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Product of Wiener and Harary Indices of Uniform Hypergraphs

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Abstract

For a connected graph G, the Wiener index W and the Harary index H are defined as $W = \sum_{u,v} d(u, v)$ and $H = \sum_{u,v} 1/d(u, v)$, respectively. In this paper, the product $W \cdot H$ is first extended to hypergraphs. We determine the unique k-uniform hypergraphs with maximum, minimum and second minimum $W \cdot H$-value, respectively.

1 Introduction

Let G be a k-uniform hypergraph with $V(G)$ and $E(G)$, where every edge contains exactly k vertices for an integer $k \geq 2$. When $k = 2$, G is an ordinary graph. The degree $d(v)$ of v is the number of edges of G that contain v. A vertex of degree one is called a pendent vertex. An edge is called a pendent edge if it contains exactly $k - 1$ vertices of degree one.

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A path P from u to v is a vertex-edge alternative sequence:$u = u_1, e_1, u_2, e_2, ..., u_s, e_s, u_{s+1} = v$ such that (a) $u_1, u_2, ..., u_{s+1}$ are distinct vertices; (b)$e_1, e_2, ..., e_s$ are distinct edges; (c) $u_i, u_{i+1} \in e_i$ for $i = 1, 2, ..., s$. The integer s is the length of the path. The path P is called a pendant path at u_1, if $d(u_1) \geq 2, d(u_i) = 2$ for $i = 2, ..., s, d(w) = 1$ for $w \in e_i \backslash \{u_i, u_{i+1}\}$ with $1 \leq i \leq s$, and $d(u_{s+1}) = 1$. The distance $d(u, v)$ between two vertices u and v in G is the minimum length of a path which connects u and v. In particular, $d(u, u) = 0$. The diameter of G is the maximum distance between all vertex pairs of G. The Wiener index $W(G)$ and the Harary index $H(G)$ of a graph G are defined as

$$W(G) = \sum_{\{u,v\} \subseteq v(G)} d(u, v), H(G) = \sum_{\{u,v\} \subseteq v(G)} \frac{1}{d(u, v)}.$$

respectively. The product of Wiener and Harary index is denoted by $W \cdot H(G)$.

The Wiener index has a long history since 1947 when Wiener introduced this parameter as the path number [12]. There are rich results for ordinary graphs and hypergraphs on this index [2,3,6,7,10]. For analogous data on Harary index see [4, 8, 9, 11, 13]. Recently, Gutman et al. [5] have obtained a lower bound for general connected graphs. Azjargal et al. [1] characterized the respective species with minimum $W \cdot H$ for all graphs of order n and size m. However, almost no results on the product $W \cdot H$ of hypergraphs have been obtained. This paper is a new attempt to study on this topic and it is likely to attract more and more attention in the nearest future. In this paper, we attempt to determine the maximum and the minimum $W \cdot H$-value for k-uniform hypergraphs.

2 Preliminaries

In this section, we introduce some definitions and lemmas, which will be used to prove our main results.

For $X \subseteq V(G)$, let $G - X$ be the sub-hypergraph of G obtained by deleting all vertices in X and all edges containing at least one vertex in X. Removing $v \in e$ from the e is called v-shrinking on e.

For $X \subseteq V(H)$ with $X \neq \varnothing$, let $G[X]$ be the sub-hypergraph induced by X, that is, $G[X]$ has vertex set X and edge set $\{e \subseteq X : e \in E(G)\}$.

For positive integers p, q, and a k-uniform hypergraph G, let $G_u(p, q)$ be the k-uniform hypergraph obtained from G by attaching two pendant paths of length p and q at u, respectively, and $G_u(p, 0)$ be the k-uniform hypergraph obtained from G by attaching a pendant path of length p at u.

We define a function of $n-1$ variables as following:

$$f\left(x_1, x_2, ..., x_{n-1}\right) = \left(\sum_{i=1}^{n-1} i x_i\right)\left(\sum_{i=1}^{n-1} \frac{x_i}{i}\right)$$

where $x_1, x_2, ..., x_{n-1}$ are nonnegative integers.

