hypergraphs and bipartite hypergraphs and obtained the extremal hypergraphs among the class of uniform, linear and general hypertrees. In addition, Wang [10] et al. also generalized the definition of Sombor index of hypergraphs and obtained several upper and lower bounds of the Sombor index of uniform hypergraphs, including those of hypertrees. They also presented a Nordhaus-Gaddum type result for the Sombor index of uniform hypergraphs. Along this direction, we continue to focus on the Sombor index(defined by Shetty and Bhat) for uniform hypergraphs, and will compare some similar results with the Sombor index defined by Wang et al.

Unless otherwise specified, the hypergraphs studied here are all linear. This paper is organized as follows. In the Section 1, some necessary notations and concepts are presented. Then in the Section 2, we obtain the extremal hypergraph with the minimum Sombor index of k-uniform hypertrees of order n with maximum degree $\Delta \geq 3$, and give the corresponding value of minimum Sombor index. Finally, in the Section 3, we obtain the extremal hypergraph with maximum (minimum) Sombor index for uniform unicyclic hypergraphs, and give the corresponding value of maximum (minimum) Sombor index.

2 Minimum Sombor index for uniform hypergraphs

In this section, we obtain the extremal hypergraph with minimum Sombor index of uniform hypertrees of order n with maximum degree $\Delta \geq 3$, and give the corresponding value of minimum Sombor index.

Let $P = ue_1u_1e_2 \cdots e_tu_t$ be a path of length t in H such that $d_H(u) \geq 3$, $d_H(u_t) = 1$ and $d_H(u_i) = 2$ for $i = 1, 2, \cdots, t-1$. Then it is called a *pendent path* in H, u and t are called the *origin* and the length of P. A vertex with degree one is called a *core vertex* and a vertex with degree larger than one is called an *intersection vertex*. We first give a hypergraph operation I that can decrease the Sombor index as follows.

Lemma 1. *Let H is a connected k-uniform linear hypergraph of order n. Let P and Q be two pendent hyperpaths with origins u and v in H, respectively. Let x be the intersection vertex in neighbor vertices of u in P and y be a core vertex of the pendent hyperedge e in Q. Denote $H^{'} = H - e_1 + e_2$, where e_1 is the hyperedge containing u and x and e_2 is the hyperedge containing x and y as depicted in Fig. 1. Then $SO(H) > SO(H^{'})$.*

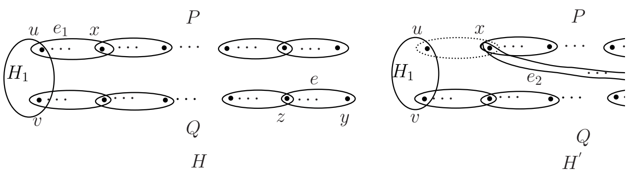

Figure 1. Hypergraph operation I

Proof. Let z be the intersection vertex of the pendent hyperedge in Q. Suppose first that $u \neq v$. Then

$$\begin{aligned}
&SO(H) - SO(H^{'}) \\
&= \sum_{e_i \in \{e_j | u \in e_j\} \setminus e_1} \sqrt{d_H(u)^2 + \sum_{w \in e_i \setminus u} d_H(w)^2} \\
&+ \sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} \\
&+ \sqrt{(k-1) + d_H(z)^2} - \sum_{e_i \in \{e_j | u \in e_j\} \setminus e_1} \sqrt{(d_H(u)-1)^2 + \sum_{w \in e_i \setminus u} d_H(w)^2} \\
&- \sqrt{d_H(x)^2 + (k-2) + 2^2} - \sqrt{(k-2) + 2^2 + d_H(z)^2} \ ,
\end{aligned} \tag{1}$$

which implies that

$$
\begin{aligned}
SO(H) - SO(H^{'}) > & \sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} + \sqrt{d_H(z)^2 + k - 1} \\
& - \sqrt{(k-2) + 2^2 + d_H(x)^2} \\
& - \sqrt{d_H(z)^2 + (k-2) + 2^2} \ .
\end{aligned} \tag{2}
$$

Also by the fact $d_H(u) \geq 3$, it follows that

$$
\begin{aligned}
& SO(H) - SO(H^{'}) > \sqrt{(k+7) + d_H(x)^2} - (\sqrt{d_H(z)^2 + (k-2) + 4} \\
& - \sqrt{d_H(z)^2 + (k-2) + 1}) - \sqrt{(k-2) + d_H(x)^2 + 2^2} \ .
\end{aligned}
$$

