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Optimization Problems for General Elliptic Sombor Index

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Abstract

Let G be a graph with vertex set V and edge set E. A topological index has the form

$$TI(G) = \sum_{uv \in E} f(d_u, d_v),$$

where $f = f(x, y)$ is a pertinently chosen function which must be symmetric and real-valued for all x, y pertaining to vertex degrees of the graph G. Particularly interesting are the Sombor index $\mathcal{SO}$ and the elliptic Sombor index $\mathcal{ESO}$, induced by the functions $f(x, y) = \sqrt{x^2 + y^2}$ and $f(x, y) = (x + y)\sqrt{x^2 + y^2}$, respectively. In this paper we solve some optimization problems for the general elliptic Sombor index $\mathcal{ESO}_\alpha$, induced by the function $f(x, y) = (x+y)^\alpha (x^2 + y^2)^{\alpha/2}$ ($\alpha \neq 0$), in particular on the set of graphs (respectively, trees) with n vertices.

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1 Introduction

In what follows, $G = (V, E)$ is a simple graph with vertex set V and edge set E. The degree of a vertex $u \in V$ is denoted by $d_u = d_u(G)$. An edge of the graph G, connecting a vertex of degree i and a vertex of degree j, is called an (i, j)-edge. The number of such edges will be denoted by $m_{i,j} = m_{i,j}(G)$.

A topological index has the form

$$TI = TI(G) = \sum_{uv \in E} f(d_u, d_v),$$

where $f = f(x, y)$ is a pertinently chosen function which must be symmetric and real-valued for all x, y pertaining to vertex degrees of the graph G. Particularly interesting is the recently created elliptic Sombor index $\mathcal{ESO}$ [6], and the Sombor index $\mathcal{SO}$ [5], induced by the functions $f(x, y) = (x + y)\sqrt{x^2 + y^2}$ and $f(x, y) = \sqrt{x^2 + y^2}$, respectively. For recent results on the Sombor index and the elliptic Sombor index we refer to [1–3, 5, 8, 10, 11, 14]. Both topological indices were conceived using geometric considerations and both showed good predictive potential [6,13].

Our main interest in this paper is to solve some optimization problems for the general elliptic Sombor index $\mathcal{ESO}_\alpha$ [11], induced by the function $f(x, y) = (x + y)^\alpha (x^2 + y^2)^{\alpha/2}$, $\alpha \in \mathbb{R} \setminus \{0\}$, in particular on the set of graphs (respectively, trees) with n vertices.

2 Extremal problems on the elliptic Sombor index and the general elliptic Sombor index

If a, b are arbitrary real numbers, the Gutman-Milovanović index is defined in [7] by

$$M_{a,b}(G) = \sum_{uv \in E(G)} (d_u d_v)^a (d_u + d_v)^b.$$

This index is a natural generalization of the first Zagreb, the general second Zagreb and the general sum-connectivity indices. This index is attracting growing interest, see e.g. [4, 9].

Notice that $M_{0,1}$ is the first Zagreb index M_1, $M_{1,0}$ is the second Zagreb index M_2, $M_{-1/2,0}$ is the Randić index R, $2M_{1/2,-1}$ is the geometric-arithmetic index GA, $\frac{1}{2}M_{-1/2,1}$ is the arithmetic-geometric index AG, $2M_{0,-1}$ is the harmonic index H, $M_{1,1}$ is the second Gourava index GO_2, $M_{\alpha,0}$ is the general second Zagreb index M_2^α, $M_{0,\beta}$ is the general sum-connectivity index χ_β, $4M_{1,-2}$ is the harmonic-arithmetic index HA, etc.

Optimization arguments using differential calculus allows to obtain the following result relating the general elliptic Sombor index and the Gutman-Milovanović index.

Theorem 1. *If $\alpha, \beta \in \mathbb{R}$ $(\alpha \neq 0)$ and G is a graph with maximum degree Δ and minimum degree δ, then*

$$k_{\alpha,\beta}\, M_{\beta,\alpha}(G) \leq \mathcal{ESO}_\alpha(G) \leq K_{\alpha,\beta}\, M_{\beta,\alpha}(G),$$

where $s = -2\beta/\alpha$,

$$k_{\alpha,\beta} := \begin{cases} (2\delta^{2s+2})^{\alpha/2}, & \text{for } \alpha > 0,\ s \geq -1, \\ (2\Delta^{2s+2})^{\alpha/2}, & \text{for } \alpha > 0,\ s < -1, \\ (2\Delta^{2s+2})^{\alpha/2}, & \text{for } \alpha < 0,\ s \geq 0, \\ \max\big\{(\Delta\delta)^s\big(\Delta^2+\delta^2\big),\, 2\Delta^{2s+2}\big\}^{\alpha/2}, & \text{for } \alpha < 0,\ -1 \leq s < 0, \\ \max\big\{(\Delta\delta)^s\big(\Delta^2+\delta^2\big),\, 2\delta^{2s+2}\big\}^{\alpha/2}, & \text{for } \alpha < 0,\ -2 < s < -1, \\ (2\delta^{2s+2})^{\alpha/2}, & \text{for } \alpha < 0,\ s \leq -2, \end{cases}$$