Lemma 2.1 [1] Let n and d be given integers such that $2 \le d \le n-1$. Then

$$f(\underbrace{x_1, ..., x_{d-1}, x_d}_{d}, 0, ..., 0) > f(\underbrace{x_1, ..., x_{d-1} + x_d}_{d-1}, 0, ..., 0),$$

Let d be the diameter of G. Suppose that $d > 2$. Denote by p_i the number of distinct pairs of vertices whose distance in G is exactly i. Then the Wiener index W and the Harary index H are represented as $W(G) = \sum_{i=1}^{d} i p_i$, $H(G) = \sum_{i=1}^{d} \frac{p_i}{i}$, respectively. In addition, we have

$$W \cdot H = f(\underbrace{p_1, ..., p_{d-1}, p_d}_{d}, 0, ..., 0),$$

Lemma 2.2 [1] Let p_k be positive integer such that $1 \le k \le d$. Then

$$\begin{aligned} f(\underbrace{p_1, ..., p_{d-1}, p_d}_{d}, 0, ..., 0) \quad &> \quad f(\underbrace{p_1, ..., p_{d-1} + p_d}_{d-1}, 0, ..., 0), \\ &> \quad ... > f(p_1, \sum_{i=2}^{d} p_i, 0, ..., 0). \end{aligned}$$

For a k-uniform hypergraph G and positive integers s and t, let $G_u(s, t)$ be the k-uniform hypergraph obtained from G by attaching two pendant paths S of length s and T of length t at u. Let $G_u^*(s+t)$ be the hypergraph obtained from G by attaching $s+t$ pendant edges.

Lemma 2.3 Let G be a connected k-uniform hypergraph with $|E(G)| \geq 1$, $u \in V(G)$. For integers $s \geq t \geq 1$, $W \cdot H(G_u(s,t)) > W \cdot H(G_u^*(s+t)$.

Proof. Let $P = (u, e_1, u_1, ..., u_{s-1}, e_s, u_s)$ and $Q = (u, e_1', v_1, ..., v_{t-1}, e_t', v_t)$ be the two pendant paths of $G_u(s,t)$ at u of lengths s and t, respectively. Let $G_u^*(s+t)$ be the hypergraph obtained from G by attaching $s+t$ pendant edges at u.

Let $A = G[V(S) \cup V(T) \cup \{u\}]$ be the sub-hypergraph of $G_u(s,t)$, and $A^* = G^*[V(S) \cup V(T) \cup \{u\}]$ be the sub-hypergraph of $G_u^*(s+t)$. Note that

$$W \cdot H_{G_u(s,t)}(G) \quad = \quad W \cdot H_{G_u^*(s+t)}(G)$$

Denote by p_i the number of distinct pairs of vertices whose distance in A is exactly i. We deduce that $p_1 = (k-1)(s+t), p_2 = (k-1)^2(s+t-1), p_3 = (k-1)^2(s+t-2), ..., p_{d-1} = 2(k-1)^2, p_d = (k-1)^2$.

$$\begin{aligned}
\sum_{i=2}^{d} p_i &= (k-1)^2(s+t-1) + (k-1)^2(s+t-2) + ... + 2(k-1)^2 \\
&+ (k-1)^2 \\
&= (k-1)^2[s+t-1 + (s+t-2) + ... + 2 + 1] \\
&= (k-1)^2 \binom{s+t}{2}
\end{aligned}$$

By Lemma 2.2, we have
$f(\underbrace{(k-1)(s+t), (k-1)^2(s+t-1), ..., 2(k-1)^2, (k-1)^2}_{d}, 0, ..., 0) >$
$f((k-1)(s+t), (k-1)^2\binom{s+t}{2}, 0, ..., 0)$.
Note that

$$W \cdot H_{G_u(s,t)}(A) > W \cdot H_{G_u^*(s+t)}(A^*)$$

Let d be the diameter of G and v be a vertex in G. Suppose that there is a path between v and u. Let m_i be the number of paths between u and v with distance i, for $1 \leq i \leq d$.

Denote by p_{m_i+j} the number of distinct pairs of vertices whose distance between v and $w \in V(A)\setminus\{u\}$ is exactly m_i+j. For $G_u(s,t)$, suppose that $s \geq t$, we deduce that $p_{m_i} = m_i(k-1)^2, p_{m_i+1} = 2m_i(k-1)^2, p_{m_i+2} = 2m_i(k-1)^2, ..., p_{m_i+t} = 2m_i(k-1)^2, p_{m_i+t+1} = m_i(k-1)^2, p_{m_i+t+2} = m_i(k-1)^2, ..., p_{m_i+s} = m_i(k-1)^2$.