Let us consider a function $f(t) = \sqrt{t^2 + 4 + a} - \sqrt{t^2 + 1 + a}$, where a is any real number, and one can easily see that $f(t)$ is decreasing on $[0, +\infty)$. Since $d_H(z) \geq 2$ and $f(t)$ is decreasing, we have

$$
\begin{aligned}
SO(H) - SO(H^{'}) > & \sqrt{k + 7 + d_H(x)^2} - \sqrt{k+6} + \sqrt{k+3} \\
& - \sqrt{k + 2 + d_H(x)^2} \ .
\end{aligned}
$$

It's easy to observe that $d_H(x) \leq 2$. If $d_H(x) = 1$, then we have $SO(H) - SO(H^{'}) > \sqrt{k+8} - \sqrt{k+6} + \sqrt{k+3} - \sqrt{k+3} > 0$. Hence $SO(H) > SO(H^{'})$. If $d_H(x) = 2$, then we have $SO(H) - SO(H^{'}) > \sqrt{k+11} - \sqrt{k+6} + \sqrt{k+3} - \sqrt{k+6}$. Next we need to prove that $\sqrt{k+11} - \sqrt{k+6} + \sqrt{k+3} - \sqrt{k+6} > 0$. Since $k > -2$, we have $k^2 + 14k + 33 > k^2 + 10k + 25 \Rightarrow \sqrt{(k+3)(k+11)} > k + 5 \Rightarrow 2\sqrt{(k+3)(k+11)} > 2k + 10 \Rightarrow 2k + 14 + 2\sqrt{(k+3)(k+11)} > 4k + 24 \Rightarrow \sqrt{k+11} + \sqrt{k+3} > 2\sqrt{k+6}$, so $\sqrt{k+11} - \sqrt{k+6} + \sqrt{k+3} - \sqrt{k+6} > 0$. Hence $SO(H) > SO(H^{'})$.

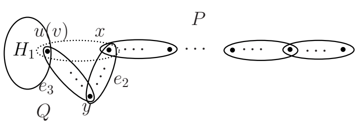

Figure 2. $H^{'}$

Following assume that $u = v$. If the length of Q is one, let e_3 (Fig. 2) be the hyperedge containing v and y, then $u = z$ and

$$\begin{aligned} &SO(H) - SO(H^{'}) \\ &= \sum_{e_i \in \{e_j | u \in e_j\} \backslash \{e_1, e_3\}} \sqrt{d_H(u)^2 + \sum_{w \in e_i \backslash u} d_H(w)^2} \\ &+ \sqrt{d_H(u)^2 + (k-1)} + \sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} \\ &- \sum_{e_i \in \{e_j | u \in e_j\} \backslash \{e_1, e_3\}} \sqrt{(d_H(u)-1)^2 + \sum_{w \in e_i \backslash u} d_H(w)^2} \\ &- \sqrt{(d_H(u)-1)^2 + (k-2) + 2^2} - \sqrt{(k-2) + 2^2 + d_H(x)^2}\ . \end{aligned} \tag{3}$$

Next we need to prove that $SO(H) - SO(H^{'}) > 0$. We get

$$\sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} > \sqrt{(k-2) + 2^2 + d_H(x)^2},$$

since $d_H(u) > 2$. We also easily know that

$$\begin{aligned} &\sum_{e_i \in \{e_j | u \in e_j\} \backslash \{e_1, e_3\}} \sqrt{d_H(u)^2 + \sum_{v \in e_i \backslash u} d_H(v)^2} \\ &> \sum_{e_i \in \{e_j | u \in e_j\} \backslash \{e_1, e_3\}} \sqrt{(d_H(u)-1)^2 + \sum_{v \in e_i \backslash u} d_H(v)^2}\ . \end{aligned}$$

Now we just need to prove that

$$\sqrt{d_H(u)^2 + (k-1)} > \sqrt{(d_H(u)-1)^2 + (k-2) + 2^2}.$$

Since $d_H(u) > 2$, we have $d_H(u)^2 + k - 1 > d_H(u)^2 - 2d_H(u) + k + 3$, so $\sqrt{d_H(u)^2 + (k-1)} > \sqrt{(d_H(u)-1)^2 + (k-2) + 2^2}$. Thus we have $SO(H) - SO(H^{'}) > 0$.