and

$$K_{\alpha,\beta} := \begin{cases} (2\Delta^{2s+2})^{\alpha/2}, & \textit{for } \alpha > 0,\ s \geq 0, \\ \max\left\{(\Delta\delta)^s\left(\Delta^2 + \delta^2\right),\ 2\Delta^{2s+2}\right\}^{\alpha/2}, & \textit{for } \alpha > 0,\ -1 \leq s < 0, \\ \max\left\{(\Delta\delta)^s\left(\Delta^2 + \delta^2\right),\ 2\delta^{2s+2}\right\}^{\alpha/2}, & \textit{for } \alpha > 0,\ -2 < s < -1, \\ (2\delta^{2s+2})^{\alpha/2}, & \textit{for } \alpha > 0,\ s \leq -2, \\ (2\delta^{2s+2})^{\alpha/2}, & \textit{for } \alpha < 0,\ s \geq -1, \\ (2\Delta^{2s+2})^{\alpha/2}, & \textit{for } \alpha < 0,\ s < -1. \end{cases}$$

The bounds are tight and they are attained on any regular graph.

Proof. For each $\delta \leq x, y \leq \Delta$, define the function $J : [\delta, \Delta] \times [\delta, \Delta] \to \mathbb{R}$ by

$$J(x, y) = (xy)^s\left(x^2 + y^2\right).$$

Thus,

$$\begin{aligned} \frac{\partial J}{\partial x}(x, y) &= sx^{s-1}y^s\left(x^2 + y^2\right) + x^s y^s 2x \\ &= x^{s-1}y^s\left(sx^2 + sy^2 + 2x^2\right) \\ &= x^{s-1}y^s\left((s + 2)x^2 + sy^2\right). \end{aligned}$$

Also,

$$\frac{\partial J}{\partial y}(x, y) = y^{s-1}x^s\left((s + 2)y^2 + sx^2\right).$$

If $s \geq 0$, then $\partial J/\partial x, \partial J/\partial y > 0$ and so,

$$2\delta^{2s+2} = J(\delta, \delta) \leq J(x, y) \leq J(\Delta, \Delta) = 2\Delta^{2s+2}$$

for any $x, y \in [\delta, \Delta]$.

If $s \leq -2$, then $\partial J/\partial x, \partial J/\partial y < 0$ and so,

$$2\Delta^{2s+2} = J(\Delta, \Delta) \leq J(x, y) \leq J(\delta, \delta) = 2\delta^{2s+2}$$

for any $x, y \in [\delta, \Delta]$.

Consider now $-1 \le s < 0$. We have $s+2 \ge -s$ and

$$\begin{aligned}\frac{\partial J}{\partial x}(x,y) &= x^{s-1}y^s\big((s+2)x^2 + sy^2\big)\\ &\ge -sx^{s-1}y^s\big(x^2 - y^2\big).\end{aligned}$$

By symmetry, we can assume that $x \ge y$. Then, $\partial J/\partial x \ge 0$ and so, $J(y,y) \le J(x,y) \le J(\Delta, y)$.

Let us define

$$a(y) = J(y,y) = 2y^{2s+2}.$$

Since $-1 \le s < 0$, the function $a(y)$ is increasing and

$$J(x,y) \ge J(y,y) = a(y) \ge a(\delta) = 2\delta^{2s+2}$$

for any $x, y \in [\delta, \Delta]$.

Define the function

$$b(y) = J(\Delta, y) = (\Delta y)^s\big(\Delta^2 + y^2\big)$$

on the interval $[\delta, \Delta]$. We have

$$b'(y) = \Delta^s y^{s-1}\big((s+2)y^2 + s\Delta^2\big).$$

Note that $b'(\Delta) = \Delta^{2s+1}2(s+1) > 0$ if $-1 < s < 0$. Since the function $(s+2)y^2 + s\Delta^2$ has at most a zero on the interval $[\delta, \Delta]$, and it is positive on $(\Delta - \varepsilon, \Delta)$ for some $\varepsilon > 0$, we conclude that b is either positive on (δ, Δ) or negative on (δ, γ) and positive on (γ, Δ) (for some $\gamma \in (\delta, \Delta)$). In both cases, $b(y) \le \max\{b(\delta), b(\Delta)\}$ and so,

$$\begin{aligned}J(x,y) \le J(\Delta, y) = b(y) &\le \max\big\{b(\delta),\, b(\Delta)\big\}\\ &= \max\big\{J(\Delta,\delta),\, J(\Delta,\Delta)\big\}\\ &= \max\big\{(\Delta\delta)^s\big(\Delta^2+\delta^2\big),\, 2\Delta^{2s+2}\big\}\end{aligned}$$

for any $x, y \in [\delta, \Delta]$. If $s = -1$, a similar argument gives the same inequality.