Denote by p'_j the number of distinct pairs of vertices whose distance between v and $w \in V(A^*)$ is exactly j. We deduce that $p'_{m_i} = m_i(k-1)^2, p'_{m_i+1} = (s+t)m_i(k-1)^2$.

$$\begin{aligned}
\sum_{j=1}^{s} p_{m_i+j} &= \underbrace{2m_i(k-1)^2 + ... + 2m_i(k-1)^2}_{t} \\
&+ \underbrace{m_i(k-1)^2 + ... + m_i(k-1)^2}_{s} \\
&= (2t+s)m_i(k-1)^2 \\
&> (t+s)m_i(k-1)^2
\end{aligned}$$

By Lemma 2.2, for $v \in V(G)$, we have

$$\sum_{u\in V(A)} d(u,v) \sum_{u\in V(A)} \frac{1}{d(u,v)} > \sum_{u\in V(A^*)} d(u,v) \sum_{u\in V(A^*)} \frac{1}{d(u,v)}$$

Further, we have the following conclusion

$$\begin{aligned}
B_1 = & \sum_{v\in V(G)\setminus\{u\}, u\in V(A)} d(u,v) \sum_{v\in V(G)\setminus\{u\}, u\in V(A)} \frac{1}{d(u,v)} \\
> & \sum_{v\in V(G)\setminus\{u\}, u\in V(A^*)} d(u,v) \sum_{v\in V(G)\setminus\{u\}, u\in V(A^*)} \frac{1}{d(u,v)} = B_2
\end{aligned}$$

Since

$$\begin{aligned}
W\cdot H(G_u(s,t)) &= W\cdot H_{G_u(s,t)}(G) + W\cdot H_{G_u(s,t)}(A) + B_1 \\
W\cdot H(G_u^*(s+t)) &= W\cdot H_{G_u^*(s+t)}(G) + W\cdot H_{G_u^*(s+t)}(A^*) + B_2
\end{aligned}$$

Thus we have $W \cdot H(G_u(s,t)) > W \cdot H(G_u^*(s+t))$. ■

For a k-uniform hypertree G of order n with m edges, if there is a disjoint partition of the vertex set $V(G) = \{u\} \cup V_1 \cup ... \cup V_m$ such that $|V_1| = ... = |V_m| = k-1$, and $E(G) = \{\{u\} \cup V_i : 1 \leq i \leq m\}$, then we call G is a k-uniform hyperstar (with center u), denoted by $S_{m,k}$.

For a k-uniform hypergraph G with $u, v \in e \in E(G)$. For positive integers s and t, let $G_{u,v}(s,t)$ be the k-uniform hypergraph obtained from G by attaching a pendant path S of length s at u and a pendant path T of length t at v. Let $G_u^*(s+t)$ be the hypergraph obtained from G by attaching a hyperstar $S_{s+t,k}$ at u.

Lemma 2.4 Let G be a connected k-uniform hypergraph with $|E(G)| \geq 2$, $u, v \in e \in E(G)$. For integers $s \geq t \geq 1$, $W \cdot H(G_{u,v}(s,t)) > W \cdot H(G_u^*(s+t))$.

Proof. For a k-uniform hypergraph G with $u, v \in e \in E(G)$, let $G_{u,v}^*(s,t)$ be the k-uniform hypergraph obtained from G by attaching a hyperstar $S_{s,k}$ at u and a hyperstar $S_{t,k}$ at v. By Lemma 2.3, we have $W \cdot H(G_{u,v}(s,t)) > W \cdot H(G_{u,v}^*(s,t))$.

Let $A = G^*[V(S_{s,k}) \cup V(S_{t,k}) \cup \{e\}]$ be the sub-hypergraph of $G_{u,v}^*(s,t)$, and $A^* = G^*[V(S_{s+t,k}) \cup \{e\}]$ be the sub-hypergraph of $G_u^*(s+t)$. Note that

$$W \cdot H_{G_{u,v}^*(s,t)}(G) \quad = \quad W \cdot H_{G_u^*(s+t)}(G)$$

Denote by p_i the number of distinct pairs of vertices whose distance in A is exactly i. For the sub-hypergraph A of $G_{u,v}^*(s,t)$, suppose that $s \geq t$, we deduce that $p_1 = (k-1)(s+t+1), p_2 = (k-1)^2[\binom{s}{2} + \binom{t}{2} + s + t], p_3 = (k-1)^2(s^t)$.