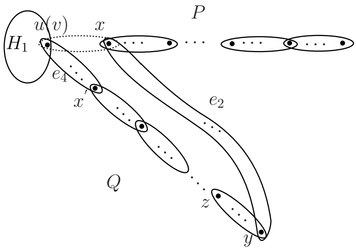

Figure 3. $H^{'}$

If the length of Q is greater than one, let $x^{'}$ be the intersection vertex in neighbor vertices of v in Q. Let e_4 (Fig. 3) be the hyperedge containing v and $x^{'}$. Hence

$$\begin{aligned}
&SO(H) - SO(H^{'}) \\
&= \sum_{e_i \in \{e_j | u \in e_j\} \setminus \{e_1, e_4\}} \sqrt{d_H(u)^2 + \sum_{w \in e_i \setminus u} d_H(w)^2} \\
&+ \sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} \\
&+ \sqrt{d_H(u)^2 + (k-2) + d_H(x^{'})^2} + \sqrt{d_H(z)^2 + k - 1} \\
&- \sum_{e_i \in \{e_j | u \in e_j\} \setminus \{e_1, e_4\}} \sqrt{(d_H(u)-1)^2 + \sum_{w \in e_i \setminus u} d_H(w)^2} \\
&- \sqrt{(d_H(u)-1)^2 + (k-2) + d_H(x^{'})^2} \\
&- \sqrt{(k-2) + 2^2 + d_H(x)^2} - \sqrt{d_H(z)^2 + (k-2) + 2^2}\ .
\end{aligned} \tag{4}$$

(4) can be scaled in the same way as (1) to obtain (2), i.e.

$$\begin{aligned}
SO(H) - SO(H^{'}) > &\sqrt{d_H(u)^2 + (k-2) + d_H(x)^2} + \sqrt{d_H(z)^2 + k - 1} \\
&- \sqrt{(k-2) + 2^2 + d_H(x)^2} \\
&- \sqrt{d_H(z)^2 + (k-2) + 2^2}\ .
\end{aligned}$$

Based on the above discussion of (2), we can get $SO(H) > SO(H^{'})$.

■

Theorem 3. *Let H be a k-uniform linear hypertree of order n with maximum degree $\Delta \geq 3$.*

(i) If $2\Delta \leq \frac{n-1}{k-1}$, then

$$SO(H) \geq \Delta(\sqrt{\Delta^2 + (k+2)} + \sqrt{k+3}\ ) + (\frac{n-1}{k-1} - 2\Delta)\sqrt{k+6} \tag{5}$$

with equality if and only if H is isomorphic to a k-uniform linear star-like hypertree of order n with maximum degree Δ, where the intersection vertices in neighbor vertices of the maximum degree vertex have degree 2.

(ii) If $2\Delta > \frac{n-1}{k-1}$, then

$$SO(H) \geq (2\Delta - \frac{n-1}{k-1})\sqrt{\Delta^2 + (k-1)} + (\frac{n-1}{k-1} - \Delta)(\sqrt{\Delta^2 + k + 2} + \sqrt{k+3}\ ) \tag{6}$$

with equality if and only if H is isomorphic to a k-uniform linear star-like hypertree of order n with maximum degree Δ in which the maximum degree vertex has exactly $2\Delta - \frac{n-1}{k-1}$ pendent hyperedges.

Proof. Let $SO(H)$ be minimum in the class of k-uniform linear hypertrees of order n with maximum degree Δ and w be a maximum degree vertex of H. Now we prove that H is isomorphic to a k-uniform linear star-like hypertree of order n with maximum degree Δ. If not there is a pendent hyperpath $ue_1u_1e_2 \cdots e_su_s$ such that $u \neq w$. Clearly there is a pendent vertex $z(\neq u_s)$ in H. Then $SO(H) > SO(H - e_1 + e_2)$ by Lemma 1, where e_1 is the hyperedge containing u and u_1 and e_2 is the hyperedge containing u_1 and z, and it contradicts the fact that $SO(H)$ is minimum. Hence H is a k-uniform linear star-like hypertree of order n with maximum degree Δ. Let t be the number of pendent hyperedges of w.

