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Some Notes on Sombor Index of Graphs

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Abstract

Several relations for the Sombor index are presented. A simpler proof of a result by Phanjoubam and Mawiong is given. An inequality connecting graph energy and Sombor index is corrected; in its proof Arizmendi's concept of vertex energy is used. An easy estimate of Sombor index in terms of eigenvalues of Sombor matrix is stated, followed by a conjecture.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are the vertex and edge sets of G, respectively. By the *order* and *size* of G, we mean the number of its vertices and edges. The edge of G, connecting the vertices u and v will be denoted by uv.

The degree d_u of a vertex $u \in V(G)$ is the number of first neighbors of u in G. The maximum and minimum degrees of G are denoted by Δ and δ, respectively.

The *Sombor index* [7,8,14,16,19] and the *first Zagreb index* [6,11,13,17]

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of a graph G are defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \qquad \text{and} \qquad M_1(G) = \sum_{uv \in E(G)} \big(d_u + d_v\big) \, .$$

The *energy* $\mathcal{E}(G)$ of the graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix [1–3, 15, 20, 21]. According to Arizmendi et al. [4, 5], the graph energy can be distributed over the vertices of G. If $\mathcal{E}_G(u)$ is the energy pertaining to the vertex u, then

$$\sum_{u \in V(G)} \mathcal{E}_G(u) = \mathcal{E}(G) \, .$$

The *Sombor matrix* $A_{SO(G)} = (s_{ij})$ of the graph G is defined by [9] $s_{ij} = \sqrt{d_{v_i}^2 + d_{v_j}^2}$ if v_i and v_j are adjacent and 0 otherwise. We denote its eigenvalues by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

In order to proceed, we need to recall a few auxiliary and earlier established results.

Theorem 1. [4] *For a graph G and a vertex $u \in V(G)$, it holds*

$$\mathcal{E}_G(u) \leq \sqrt{d_u} \, .$$

Theorem 2. [4] *Let G be a graph with vertex covering set C. Then*

$$\sum_{u \in C} \mathcal{E}_G(u) \geq \frac{1}{2} \, \mathcal{E}(G) \, .$$

Lemma 1. *Let $x, y \geq 2$ be two real numbers. Then*

$$\sqrt{x^2 + y^2} \geq \sqrt{x} + \sqrt{y} \, . \tag{1}$$

Equality holds if and only if $x = y = 2$.

Proof. Observe first that both $x(x-2)$ and $y(y-2)$ are non-negative, implying $x^2 - x \geq x$ and $y^2 - y \geq y$. Therefore, $x^2 + y^2 - x - y \geq x + y$ and $x + y \geq 2\sqrt{xy}$ (by the inequality between the arithmetic and geometric means). Consequently, $x^2 + y^2 \geq x + y + 2\sqrt{xy} = (\sqrt{x} + \sqrt{y})^2$ and inequality

(1) holds.

Since $x(x-2)$ and $y(y-2)$ are equal to zero only for $x = y = 2$, the condition for equality in (1) follows. ■

Note that equality in (1) holds also for (x, y) equal to (0,0), (1,0), or (0,1).

2 Main results

In Theorem 2.6 of [18], the authors proved that for an arbitrary graph of size m, $SO(G) \leq \sqrt{m\,\Delta\,M_1(G)}$. We start this article by giving a shorter proof for this result.

Theorem 3. [18] *Let G be a graph of size m. Then $SO(G) \leq \sqrt{m\,\Delta\,M_1(G)}$.*

Proof. By the Cauchy–Schwartz inequality, we have

$$\begin{aligned} SO(G) &= \sum_{xy \in E(G)} \sqrt{d_x^2 + d_y^2} \leq \sum_{xy \in E(G)} \sqrt{\Delta(d_x + d_y)} \\ &\leq \sqrt{\sum_{xy \in E(G)} \Delta} \sqrt{\sum_{xy \in E(G)} (d_x + d_y)} \end{aligned}$$

and the result follows. ■

In Theorem 5 of [12] it was claimed that if C is a vertex-covering set of a graph G, then

$$\mathcal{E}(G) \leq \frac{2}{\sqrt{\delta}}\,SO(G) - \frac{\sqrt{2}\,\Delta^2}{\sqrt{\delta}}\,|C|\,.$$

A mistake has occurred in the last line of the respective proof. In the next theorem, we state an improved version of this result.

Theorem 4. *Let G be a connected graph and C its vertex-covering set. Then*

$$\mathcal{E}(G) \leq \frac{2}{\delta\sqrt{\delta}}\,SO(G) + 4\,\frac{\sqrt{\delta}-1}{\delta}|E(G[C])| \tag{2}$$

where $|E(G[C])|$ is the number of edges of the subgraph of G indices by C, that is the number of edges of G whose both endpoints belong to C.

Proof. If G is a graph of size 0 or 1, then clearly the inequality holds. Therefore, we assume that the size of G is greater than one.