Denote by p_i' the number of distinct pairs of vertices whose distance in A^* is exactly i. We deduce that $p_1' = (k-1)(s+t+1), p_2' = (k-1)^2[\binom{s+t}{2} + s + t]$.

$$\sum_{i=2}^{d} p_i \quad = \quad (k-1)^2 \left[\binom{s}{2} + \binom{t}{2} + s + t \right] + (k-1)^2(s^t)$$

$$
\begin{aligned}
&= (k-1)^2\left[\binom{s}{2}+\binom{t}{2}+s+t+s^t\right] \\
&> (k-1)^2\left[\binom{s+t}{2}+s+t\right]
\end{aligned}
$$

By Lemma 2.2, we have

$$
f\left((k-1)(s+t+1),(k-1)^2\left[\binom{s}{2}+\binom{t}{2}+s+t\right],(k-1)^2(s^t),0,...,0\right)
$$
$$
> f\left((k-1)(s+t+1),(k-1)^2\left[\binom{s+t}{2}+s+t\right],0,...,0\right).
$$

Note that

$$
W\cdot H_{G_u^*(s,t)}(A) > W\cdot H_{G_u^*(s+t)}(A^*)
$$

Let d be the diameter of G and w be a vertex in G. Suppose that there is a path between w and u. Let m_i be the number of paths between w and u with distance i, for $1\le i\le d$. Suppose that there is a path between w and v. Let n_j be the number of paths between w and v with distance j, for $1\le j\le d$.

Denote by p_k the number of distinct pairs of vertices whose distance between w and $w'\in V(A)\setminus\{u,v\}$ is exactly k. We deduce that $p_{m_i+1} = m_i(s+1)(k-1)^2, p_{m_i+2} = m_i t(k-1)^2, p_{n_j+1} = n_j(t+1)(k-1)^2, p_{n_j+2} = n_j s(k-1)^2$.

Denote by p'_k the number of distinct pairs of vertices whose distance between w and $w'\in A^*\setminus\{u,v\}$ is exactly k. We deduce that $p'_{m_i+1} = m_i(s+t+1)(k-1)^2, p'_{n_j+1} = n_j(k-1)^2, p'_{n_j+2} = n_j(s+t)(k-1)^2$.

$$
\begin{aligned}
\sum_{i=2}^{d} p_i &= m_i(s+1)(k-1)^2 + m_i t(k-1)^2 + n_j(t+1)(k-1)^2 \\
&+ n_j s(k-1)^2 \\
&= (k-1)^2[(m_i+n_j)s + (m_i+n_j)t + m_i + n_j)] \\
&= (k-1)^2[m_i(s+t+1) + n_j + n_j(s+t)]
\end{aligned}
$$

By Lemma 2.2, for $w \in V(G)$, we have

$$\sum_{w' \in V(A)} d(w,w') \sum_{w' \in V(A)} \frac{1}{d(w,w')} = \sum_{w' \in V(A^*)} d(w,w') \sum_{w' \in V(A^*)} \frac{1}{d(w,w')}$$

Further, we have the following conclusion

$$C_1 = \sum_{w \in V(G)\setminus V(e), w' \in V(A)} d(w,w') \sum_{w \in V(G)\setminus V(e), w' \in V(A)} \frac{1}{d(w,w')}$$
$$= \sum_{w \in V(G)\setminus V(e), w' \in V(A^*)} d(w,w') \sum_{w \in V(G)\setminus V(e), w' \in V(A^*)} \frac{1}{d(w,w')} = C_2$$

Since

$$\begin{aligned} W \cdot H(G_u^*(s,t)) &= W \cdot H_{G_u^*(s,t)}(G) + W \cdot H_{G_u^*(s,t)}(A) + C_1 \\ W \cdot H(G_u^*(s+t)) &= W \cdot H_{G_u^*(s+t)}(G) + W \cdot H_{G_u^*(s+t)}(A^*) + C_2 \end{aligned}$$

Then we have $W \cdot H(G_u^*(s,t)) > W \cdot H(G_u^*(s+t))$.
Thus $W \cdot H(G_{u,v}(s,t)) > W \cdot H(G_u^*(s+t))$. ■

3 Hypertrees with small $W \cdot H$-value

In this section, we will obtain the unique k-uniform hypertrees with minimum, second minimum $W \cdot H$, respectively.