If $t = \Delta$, then $SO(H) = \Delta\sqrt{\Delta^2 + (k-1)}$.

If $t < \Delta$, then $SO(H) = t\sqrt{\Delta^2 + (k-1)} + (\Delta - t)\sqrt{\Delta^2 + 2^2 + (k-2)} + (\Delta - t)\sqrt{2^2 + (k-1)}$. When $a = k-2$ in $f(t)$, then $SO(H) = t(f(2) - f(\Delta)) + \Delta(\sqrt{\Delta^2 + k + 2} + \sqrt{k+3}) - t\sqrt{k+6}$.

If $t = 0$, then $SO(H) = \Delta(\sqrt{\Delta^2 + k + 2} + \sqrt{k+3}\ ) + (\frac{n-1}{k-1} - 2\Delta)\sqrt{k+6}$.

Since $t = 2\Delta - \frac{n-1}{k-1}$, then we have

$$SO(H) = t(f(2) - f(\Delta)) + \Delta(\sqrt{\Delta^2 + (k+2)} + \sqrt{k+3}\ ) + (\frac{n-1}{k-1} - 2\Delta)\sqrt{k+6}. \tag{7}$$

Since f is a decreasing function and $\Delta \geq 3$, we have $f(2) > f(\Delta)$. However, $\frac{n-1}{k-1}$ represents the number of hyperedges in H, and the magnitude of 2Δ cannot be determined. Therefore we distinguish the following two cases.

(i) If $2\Delta \leq \frac{n-1}{k-1}$, then $t = 0$. Hence from (7), we easily get the inequality (5) and with equality if and only if H is isomorphic to a k-uniform linear star-like hypertree of order n with maximum degree Δ, where the intersection vertices in neighbor vertices of the maximum degree vertex have degree 2.

(ii) If $2\Delta > \frac{n-1}{k-1}$, then we easily get the inequality (6) from (7) and with equality if and only if H is isomorphic to a k-uniform linear star-like hypertree of order n with maximum degree Δ in which the maximum degree vertex has exactly $2\Delta - \frac{n-1}{k-1}$ pendent hyperedges. ■

From the above Theorem 3, we easily obtain the following result in [5].

Corollary. [5] *Let G be a connected graph of order n with maximum degree $\Delta \geq 3$.*

(i) If $2\Delta \leq n-1$ then

$$SO(G) \geq \Delta(\sqrt{\Delta^2 + 4} + \sqrt{5}\ ) + 2(n - 2\Delta - 1)\sqrt{2}$$

with equality if and only if G is isomorphic to a star-like tree of order n with maximum degree Δ in which all neighbors of the maximum degree vertex have degree two.

(ii) If $2\Delta > n-1$ then

$$SO(G) \geq (n - 1 - \Delta)(\sqrt{\Delta^2 + 4} + \sqrt{5}\ ) + (2\Delta - n + 1)\sqrt{\Delta^2 + 1}$$

with equality if and only if G is isomorphic to a star-like tree of order n with maximum degree Δ in which the maximum degree vertex has exactly $2\Delta - n + 1$ pendent neighbors.

Let C_n be a k-uniform linear cycle of order n. Denote by $C_{n,1}$ the hypergraph obtained by attaching one pendent hyperedge to an intersection vertex of $C_{n-(k-1)}$.

Theorem 4. *Let $SO(H)$ be minimum in the class of connected k-uniform linear unicyclic hypergraphs of order n, where the length of the cycle is g. If H is different from C_n, then H is isomorphic to the unicyclic hypergraph, where the length of the cycle is g, that has exactly one pendent path.*

Proof. If $g = \frac{n}{k-1} - 1$, then H is isomorphic to $C_{n,1}$ and hence the theorem holds. Let $g \leq \frac{n}{k-1} - 2$ and H is not isomorphic to the unicyclic hypergraph that has exactly one pendent path of length at least two. Then repeatedly using the operation in Lemma 1, we get the required result. ■

The following result easily follows from Theorem 4.