Finally, consider the case $-2 < s < -1$. We have $s+2 < -s$ and

$$\begin{aligned}\frac{\partial J}{\partial x}(x,y) &= x^{s-1}y^s\big((s+2)x^2+sy^2\big)\\ &< -sx^{s-1}y^s\big(x^2-y^2\big).\end{aligned}$$

By symmetry, we can assume that $x \le y$. Then, $\partial J/\partial x < 0$ and so, $J(y,y) \le J(x,y) \le J(\delta,y)$.

Let us consider

$$a(y) = J(y,y) = 2y^{2s+2}.$$

Since $-2 < s < -1$, the function $a(y)$ is decreasing and

$$J(x,y) \ge J(y,y) = a(y) \ge a(\Delta) = 2\Delta^{2s+2}$$

for any $x, y \in [\delta, \Delta]$.

Consider the function

$$c(y) = J(\delta,y) = (\delta y)^s\big(\delta^2+y^2\big)$$

on $[\delta,\Delta]$. We have

$$c'(y) = \delta^s y^{s-1}\big((s+2)y^2+s\delta^2\big).$$

Note that $c'(\delta) = \delta^{2s+1}2(s+1) < 0$. Since the function $(s+2)y^2+s\delta^2$ has at most a zero on the interval $[\delta,\Delta]$, and it is negative on $(\delta,\delta+\varepsilon)$ for some $\varepsilon > 0$, we conclude that b is either negative on (δ,Δ) or negative on (δ,γ) and positive on (γ,Δ) (for some $\gamma \in (\delta,\Delta)$). In both cases, $c(y) \le \max\{c(\delta), c(\Delta)\}$ and so,

$$\begin{aligned}J(x,y) \le J(\delta,y) = c(y) &\le \max\big\{c(\Delta),\, c(\delta)\big\}\\ &= \max\big\{J(\Delta,\delta),\, J(\delta,\delta)\big\}\\ &= \max\big\{(\Delta\delta)^s\big(\Delta^2+\delta^2\big),\, 2\delta^{2s+2}\big\}\end{aligned}$$

for any $x, y \in [\delta, \Delta]$.

Let us define

$$a_s := \begin{cases} 2\delta^{2s+2}, & \text{for } s \ge -1, \\ 2\Delta^{2s+2}, & \text{for } s < -1, \end{cases}$$

and

$$A_s := \begin{cases} 2\Delta^{2s+2}, & \text{for } s \ge 0, \\ \max\left\{(\Delta\delta)^s\left(\Delta^2+\delta^2\right),\, 2\Delta^{2s+2}\right\}, & \text{for } -1 \le s < 0, \\ \max\left\{(\Delta\delta)^s\left(\Delta^2+\delta^2\right),\, 2\delta^{2s+2}\right\}, & \text{for } -2 < s < -1, \\ 2\delta^{2s+2}, & \text{for } s \le -2. \end{cases}$$

Consequently,

$$a_s \le (xy)^s\left(x^2+y^2\right) = J(x,y) \le A_s$$

for every $s \in \mathbb{R}$ and $\delta \le x, y \le \Delta$. If $\alpha > 0$, then

$$a_s^{\alpha/2} \le (xy)^{s\alpha/2}\left(x^2+y^2\right)^{\alpha/2} \le A_s^{\alpha/2},$$
$$a_{-2\beta/\alpha}^{\alpha/2} \le (xy)^{-\beta}\left(x^2+y^2\right)^{\alpha/2} \le A_{-2\beta/\alpha}^{\alpha/2},$$

and if $\alpha < 0$, then we obtain the converse inequalities. Note that

$$k_{\alpha,\beta} = \begin{cases} a_{-2\beta/\alpha}^{\alpha/2}, & \text{for } \alpha > 0, \\ A_{-2\beta/\alpha}^{\alpha/2}, & \text{for } \alpha < 0, \end{cases} \qquad K_{\alpha,\beta} = \begin{cases} A_{-2\beta/\alpha}^{\alpha/2}, & \text{for } \alpha > 0, \\ a_{-2\beta/\alpha}^{\alpha/2}, & \text{for } \alpha < 0. \end{cases}$$

Hence,

$$k_{\alpha,\beta} \le (xy)^{-\beta}\left(x^2+y^2\right)^{\alpha/2} \le K_{\alpha,\beta},$$

for every $\alpha, \beta \in \mathbb{R}$ ($\alpha \ne 0$) and $\delta \le x, y \le \Delta$. Thus,

$$k_{\alpha,\beta}(d_u d_v)^{\beta}(d_u+d_v)^{\alpha} \le (d_u+d_v)^{\alpha}\left(d_u^2+d_v^2\right)^{\alpha/2} \le K_{\alpha,\beta}(d_u d_v)^{\beta}(d_u+d_v)^{\alpha},$$

for every $\alpha, \beta \in \mathbb{R}$ ($\alpha \ne 0$) and $uv \in E(G)$. Therefore,

$$k_{\alpha,\beta}\, M_{\beta,\alpha}(G) \le \mathcal{ESO}_\alpha(G) \le K_{\alpha,\beta}\, M_{\beta,\alpha}(G).$$