First note that $\frac{1}{2}\,\mathcal{E}(G) \leq \sum\limits_{i\in C} \sqrt{d_i}$, by Theorems 1 and 2. Also, since C is a vertex-covering set, every edge of G has at least one side in C. So we have:

$$\frac{1}{2}\mathcal{E}(G) \leq \sum_{i\in C} \sqrt{d_i} = \sum_{\substack{ij\in E(G)\\ i\in C, j\in V\setminus C}} \frac{\sqrt{d_i}}{d_i} + \sum_{\substack{ij\in E(G)\\ i,j\in C}} \left(\frac{\sqrt{d_i}}{d_i} + \frac{\sqrt{d_j}}{d_j}\right)$$

$$= \sum_{\substack{ij\in E(G)\\ i\in C, j\in V\setminus C}} \frac{\sqrt{d_i}}{d_i} + \frac{1}{\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i,j\in C}} \left(\frac{\sqrt{d_i}}{d_i} + \frac{\sqrt{d_j}}{d_j}\right)$$

$$+ \frac{\sqrt{\delta}-1}{\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i,j\in C}} \left(\frac{\sqrt{d_i}}{d_i} + \frac{\sqrt{d_j}}{d_j}\right)$$

$$\leq \frac{1}{\delta} \sum_{\substack{ij\in E(G)\\ i\in C, j\in V\setminus C}} \sqrt{\frac{d_i^2}{d_i} + \frac{d_j^2}{d_j}} + \frac{1}{\delta\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i,j\in C}} (\sqrt{d_i} + \sqrt{d_j})$$

$$+ \frac{\sqrt{\delta}-1}{\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i,j\in C}} \left(\frac{1}{\sqrt{d_i}} + \frac{1}{\sqrt{d_i}}\right).$$

Also, $d_u \geq 2$ for any vertex $u \in C$. Therefore, by Lemma 1,

$$\frac{1}{2}\mathcal{E}(G) \leq \frac{1}{\delta\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i\in C, j\in V\setminus C}} \sqrt{d_i^2 + d_j^2} + \frac{1}{\delta\sqrt{\delta}} \sum_{\substack{ij\in E(G)\\ i,j\in C}} \sqrt{d_i^2 + d_j^2}$$

$$+ 2\frac{\sqrt{\delta}-1}{\delta}|E(G[C])| = \frac{1}{\delta\sqrt{\delta}} SO(G) + 2\frac{\sqrt{\delta}-1}{\delta}|E(G[C])|$$

from which Eq. (2) straightforwardly follows. ∎

Denote by K_2 the connected graph of order 2. Also, denote by $H_{m,n}$ the graph consisting of m copies of K_2 and $n-2m$ isolated vertices. Note that $\sigma_1(K_2)=\sqrt{2}$ and therefore also $\sigma_1(H_{m,n})=\sqrt{2}$.

Theorem 5. *Let G be a graph of size $m>0$ and order n, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the eigenvalues of its Sombor matrix. Then*

$$m\,\sigma_1 \geq SO(G)\,. \tag{3}$$

If G is connected, then equality in (3) holds if and only if $G \cong K_2$, i.e., if G is the complete graph of order 2. In the general case, equality holds if and only if $G \cong H_{m,n}$, when $SO(G)=\sqrt{2}\,m$.

Note that in a trivial manner, relation (3) holds also for $m=0$, since for edgeless graphs, $SO(G)=0$.

Proof. Let $e=v_iv_j$ be an arbitrary edge of G and $X=[x_k]$ be an n-dimensional (0,1)-row-vector, where $x_k=1$ if and only if $k=i$ and $k=j$, and zero otherwise. By the Rayleigh–Ritz variational principle,

$$\frac{X^T\,A_{SO(G)}X}{X^T\,X} \leq \sigma_1$$

and therefore $\sqrt{d_i^2+d_j^2} \leq \sigma_1$. Summation over all edges yields (3).

Equality in (3) will hold if X is the eigenvector of $A_{SO(G)}$ corresponding to the eigenvalue σ_1.

Assume first that G is connected and that equality in (3) holds. Then by the Perron–Frobenius theorem, all components of X must be positive-valued. Therefore, X must be of dimension $n=2$, and therefore it must be $G \cong K_2$.

If G is not connected, then all components of G must be K_2 or isolated vertices, i.e., $G \cong H_{m,n}$. ■

Note that if G is a bipartite graph, then $\sigma_1=-\sigma_n$. If G is a connected non-bipartite graph, then $\sigma_1>-\sigma_n$, see [9, 10].

Conjecture 1. *(a) Using the same notation as in Theorem 5, we hypothesize that*

$$m\,|\sigma_n| = -m\,\sigma_n \geq SO(G)\,. \tag{4}$$

(b) If G is connected, then equality in (4) holds if and only if G is the complete graph. Then $\sigma_n = -\sqrt{2}(n-1)$ and $SO(G) = \sqrt{2}(n-1)\,\frac{n(n-1)}{2}$. In the general case, equality holds if and only if G consists of mutually isomorphic complete graphs and some (or no) isolated vertices.

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