Theorem 5. *Let H be a connected k-uniform linear unicyclic hypergraph of order n which is different form C_n. Then $SO(C_n) < SO(H)$.*

Proof. Let g be the length of the cycle in H. Suppose that $SO(H)$ is minimum in the class of linear k-uniform unicyclic hypergraphs of length g order n. Since H is different form C_n, H is isomorphic to the k-uniform linear unicyclic of length g order n that has exactly one pendent path by Theorem 4. If $g = \frac{n}{k-1} - 1$, then H is isomorphic to $C_{n,1}$ and it follows that

$$\begin{aligned} SO(C_{n,1}) &= (g-2)\sqrt{2^2+2^2+k-2} + 2\sqrt{2^2+3^2+k-2} \\ &\quad + \sqrt{2^2+k-1} \\ &= (\frac{n}{k-1} - 3)\sqrt{k+6} + 2\sqrt{k+11} + \sqrt{k+3}. \end{aligned}$$

Next we need to prove that $SO(C_{n,1}) > SO(C_n)$, where

$SO(C_n) = \frac{n}{k-1}\sqrt{2^2+2^2+(k-2)} = \frac{n}{k-1}\sqrt{k+6}$.

We can easily know that $\sqrt{k+11} > \sqrt{k+6}$. Since $k > -2$, we have $\sqrt{k+11} + \sqrt{k+3} > 2\sqrt{k+6}$. So we have $SO(C_{n,1}) > SO(C_n)$.

If $g \leq \frac{n}{k-1} - 2$, then H is isomorphic to the k-uniform linear unicyclic of length g order n that has exactly one pendent path of length at least two and it follows that

$$SO(H) = (\frac{n}{k-1} - 4)\sqrt{2^2 + 2^2 + k - 2} + 3\sqrt{2^2 + 3^2 + k - 2}$$
$$+ \sqrt{2^2 + k - 1} = (\frac{n}{k-1} - 4)\sqrt{k+6} + 3\sqrt{k+11} + \sqrt{k+3} \ .$$

Next we need to prove that $SO(H) > SO(C_n) = \frac{n}{k-1}\sqrt{k+6}$. Now we just need to prove that $3\sqrt{k+11} + \sqrt{k+3} - 4\sqrt{k+6} > 0$. Since $k > -2$, we have $3\sqrt{k+11} + \sqrt{k+3} > 4\sqrt{k+6}$, so $-4\sqrt{k+6} + 3\sqrt{k+11} + \sqrt{k+3} > 0$. So we have $SO(C_n) < SO(H)$. ■

3 Maximum Sombor index for uniform hypergraphs

In this section, we obtain the extremal hypergraph with maximum Sombor index for k-uniform unicyclic hypergraphs of order n, and give the corresponding value of maximum Sombor index.

We first give a hypergraph operation II that can increase the Sombor index as follows.

Lemma 2. *Let H be a connected k-uniform linear hypergraph of order n. Let e be a non-pendent hyperedge that is not on any cycles in H, where e contains vertices u and v in H. Denote by H' the hypergraph obtained by the contraction of e onto its endpoint $u \in e$ and adding a pendent hyperedge e' to u and adding another $k-1$ vertices to e'. Then $SO(H) < SO(H')$.*

Proof. Let $M = \{e_i | u \in e_i\} \backslash \{e\} = \{e_1, e_2, \cdots, e_s\}$, where $\{u, u_i\} \subseteq \{e_i\}$, $i = 1, 2, \cdots, s$. Let $N = \{e_i | v \in e_i\} \backslash \{e\} = \{e_1', e_2', \cdots, e_t'\}$, where $\{v, v_i\} \subseteq \{e_i'\}$, $i = 1, 2, \cdots, t$. Then $d_H(u) = s + 1$ and $d_H(v) = t + 1$. Since e is a non-pendent hyperedge that is not on any cycles in H, we have $st > 0$. Then

$$SO(H') - SO(H)$$
$$= \sum_{i=1}^{s} \sqrt{(s+t+1)^2 + (k-2) + d_H(u_i)^2}$$

$$+\sum_{j=1}^{t}\sqrt{(s+t+1)^2+(k-2)+d_H(v_j)^2}$$
$$+\sqrt{(s+t+1)^2+(k-1)}-\sum_{i=1}^{s}\sqrt{(s+1)^2+(k-2)+d_H(u_i)^2}$$
$$-\sum_{j=1}^{t}\sqrt{(t+1)^2+(k-2)+d_H(v_j)^2}-\sqrt{(s+1)^2+(k-2)+(t+1)^2}$$
$$>\sqrt{(s+t+1)^2+(k-1)}-\sqrt{(s+1)^2+(k-2)+(t+1)^2}\ .$$