Finally, we are going to show that the bounds are tight and they are attained on any regular graph. If G is a δ-regular graph with m edges, then $\Delta = \delta$, $k_{\alpha,\beta} = K_{\alpha,\beta} = (2\delta^{2s+2})^{\alpha/2} = 2^{\alpha/2}\delta^{s\alpha+\alpha} = 2^{\alpha/2}\delta^{-2\beta+\alpha}$ and

$$k_{\alpha,\beta}\, M_{\beta,\alpha}(G) = 2^{\alpha/2}\delta^{-2\beta+\alpha}\delta^{2\beta}2^{\alpha}\delta^{\alpha}m = 2^{\alpha}\delta^{\alpha}2^{\alpha/2}\delta^{\alpha}m = \mathcal{ESO}_{\alpha}(G).$$

■

Since $M_{0,1}$ is the first Zagreb index M_1, $\frac{1}{2}M_{-1/2,1}$ is the arithmetic-geometric index AG, $M_{1,1}$ is the second Gourava index GO_2, $M_{0,\alpha}$ is the general sum-connectivity index χ_α, Theorem 1 has the following consequence.

Corollary 1. *If $\alpha \in \mathbb{R} \setminus \{0\}$ and G is a graph with maximum degree Δ and minimum degree δ, then*

$$\begin{gathered} k_\alpha\, \chi_\alpha(G) \le \mathcal{ESO}_\alpha(G) \le K_\alpha\, \chi_\alpha(G), \\ \sqrt{2}\,\delta\, M_1(G) \le \mathcal{ESO}(G) \le \sqrt{2}\,\Delta\, M_1(G), \\ \frac{\sqrt{2}}{\Delta}\, GO_2(G) \le \mathcal{ESO}(G) \le \frac{\sqrt{2}}{\delta}\, GO_2(G), \\ 2\sqrt{2}\,\delta^2 AG(G) \le \mathcal{ESO}(G) \le 2\sqrt{2}\,\Delta^2 AG(G), \end{gathered}$$

where

$$k_\alpha := \begin{cases} 2^{\alpha/2}\delta^\alpha, & \text{for } \alpha > 0, \\ 2^{\alpha/2}\Delta^\alpha, & \text{for } \alpha < 0, \end{cases} \qquad K_\alpha := \begin{cases} 2^{\alpha/2}\Delta^\alpha, & \text{for } \alpha > 0, \\ 2^{\alpha/2}\delta^\alpha, & \text{for } \alpha < 0. \end{cases}$$

The bounds are tight and they are attained on any regular graph.

Consider any topological index defined as

$$TI(G) = \sum_{uv \in E(G)} f(d_u, d_v), \tag{1}$$

where $f(x, y)$ is any non-negative symmetric function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, \infty)$.

We say that the index TI defined by (1) belongs to $\mathcal{F}_1$ if f is a positive function that is strictly increasing in each variable.

Considering the index TI in these classes allows to study many indices in a unified way.

It is clear that $TI \in \mathcal{F}_1$ for:

• $f(x,y) = (x^a + y^a)^{-1}$ with $a < 0$ (variable inverse sum deg index),

• $f(x,y) = \log^a x + \log^a y$ with $a > 0$ (variable sum lodeg index, for graphs without isolated edges),

• $f(x,y) = a^x + a^y$ with $a > 1$ (variable sum exdeg index),

• $f(x,y) = x^{a-1} + y^{a-1}$ with $a > 1$ (variable first Zagreb index),

• $f(x,y) = (xy)^a$ with $a > 0$ (variable second Zagreb index),

• $f(x,y) = (x+y)^a$ with $a > 0$ (variable sum connectivity index),

• $f(x,y) = x+y+xy$ and $f(x,y) = x^2y+xy^2$ (first and second Gourava indices, respectively),

• $f(x,y) = (x+y+xy)^2$ and $f(x,y) = (x^2y+xy^2)^2$ (first and second hyper-Gourava indices, respectively),

• $f(x,y) = (xy)^\alpha (x+y)^\beta$ with $\alpha, \beta > 0$ (Gutman-Milovanović index),

• $f(x,y) = \sqrt{x^2+y^2}$ (Sombor index),

• $f(x,y) = (x+y)\sqrt{x^2+y^2}$ (elliptic Sombor index),

• $f(x,y) = (x+y)^\alpha (x^2+y^2)^{\alpha/2}$ with $\alpha > 0$ (general elliptic Sombor index).

Given an integer $n \geq 2$, let $\mathcal{G}(n)$ (respectively, $\mathcal{G}_c(n)$) be the set of graphs (respectively, connected graphs) with n vertices. In [12] appear the two following results.