Theorem 3.1 For $m \geq 1$, let T be a k-uniform hypertree with m edges. Then $W \cdot H(T) \geq W \cdot H(S_{m,k})$ with equality if and only if $T \cong S_{m,k}$.

Proof. It is trivial if $m \leq 2$. Suppose that $m \geq 3$. Let T be a k-uniform hypertree with minimum $W \cdot H$-value.

Let d be the diameter of T. Obviously, $d \geq 2$. Suppose that $d \geq 3$. Let $P = (v_0, e_1, v_1, ..., v_{d-1}, e_d, v_d)$ be a diametral path of T and $e_{d-1} = \{v_{d-2}, w_1, ..., w_{k-2}, w_{k-1}\}$, where $w_{k-1} = v_{d-1}$. The edges associated with w_i are only pendant edges if exit, and there may be many pendant edges at w_i, forming a star with centre w_i for $1 \leq i \leq k-1$.

Let S_i be the star with centre w_i for $1 \leq i \leq k-1$. Denote by m_i the number of edges in S_i for $1 \leq i \leq k-1$. Let $G = T[V(T) \setminus V(S_1 \cup S_2) \cup$

$\{w_1\} \cup \{w_2\}]$, then $G_{w_1,w_2}(m_1, m_2) \cong T$. By moving edges in S_2 from w_2 to w_1, we get a k-uniform graph $G_{w_1}(m_1 + m_2)$. By Lemma 2.4, we have $G_{w_1,w_2}(m_1, m_2) > G_{w_1}(m_1 + m_2)$.

Repeat the process above, we get a k-uniform graph $G'_{w_1}\left(\sum_{i=1}^{k-1} m_i\right)$ with $G' = T[V(T) \setminus V(S_i) \cup \{w_i\}]$ for $1 \le i \le k-1$. By the prove of Lemma 2.4, we have $G_{w_1}(m_1 + m_2) > G'_{w_1}\left(\sum_{i=1}^{k-1} m_i\right)$.

Let $E = E(S_1 \cup S_2 ... \cup S_{k-1})$ for $1 \le i \le k-1$. By moving each edge in E from w_1 to v_{d-2}, we get a k-uniform hypertree T'. By Lemma 2.4, then $W \cdot H\left(G'_{w_1}\left(\sum_{i=1}^{k-1} m_i\right)\right) > W \cdot H(T')$,thus $W \cdot H(T) > W \cdot H(T')$, a contradiction.

Thus $d = 2$, implying that $T \cong S_{m,k}$. ■

When $k = 2$, the conclusion of Theorem 3.1 is exactly the result of [5].

Corollary 3.1 [5] For a 2-uniform hypertree T with m edges. Then $W \cdot H(T) \ge W \cdot H(S_{m,2})$ with equality if and only if $T \cong S_{m,2}$.

For $m \ge 3$ and $1 \le a \le m-1$, let $D_{m,k,a}$ be the k-uniform hypertree obtained from vertex-disjoint $S_{a,k}$ with center u and $S_{m-a,k}$ with center v by adding $k-2$ new vertices $w_1, ..., w_{k-2}$ and an edge $e = \{u, v, w_1, ..., w_{k-2}\}$.

Theorem 3.2 For $m \ge 3$, let T be a k-uniform hypertree with m edges. Suppose that $T \ncong S_{m,k}$. Then $W \cdot H(T) > W \cdot H(D_{m,k,2})$ with equality if and only if $T \cong D_{m,k,2}$.

Proof. It is trivial if $m \le 3$. Suppose that $m \ge 4$. Let T be a k-uniform hypergraph with m edges nonisomorphic to $S_{m,k}$ with minimum $W \cdot H$-value.

Let d be the diameter of T. Since $T \ncong S_{m,k}$,we have $d \ge 3$. By similar argument as in the proof of Theorem 3.1, we have $d = 3$. Then $W \cdot H(T) > W \cdot H(D_{m,k,a})$.

For the k-uniform hypergraph $D_{m,k,a}$, suppose that $a \ge 3$ and $a - 1 \ge m - a$. Denote by p_i the number of distinct pairs of vertices whose distance in $D_{m,k,a}$ is exactly i. We deduce that $p_1 = m(k-1), p_2 = \left[\binom{a}{2} + \binom{m-a+1}{2}\right](k-1)^2, p_3 = (a-1)^{m-a}(k-1)^2$.