So we have $SO(H) < SO(H^{'})$. ■

Lemma 3. *Let H be a connected k-uniform linear hypergraph of order n with m hyperedges that are not on any cycles in H. If $SO(H)$ is maximum in the class of connected k-uniform linear hypergraphs of order n with m hyperedges that are not on any cycles in H, then all m hyperedges that are not on any cycles in H are pendent.*

Proof. Suppose, on the contrary, that H contains a non-pendent hyperedge e that is not on any cycles in H. Let $H^{'}$ be the hypergraph obtained by the contraction of e ($u \in e$) onto the vertex u and adding a pendent hyperedge $e^{'}$ to u and adding another $k-1$ vertices to $e^{'}$. Then $SO(H) < SO(H^{'})$ by Lemma 2. Therefore, we have a contradiction to the assumption that $SO(H)$ is maximum in the class of k-uniform linear connected hypergraphs of order n with m hyperedges that are not on any cycles in H. ■

Particularly, if the hypergraph is a k-uniform hypertree, by Lemma 2 we can get the following result of [9].

Corollary. [9] *Let $\mathcal{T}$ be a k-uniform hypertree with m hyperedges. Then*

$$SO(\mathcal{T}) \leq m\sqrt{m^2+k-1}$$

with equality if and only if $\mathcal{T}$ is k-uniform hyperstar.

Remark. Compare to the generalized definition of Sombor index for hypergraphs [10], we see that the extremal hypergraph of a k-uniform hypertree is consistent with the one under this definition.

Lemma 4. *Let H be a connected k-uniform linear unicyclic hypergraph of order $n = g(k-1)+m(k-1)$, $k \geq 1$ vertices with $m \geq 3$ pendent hyperedges, where the length of the cycle is g. $H^{'}$ is the hypergraph obtained from H by attaching one pendent hyperedge on vertex u_i to vertex u_j, $1 \leq i, j \leq g$. Then $SO(H^{'}) > SO(H)$.*

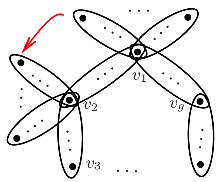

Figure 4. $H^{'}$

Proof. We assume, without loss of generality, that $u_i = u_1, u_j = u_2$ as depicted in Fig. 4. Denote by $u_1, u_2, \cdots, u_g$ the intersection vertices of the cycle in H. For simplicity's sake we denote $d_H(u_i) = d_i, i = 1, 2, \cdots, g$. Let $d_1 \leq d_2 \leq \cdots \leq d_g$. Then

$$
\begin{aligned}
& SO(H) - SO(H^{'}) \\
&= \sqrt{d_1^2 + d_2^2 + k - 2} + \cdots + \sqrt{d_g^2 + d_1^2 + k - 2} \\
&+ (d_1 - 2)\sqrt{d_1^2 + k - 1} + \cdots + (d_g - 2)\sqrt{d_g^2 + k - 1} \\
&- \sqrt{(d_1 - 1)^2 + (d_2 + 1)^2 + k - 2} - \sqrt{(d_2 + 1)^2 + d_3^2 + k - 2} \\
&- \cdots - \sqrt{d_g^2 + (d_1 - 1)^2 + k - 2} \\
&- (d_1 - 3)\sqrt{(d_1 - 1)^2 + k - 1} - (d_2 - 1)\sqrt{(d_2 + 1)^2 + k - 1} - \cdots \\
&= \sqrt{d_1^2 + d_2^2 + k - 2} + \sqrt{d_2^2 + d_3^2 + k - 2} + \sqrt{d_g^2 + d_1^2 + k - 2} \\
&- \sqrt{(d_1 - 1)^2 + (d_2 + 1)^2 + k - 2} - \sqrt{(d_2 + 1)^2 + d_3^2 + k - 2} \\
&- \sqrt{d_g^2 + (d_1 - 1)^2 + k - 2} + (d_1 - 2)\sqrt{d_1^2 + k - 1} \\
&+ (d_2 - 2)\sqrt{d_2^2 + k - 1} - (d_1 - 3)\sqrt{(d_1 - 1)^2 + k - 1}
\end{aligned}
$$