Proposition 2. *Consider $TI \in \mathcal{F}_1$ and an integer $n \geq 2$.*

(1) *The only graph that maximizes the TI index in $\mathcal{G}_c(n)$ or $\mathcal{G}(n)$ is the complete graph K_n.*

(2) *If a graph minimizes the TI index in $\mathcal{G}_c(n)$, then it is a tree.*

(3) *If n is even, then the only graph that minimizes the TI index in $\mathcal{G}(n)$ is the union of $n/2$ paths P_2. If n is odd, then the only graph that minimizes the TI index in $\mathcal{G}(n)$ is the union of $(n-3)/2$ paths P_2 with a path P_3.*

Corollary 2. *Let G be a graph with n vertices and $TI \in \mathcal{F}_1$.*

(1) *Then,*

$$TI(G) \leq \frac{1}{2} n(n-1) f(n-1, n-1),$$

and the equality in the bound is attained if and only if G is the complete graph K_n.

(2) *If n is even, then*

$$TI(G) \geq \frac{1}{2}\, nf(1,1),$$

and the equality in the bound is attained if and only if G is the union of $n/2$ path graphs P_2.

(3) *If n is odd, then*

$$TI(G) \geq \frac{1}{2}\,(n-3)f(1,1) + 2f(1,2)\,,$$

and the equality in the bound is attained if and only if G is the union of $(n-3)/2$ path graphs P_2 and a path graph P_3.

Since $TI = \mathcal{ESO}_\alpha$ if $f(x,y) = (x+y)^\alpha (x^2+y^2)^{\alpha/2}$, we have $\mathcal{ESO}_\alpha \in \mathcal{F}_1$ for every $\alpha > 0$. Hence, Proposition 2 and Corollary 2 have the following consequences.

Proposition 3. *Consider $\alpha > 0$ and an integer $n \geq 2$.*

(1) *The only graph that maximizes $\mathcal{ESO}_\alpha$ in $\mathcal{G}_c(n)$ or $\mathcal{G}(n)$ is the complete graph K_n.*

(2) *If a graph minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{G}_c(n)$, then it is a tree.*

(3) *If n is even, then the only graph that minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{G}(n)$ is the union of $n/2$ paths P_2. If n is odd, then the only graph that minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{G}(n)$ is the union of $(n-3)/2$ paths P_2 with a path P_3.*

Proposition 4. *Let G be a graph with n vertices and $\alpha > 0$.*

(1) *Then,*

$$\mathcal{ESO}_\alpha(G) \leq \sqrt{2}\, n(n-1)^3,$$

and the equality in the bound is attained if and only if G is the complete graph K_n.

(2) *If n is even, then*

$$\mathcal{ESO}_\alpha(G) \geq \sqrt{2}\, n,$$

and the equality in the bound is attained if and only if G is the union of $n/2$ path graphs P_2.

(3) *If n is odd, then*

$$\mathcal{ESO}_\alpha(G) \geq \sqrt{2}\,(n-3) + 6\sqrt{5}\,,$$

and the equality in the bound is attained if and only if G is the union of $(n-3)/2$ path graphs P_2 and a path graph P_3.

We are going to show two graph transformations that allow to obtain graphs with smaller $\mathcal{ESO}_\alpha$.

We need some previous results.

Lemma 1. *Let $0 < a < 1 < A$. If $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(\alpha) = 2 - a^\alpha - A^\alpha$ and $h(1) > 0$, then $h(\alpha) > 0$ for every $\alpha \in (0, 1]$.*

Proof. We have $h'(\alpha) = -a^\alpha \log a - A^\alpha \log A = 0$ if and only if

$$\left(\frac{A}{a}\right)^\alpha = \frac{-\log a}{\log A} \qquad \Longleftrightarrow \qquad \alpha = \frac{\log \frac{-\log a}{\log A}}{\log \frac{A}{a}} =: \alpha_1.$$

Hence, $h' > 0$ on $(-\infty, \alpha_1)$ and $h' < 0$ on (α_1, ∞).

Since $\lim_{t\to-\infty} h(t) = -\infty$, $h(0) = 0$, $h(1) > 0$ and $\lim_{t\to\infty} h(t) = -\infty$, we have $h(\alpha) > 0$ for every $\alpha \in (0, 1]$. ∎

Proposition 5. *Let G be a graph of n vertices and $0 < \alpha \leq 1$. Assume that the vertices $u, v, w, x \in V(G)$ satisfy the following properties: uv, wx are different pendent edges (although it is possible to have $w = v$) with $d_v \geq 3$ and $2 \leq d_w \leq d_v$. Let G' be the graph with n vertices obtained form G by deleting the edge uv and adding a pendent edge to x. Then, $\mathcal{ESO}_\alpha(G') < \mathcal{ESO}_\alpha(G)$.*

Proof. Assume first that $w \neq v$.