Denote by p_i' the number of distinct pairs of vertices whose distance in $D_{m,k,2}$ is exactly i. We deduce that $p_1 = m(k-1), p_2 = \left[\binom{m-2}{2}+1\right](k-1)^2, p_3 = (m-2)(k-1)^2$.

$$\begin{aligned}
\sum_{i=2}^{3} p_i &= \left[\binom{a}{2}+\binom{m-a+1}{2}\right](k-1)^2 + (a-1)^{m-a}(k-1)^2 \\
&= (k-1)^2\left[\binom{a}{2}+\binom{m-a+1}{2}+(a-1)^{m-a}\right] \\
&> (k-1)^2\left[\binom{m-2}{2}+1+(m-2)\right]
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&f\left((k-1)m, (k-1)^2\left[\binom{a}{2}+\binom{m-a+1}{2}\right], (k-1)^2(a-1)^{m-a}, 0, ..., 0\right) \\
&\qquad > f\left((k-1)m, (k-1)^2\left[\binom{m-2}{2}+1\right], (k-1)^2(m-2), 0, ..., 0\right).
\end{aligned}$$

Note that

$$W\cdot H(D_{m,k,a}) \quad > \quad W\cdot H(D_{m,k,2})$$

Since $W\cdot H(T)>W\cdot H(D_{m,k,a})$ and $T \ncong S_{m,k}$, we have $W\cdot H(T)>W\cdot H(D_{m,k,2})$. ■

4 Hypertrees with large $W\cdot H$-value

In this section, we shall present the unique $k-uniform$ hypertrees with maximum $W\cdot H$-value.

For a k-uniform hypertree G with $V(G)=\{v_1,...,v_k,...,v_{(m-1)(k-1)+k}\}$, if $E(G)=\{e_1,...,e_m\}$, where $e_i=\{v_{(i-1)(k-1)+1},...,v_{(i-1)(k-1)+k}\}$ for $i=1,...,m$, then we call G a k-uniform loose path, denoted by $P_{m,k}$.

For positive integers $\triangle$,m ,with $1\leq\triangle\leq m$. Let $B_{m,k}^{\triangle}$ be the k-uniform hypertree obtained from vertex-disjoint hyperstar $S_{\triangle,k}$ with center u and loose path $P_{m-\triangle,k}$ with an end vertex v by identifying u and v. In particular, $B_{m,k}^{\triangle}\cong P_{m,k}$ if $\triangle=1,2$.

Lemma 4.1 Let T be a k-uniform hypertree with m edges and maximum degree $\triangle$, where $1 \leq \triangle \leq m$. Then $W \cdot H(T) \leq W \cdot H(B_{m,k}^{\triangle})$ with equality if and if $T \cong B_{m,k}^{\triangle}$.

Proof. It is trivial if $\triangle = 1$. Suppose that $\triangle \geq 2$. Let T be a k-uniform hypertree with m edges and maximum degree $\triangle$ having maximum $W \cdot H$-value.

Let u be a vertex of T with degree $\triangle$.

Case1. $\triangle \geq 3$.

Suppose that there are at least two vertices of degree at least 3 in T. Choose a vertex v of degree at least 3 such that $d(u, v)$ is as large as possible. Let $T_1', T_2', ..., T_{d(v)}'$ be the components by applying the v-shrinking on $E(v)$ and $T_i = T[V(T_i') \cup \{v\}]$ for $1 \leq i \leq d(v)$. So $T_1, T_2, ..., T_{d(v)}$ are the sub-hypergraphs of T. Suppose without loss of generality that $u \in V(T_1)$. Suppose that T_i is not a pendant path at v for $2 \leq i \leq d(v)$. Then there is at least one edge in T_i with at least three vertices of degree 2. We choose such an edge $e = \{w_1, w_2, ..., w_k\}$ by requiring that $d(v, w_1)$ is as large as possible, where $d(v, w_1) = d(v, w_j) - 1$ for $2 \leq j \leq k$. Then there are two pendant paths at different vertices of e, say P at w_p and Q at w_q, where $2 \leq p < q \leq k$. Let p and q with $p, q \geq 1$ be the length of P and Q,respectively. Then $T \cong G_{w_p,w_q}(p, q)$ with $G = T[V(T) \setminus V(P \cup Q) \cup \{w_p, w_q\}]$. Note that $d(w_p) = d(w_q) = 1$ in $G_{w_p,w_q}(p, q)$. Suppose without loss of generality that $p \geq q$. Obviously, $T' = G_{w_p,w_q}(p + q, 0)$ is a k-uniform hypertree with maximum degree $\triangle$. By the prove of Lemma 2.4, we have $W \cdot H(T) < W \cdot H(T')$, a contradiction.