$$-(d_2-1)\sqrt{(d_2+1)^2+k-1}. \tag{8}$$

We will show that $SO(H)-SO(H^{'})<0$ by showing that

$$\sqrt{d_1^2+d_2^2+k-2}<\sqrt{(d_1-1)^2+(d_2+1)^2+k-2}\ , \tag{9}$$

$$\begin{aligned}&\sqrt{d_2^2+d_3^2+k-2}+\sqrt{d_g^2+d_1^2+k-2}\\ &<\sqrt{(d_2+1)^2+d_3^2+k-2}+\sqrt{d_g^2+(d_1-1)^2+k-2}\ ,\end{aligned} \tag{10}$$

$$\begin{aligned}&(d_1-2)\sqrt{d_1^2+k-1}+(d_2-2)\sqrt{d_2^2+k-1}\\ &(d_1-3)\sqrt{(d_1-1)^2+k-1}+(d_2-1)\sqrt{(d_2+1)^2+k-1}\ ,\end{aligned} \tag{11}$$

as follows.

First we prove (9), that is, prove $d_1^2+d_2^2+k-2<d_1^2+d_2^2-2d_1+2d_2+k$. The inequality $d_1^2+d_2^2+k-2<d_1^2+d_2^2-2d_1+2d_2+k$ is equivalent to $d_1-d_2<1$. Since $d_1\leq d_2$, we have $d_1-d_2<1$. So (9) holds.

Next we prove (10). Let $d_1=x$, $d_2=y$, $d_3=z$, $d_g=w$, $k-2=m$. The inequality (10) is equivalent to

$$\begin{aligned}&\sqrt{y^2+z^2+m}+\sqrt{w^2+x^2+m}<\sqrt{w^2+(x-1)^2+m}\\ &+\sqrt{(y+1)^2+z^2+m},\end{aligned}$$

which is

$$\begin{aligned}&\sqrt{y^2+z^2+m}-\sqrt{(y+1)^2+z^2+m}\\ &-(\sqrt{w^2+(x-1)^2+m}-\sqrt{w^2+x^2+m}\ )<0.\end{aligned} \tag{12}$$

Set binary function $P(a,b)=\sqrt{a^2+b^2+m}-\sqrt{(a+1)^2+b^2+m}\ $, where m is any real number. So the inequality (12) is equivalent to

$$P(y,z)-P(x-1,w)<0. \tag{13}$$

It's easy to verify that $P_a^{'}<0$ and $P_b^{'}>0$, where $P_a^{'}$ and $P_b^{'}$ stand for partial derivative. Since $x-1\leq y\leq z\leq w$, we have (13) holds. So (10) holds.

Finally, we prove (11). Let $d_1 = x$, $d_2 = y$, $k - 1 = t$. The inequality (11) is equivalent to

$$\begin{aligned}&(x-2)\sqrt{x^2+t}-(x-3)\sqrt{(x-1)^2+t}\\&-[\,(y-1)\sqrt{(y+1)^2+t}-(y-2)\sqrt{y^2+t}\,]<0.\end{aligned} \tag{14}$$

Set the function $Q(a) = (a-2)\sqrt{a^2+t} - (a-3)\sqrt{(a-1)^2+t}$, where t is any real number, and one can easily see that $Q(a)$ is increasing on $[0, +\infty)$. Then the inequality (14) is equivalent to

$$Q(x) - Q(y+1) < 0. \tag{15}$$

Since $x \leq y + 1$, so (15) holds. So (11) holds. ■

Theorem 6. *Let H be a connected k-uniform linear unicyclic hypergraph on $n = g(k-1) + m(k-1)$ vertices with $m \geq 3$ hyperedges that are not on the cycle in H and the girth is g. Then*

$$SO(H) \leq 2\sqrt{(m+2)^2+k+2} + (g-2)\sqrt{k+6} + m\sqrt{(m+2)^2+k-1} \tag{16}$$

with equality if and only if H is isomorphic to the hypergraph obtained by attaching m pendent hyperedges to an intersection vertex of the cycle in H.