Since $d_u = 1$, a computation gives

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&= \sum_{z\in N(v)\setminus\{u\}} (d_v + d_z)^\alpha (d_v^2 + d_z^2)^{\alpha/2} + (d_v + 1)^\alpha (d_v^2 + 1)^{\alpha/2} \\
&\quad + (d_w + 1)^\alpha (d_w^2 + 1)^{\alpha/2} - \sum_{z\in N(v)\setminus\{u\}} (d_v - 1 + d_z)^\alpha ((d_v - 1)^2 + d_z^2)^{\alpha/2} \\
&\quad - 3^\alpha 5^{\alpha/2} - (d_w + 2)^\alpha (d_w^2 + 4)^{\alpha/2} \\
&> (d_v + 1)^\alpha (d_v^2 + 1)^{\alpha/2} + (d_w + 1)^\alpha (d_w^2 + 1)^{\alpha/2} \\
&\quad - 3^\alpha 5^{\alpha/2} - (d_w + 2)^\alpha (d_w^2 + 4)^{\alpha/2}.
\end{aligned}$$

If $d_w = 2$, since $d_v \geq 3$, we obtain

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&> (d_v + 1)^\alpha (d_v^2 + 1)^{\alpha/2} + (d_w + 1)^\alpha (d_w^2 + 1)^{\alpha/2} \\
&\quad - 3^\alpha 5^{\alpha/2} - (d_w + 2)^\alpha (d_w^2 + 4)^{\alpha/2} \\
&= (d_v + 1)^\alpha (d_v^2 + 1)^{\alpha/2} + 3^\alpha 5^{\alpha/2} - 3^\alpha 5^{\alpha/2} - 8^\alpha 2^{\alpha/2} \\
&\geq 4^\alpha 10^{\alpha/2} - 8^\alpha 2^{\alpha/2} = 4^\alpha 2^{\alpha/2} \left(5^{\alpha/2} - 2^\alpha\right) > 0.
\end{aligned}$$

Now, we are going to prove that the function

$$g(t) = 2(t + 1)^\alpha (t^2 + 1)^{\alpha/2} - (t + 2)^\alpha (t^2 + 4)^{\alpha/2}$$

is increasing on $[0, \infty)$ for any $0 < \alpha \leq 1$.

Let us check first that

$$2(t + 1)^{\alpha-1} (t^2 + 1)^{\alpha/2} > (t + 2)^{\alpha-1} (t^2 + 4)^{\alpha/2}$$

for $t \geq 0$ and $0 < \alpha \leq 1$. We have

$$\begin{aligned}&4t^2+4 > t^2+4 \quad \Rightarrow \quad 2 > \sqrt{\frac{t^2+4}{t^2+1}} \quad \Rightarrow\\ &\frac{\log\left(2\,\frac{t+2}{t+1}\right)}{\log\left(\frac{t+2}{t+1}\sqrt{\frac{t^2+4}{t^2+1}}\right)} > 1 \geq \alpha \quad \Rightarrow \quad 2\,\frac{t+2}{t+1} > \left(\frac{t+2}{t+1}\sqrt{\frac{t^2+4}{t^2+1}}\right)^{\alpha} \quad \Rightarrow\\ &2(t+1)^{\alpha-1}(t^2+1)^{\alpha/2} > (t+2)^{\alpha-1}(t^2+4)^{\alpha/2}\end{aligned}$$

for $t \geq 0$ and $0 < \alpha \leq 1$.

Let us check now that

$$2t(t+1)^{\alpha}(t^2+1)^{\alpha/2-1} \geq t(t+2)^{\alpha}(t^2+4)^{\alpha/2-1}$$

for $t \geq 0$ and $0 < \alpha \leq 1$. We have

$$\begin{aligned}&2t+2 > t+2 \quad \Rightarrow \quad 2 > \frac{t+2}{t+1} \quad \Rightarrow\\ &\frac{\log\left(2\,\frac{t^2+4}{t^2+1}\right)}{\log\left(\frac{t+2}{t+1}\sqrt{\frac{t^2+4}{t^2+1}}\right)} > 1 \geq \alpha \quad \Rightarrow \quad 2\,\frac{t^2+4}{t^2+1} > \left(\frac{t+2}{t+1}\sqrt{\frac{t^2+4}{t^2+1}}\right)^{\alpha} \quad \Rightarrow\\ &2(t+1)^{\alpha}(t^2+1)^{\alpha/2-1} > (t+2)^{\alpha}(t^2+4)^{\alpha/2-1} \quad \Rightarrow\\ &2t(t+1)^{\alpha}(t^2+1)^{\alpha/2-1} \geq t(t+2)^{\alpha}(t^2+4)^{\alpha/2-1}\end{aligned}$$

for $t \geq 0$ and $0 < \alpha \leq 1$.

Since

$$\begin{aligned}2(t+1)^{\alpha-1}(t^2+1)^{\alpha/2} &> (t+2)^{\alpha-1}(t^2+4)^{\alpha/2},\\ 2t(t+1)^{\alpha}(t^2+1)^{\alpha/2-1} &\geq t(t+2)^{\alpha}(t^2+4)^{\alpha/2-1},\end{aligned}$$

for $t \geq 0$ and $0 < \alpha \leq 1$, we have

$$\begin{aligned}g'(t) = {}& 2\alpha(t+1)^{\alpha-1}(t^2+1)^{\alpha/2} + 2\alpha t(t+1)^{\alpha}(t^2+1)^{\alpha/2-1}\\ &- \alpha(t+2)^{\alpha-1}(t^2+4)^{\alpha/2} - \alpha t(t+2)^{\alpha}(t^2+4)^{\alpha/2-1} > 0\end{aligned}$$

for every $t \geq 0$ and $0 < \alpha \leq 1$ and so, g is increasing on $[0, \infty)$.