Thus T_i is a pendant path at v for $2 \leq i \leq d(v)$. Let l_i be the lengths of the pendant path T_i at v, where $2 \leq i \leq d(v)$ and $l_i \geq 1$. Suppose without loss of generality that $l_2 \geq l_3$. Then $T \cong G_v'(l_2, l_3)$, where $G' = T[V(T) \setminus V(T_2 \cup T_3) \cup \{v\}]$. Note that $T'' = G_v'(l_2 + l_3, 0)$ is a k-uniform hypertree with maximum degree $\triangle$. By the prove of Lemma 2.3, $W \cdot H(T') < W \cdot H(T'')$, a contradiction. Thus u is the unique vertex of degree at least 3 in T.

Let $G_1', G_2', ..., G_{\triangle}'$ be the components by applying the u-shrinking on $E(u)$ and $G_i = T[V(G_i') \cup \{u\}]$ for $1 \leq i \leq \triangle$. So $G_1, G_2, ..., G_{\triangle}$ are the sub-hypergraphs of T.

By similar argument as above, G_i is a pendant path at u for $1 \leq i \leq \triangle$. Suppose that there are at least two pendant paths of length at least 2 at u, say G_i and G_j are such two paths with lengths s and t respectively, where $1 \leq i < j \leq \triangle$. Then $T \cong G_u(s,t)$ with $G = T[V(T) \setminus V(G_i \cup G_j)]$. Suppose without loss of generality that $s \geq t$. Then $T^* = G_u(s+1, t-1)$ is a k-uniform hypertree with maximum degree $\triangle$. By the prove of Lemma 2.3, we have $W \cdot H(T) < W \cdot H(T^*)$, a contradiction. Thus there is at most one pendant path of length at least 1, implying that $T \cong B^{\triangle}_{m,k}$.

Case2. $\triangle = 2$. Suppose that $T \ncong B^2_{m,k}$.Then there is an edge in T with at least three vertices of degree 2. We choose such an edge $e = \{w_1, ..., w_k\}$ in T by requiring that $d(u, w_1)$ is as large as possible, where $d(u, w_1) = d(u, w_j) - 1$ for $2 \leq j \leq k$. Then there are two pendent paths at different vertices of e, say P at w_p and Q at w_q, where $2 \leq q < p \leq k$. Let p and q with $p, q \geq 1$ be the lengths of P and Q, respectively.Then $T \cong G_{w_p,w_q}(p,q)$ with $G = T[V(T) \setminus V(P \cup Q) \cup \{w_p, w_q\}]$. Note that $d(w_p) = d(w_q) = 1$. Suppose without loss of generality that $p \geq q$. Obviously, $T' = G_{w_p,w_q}(p+q, 0)$ is a k-uniform hypertree with maximum degree 2. By the prove of Lemma 2.3 ,we have $W \cdot H(T) < W \cdot H(T')$, a contradiction. Thus there are at most two vertices of degree 2 in each edge, implying that $T \cong B^2_{m,k}$. Combining Cases 1 and 2, we complete the proof. ■

Theorem 4.2 For $m \geq 1$, let T be a k-uniform hypertree with m edges and n vertices. Then $W \cdot H(T) \leq W \cdot H(P_{m,k})$ with equality if and only if $T \cong P_{m,k}$.

Proof. It is trivial if $m = 1, 2$. Suppose that $m \geq 3$. Let T be a k-uniform hypertree with m edges with maximum $W \cdot H$-value. Let $\triangle$ be the maximum degree of T. Then by Theorem 4.1, $T \cong B^{\triangle}_{m,k}$. Suppose that $\triangle \geq 3$, then by Lemma 2.3, we have $W \cdot H\left(B^{\triangle}_{m,k}\right) < W \cdot H\left(B^{\triangle-1}_{m,k}\right)$, a contradiction. Then $\triangle = 2$, and thus $T \cong B^2_{m,k} \cong P_{m,k}$. ■

When $k = 2$, the conclusion of Theorem 4.2 is exactly the result of [5].

Corollary 4.1 [5] For a 2-uniform hypertree T with m edges. Then $W \cdot H(T) \leq W \cdot H(P_{m,2})$ with equality if and only if $T \cong P_{m,2}$.

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