Proof. Let C_g be the cycle in H. Denote by $U_{n,m}$ the hypergraph obtained by attaching m pendent hyperedges to an intersection vertex of C_g. If H is isomorphic to $U_{n,m}$, then the equality holds in (16). Suppose that H is not isomorphic to $U_{n,m}$ and $SO(H)$ is maximum among all k-uniform linear unicyclic hypergraphs on $n = g(k-1) + m(k-1)$, $k \geq 1$ vertices with $m \geq 3$ hyperedges that are not on C_g, where the length of C_g is g. Then by Lemma 3, H is isomorphic to a hypergraph such that each pendent hyperedge is attached to the unique cycle.

Denote by $u_1, u_2, \cdots, u_g$ the intersection vertices of C_g. For simplicity's sake we denote $d_H(u_i) = d_i, i = 1, 2, \cdots, g$. Let $d_1 \leq d_2 \leq \cdots \leq d_g$. Let m be the number of pendent hyperedges in H. Then, we have $2 \leq d_i \leq m+2$, $d_1 + d_2 + \cdots + d_g = 2g + m$.

Now let $H^{'}$ be the hypergraph obtained from H by attaching one pendent hyperedge on vertex u_i to vertex u_j, $1 \leq i, j \leq g$. We assume, without loss of generality, that $u_i = u_1, u_j = u_2$. We have $SO(H) < SO(H^{'})$ by Lemma 4.

In a similar way, we attach one pendent hyperedge on vertex u_i to vertex u_j, $1 \leq i, j \leq g$, where $d_i \leq d_j$. Continuing with the operation, we obtain that H is isomorphic to $U_{n,m}$. ■

We know that a connected graph is obtained from a k-uniform connected hypergraph where $k = 2$. From the above Theorem, we easily obtain the following result in [5].

Corollary. [5] *Let G be a unicyclic graph of order n with girth g. Then*

$$SO(G) \leq 2\sqrt{(n-g+2)^2+4} + (n-g)\sqrt{(n-g+2)^2+1} + 2\sqrt{2}(g-2)$$

with equality if and only if G is isomorphic to the graph obtained by attaching $n-g$ pendent edges to a vertex of C_g.

4 Conclusion

By the definition of the Sombor index given by Shetty and Bhat [9], we mainly consider the Sombor indices of uniform hypergraphs by hypergraph operations. Among all uniform hypertrees with maximum degree $\Delta \geq 3$, the extremal hypergraph with minimum Sombor index is obtained, and the corresponding value of minimum Sombor index is also obtained. In addition, for uniform unicyclic hypergraphs the extremal hypergraph with maximum(minimum) Sombor index is given, and the corresponding values for maximum(minimum) Sombor index are also given. We note that for uniform hypertree with given hyperedges the extremal hypergraph with maximum Sombor index is consistent with the one under the definition of the Sombor index for hypergraphs given by Wang et al. [10].

***Acknowledgment*:** The authors would like to thank the anonymous referees for their constructive corrections and valuable comments on this paper, which have considerably improved the presentation of this paper. This work was supported by Natural Science Foundation of Hunan Province (Grant Nos. 2023JJ40424), the National Natural Science Foundation of China (Grant Nos. 12161073) and the Postdoctoral Research Foundation of China (Grant Nos. 2023M74
1147).

References

[1] A. Alidadi, A. Parsian, H. Arianpoor, The minimum Sombor index for unicyclic graphs with fixed diameter, *MATCH Commun. Math. Comput. Chem.* **88** (2022) 561–572.

[2] H. Chen, W. Li, J. Wang, Extremal values on the Sombor index of trees, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 23–49.

[3] K. Das, I. Gutman, On Sombor index of trees, *Appl. Math. Comput.* **412** (2022) 126–575.

[4] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 11–16.

[5] B. Horoldagva, C. Xu, On Sombor index of graphs, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 703–713.

[6] S. Hu, L. Qi, J. Shao, Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues, *Lin. Algebra Appl.* **439** (2013) 2980–2998.

[7] S. Li, Z. Wang, M. Zhang, On the extremal Sombor index of trees with a given diameter, *Appl. Math. Comput.* **416** (2022) #126731.

[8] M. Sepehr, N. Rad, On graphs with integer Sombor indices, *Commun. Comb. Optim.* **4** (2024) 693–705.

[9] S. Shetty, K. Bhat, Sombor index of hypergraphs, *MATCH Commun. Math. Comput. Chem.* **91** (2024) 235–254.

[10] X. Wang, M. Wang, Sombor index of uniform hypergraphs, *AIMS Math.* **9** (2024) 30174–30185.