If $d_w \geq 3$, we have $g(d_w) \geq g(3)$ and

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&> (d_v+1)^\alpha(d_v^2+1)^{\alpha/2} + (d_w+1)^\alpha(d_w^2+1)^{\alpha/2} \\
&\quad - 3^\alpha 5^{\alpha/2} - (d_w+2)^\alpha(d_w^2+4)^{\alpha/2} \\
&\geq 2(d_w+1)^\alpha(d_w^2+1)^{\alpha/2} - 3^\alpha 5^{\alpha/2} - (d_w+2)^\alpha(d_w^2+4)^{\alpha/2} \\
&\geq 2\cdot 4^\alpha 10^{\alpha/2} - 3^\alpha 5^{\alpha/2} - 5^\alpha 13^{\alpha/2} \\
&= 4^\alpha 10^{\alpha/2}\left(2 - \left(\frac{3}{4\sqrt{2}}\right)^\alpha - \left(\frac{5\sqrt{13}}{4\sqrt{10}}\right)^\alpha\right) \\
&=: 4^\alpha 10^{\alpha/2} h(\alpha).
\end{aligned}$$

Since

$$h(1) = 2 - \frac{3}{4\sqrt{2}} - \frac{5\sqrt{13}}{4\sqrt{10}} > 0,$$

Lemma 1 gives that $h(\alpha) > 0$ for every $\alpha \in (0,1]$, and we conclude that $\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') > 0$ for any $\alpha \in (0,1]$.

Assume now that $w = v$. A computation gives

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&= \sum_{z\in N(v)\setminus\{u,x\}} (d_v+d_z)^\alpha(d_v^2+d_z^2)^{\alpha/2} + 2(d_v+1)^\alpha(d_v^2+1)^{\alpha/2} \\
&\quad - \sum_{z\in N(v)\setminus\{u,x\}} (d_v-1+d_z)^\alpha((d_v-1)^2+d_z^2)^{\alpha/2} \\
&\quad - 3^\alpha 5^{\alpha/2} - (d_v+1)^\alpha((d_v-1)^2+4)^{\alpha/2} \\
&> 2(d_v+1)^\alpha(d_v^2+1)^{\alpha/2} - 3^\alpha 5^{\alpha/2} - (d_v+1)^\alpha((d_v-1)^2+4)^{\alpha/2} \\
&> 2(d_v+1)^\alpha(d_v^2+1)^{\alpha/2} - 3^\alpha 5^{\alpha/2} - (d_v+2)^\alpha(d_v^2+4)^{\alpha/2}.
\end{aligned}$$

Since $d_v \geq 3$, the previous argument implies

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&> 2(d_v+1)^\alpha(d_v^2+1)^{\alpha/2} - 3^\alpha 5^{\alpha/2} - (d_v+2)^\alpha(d_v^2+4)^{\alpha/2} \\
&\geq 2\cdot 4^\alpha 10^{\alpha/2} - 3^\alpha 5^{\alpha/2} - 5^\alpha 13^{\alpha/2} > 0
\end{aligned}$$

for any $\alpha \in (0,1]$. ■

Lemma 2. *Let* $0 < a < 1 < A$ *with* $aA < 1$ *and let* $H : \mathbb{R} \to \mathbb{R}$ *be the function defined by* $H(\alpha) = a^\alpha + A^\alpha - 2$. *Then, there exists* $\alpha_0 > 0$ *such that* $H < 0$ *on* $(0, \alpha_0)$ *and* $H \geq 0$ *otherwise.*

Proof. We have $H'(\alpha) = a^\alpha \log a + A^\alpha \log A = 0$ if and only if

$$\left(\frac{A}{a}\right)^\alpha = \frac{-\log a}{\log A} \qquad \Longleftrightarrow \qquad \alpha = \frac{\log \frac{-\log a}{\log A}}{\log \frac{A}{a}} =: \alpha_1.$$

Since $aA < 1$, we conclude that $\log A < -\log a$ and so, $\alpha_1 > 0$. Hence, $H' < 0$ on $(-\infty, \alpha_1)$ and $H' > 0$ on (α_1, ∞).

Since $\lim_{t \to -\infty} H(t) = \infty$, $H(0) = 0$, $H'(\alpha_0) = 0$ and $\lim_{t \to \infty} H(t) = \infty$, there exists a unique positive zero α_0 of H and there is no negative zero of H.

Consequently, $H < 0$ on $(0, \alpha_0)$ and $H \geq 0$ otherwise. ∎

Definition 1. Let α_0 be the unique positive solution of the equation

$$\left(\frac{5\sqrt{13}}{8\sqrt{2}}\right)^\alpha + \left(\frac{3\sqrt{5}}{8\sqrt{2}}\right)^\alpha = 2.$$

Lemma 3. *This constant* α_0 *belongs to the interval* $(0, 1)$ *and the function*

$$H(\alpha) = \left(\frac{5\sqrt{13}}{8\sqrt{2}}\right)^\alpha + \left(\frac{3\sqrt{5}}{8\sqrt{2}}\right)^\alpha - 2$$

satisfies $H < 0$ *on* $(0, \alpha_0)$ *and* $H \geq 0$ *on* $[\alpha_0, \infty)$.

Proof. This function H is the one in Lemma 2, with

$$a = \frac{3\sqrt{5}}{8\sqrt{2}}, \qquad A = \frac{5\sqrt{13}}{8\sqrt{2}}.$$

Since

$$aA = \frac{3\sqrt{5}}{8\sqrt{2}}\,\frac{5\sqrt{13}}{8\sqrt{2}} = \frac{15\sqrt{65}}{128} < 1,$$

Lemma 2 gives that $H(\alpha) < 0$ for any $0 < \alpha < \alpha_0$ and $H(\alpha) \geq 0$ for every

$\alpha \geq \alpha_0$. Since

$$H(1) = \frac{5\sqrt{13}}{8\sqrt{2}} + \frac{3\sqrt{5}}{8\sqrt{2}} - 2 > 0,$$

we have $0 < \alpha_0 < 1$. ■

An induced path with vertices $u_1, u_2, \ldots, u_n$ $(n \geq 3)$ of a graph G is called a *pendent path* at u_1 of G, if $d_{u_2} = \cdots = d_{u_{n-1}} = 2$ and $d_{u_n} = 1$ (there is no requirement on the degree of u_1).

Proposition 6. *Let $\alpha \geq \alpha_0$ and let G be a graph of n vertices with two pendent paths P and Q, such that P starts at a vertex v with $d_v \geq 3$. Let G' be the graph with n vertices obtained form G by deleting P and pasting it at the pendent vertex in Q. Then, $\mathcal{ESO}_\alpha(G') < \mathcal{ESO}_\alpha(G)$.*

Proof. Let u be the vertex in P which is incident to v. Since $d_v \geq 3$ and $d_u = 2$, a computation gives

$$\begin{aligned}
&\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') \\
&= \sum_{z \in N(v)\setminus\{u\}} (d_v + d_z)^\alpha (d_v^2 + d_z^2)^{\alpha/2} + (d_v + 2)^\alpha (d_v^2 + 4)^{\alpha/2} + 3^\alpha 5^{\alpha/2} \\
&\quad - \sum_{z \in N(v)\setminus\{u\}} (d_v - 1 + d_z)^\alpha ((d_v - 1)^2 + d_z^2)^{\alpha/2} - 8^\alpha 2^{\alpha/2} - 8^\alpha 2^{\alpha/2} \\
&> (d_v + 2)^\alpha (d_v^2 + 4)^{\alpha/2} + 3^\alpha 5^{\alpha/2} - 2 \cdot 8^\alpha 2^{\alpha/2} \\
&\geq 5^\alpha 13^{\alpha/2} + 3^\alpha 5^{\alpha/2} - 2 \cdot 8^\alpha 2^{\alpha/2} \\
&= 8^\alpha 2^{\alpha/2} \left(\left(\frac{5\sqrt{13}}{8\sqrt{2}} \right)^\alpha + \left(\frac{3\sqrt{5}}{8\sqrt{2}} \right)^\alpha - 2 \right) \\
&= 8^\alpha 2^{\alpha/2} H(\alpha),
\end{aligned}$$

where H is the function in Lemma 3. Hence, $H(\alpha) \geq 0$ for every $\alpha \geq \alpha_0$.

Therefore, $\mathcal{ESO}_\alpha(G) - \mathcal{ESO}_\alpha(G') > 0$ for any $\alpha \geq \alpha_0$. ■

Motivated by Proposition 3, we are going to optimize $\mathcal{ESO}_\alpha$ on the set $\mathcal{T}(n)$ of trees with n vertices. The corresponding results for $\mathcal{ESO}$ appear in [6].

Theorem 7. *Consider an integer $n \geq 2$ and $\alpha \geq \alpha_0$. The only graph that minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{T}(n)$ is the path graph P_n, and every tree $T \in \mathcal{T}(n)$ satisfies*

$$\mathcal{ESO}_\alpha(T) \geq (n-3)\left(8\sqrt{2}\right)^\alpha + 2\left(3\sqrt{5}\right)^\alpha.$$

Proof. If $f(x,y) = (x+y)^\alpha (x^2+y^2)^{\alpha/2}$, the general elliptic Sombor index of any graph G is

$$\mathcal{ESO}_\alpha(G) = \sum_{uv \in E(G)} f(d_u, d_v).$$

Hence,

$$\mathcal{ESO}_\alpha(P_n) = (n-3)f(2,2) + 2f(1,2) = (n-3)\left(8\sqrt{2}\right)^\alpha + 2\left(3\sqrt{5}\right)^\alpha.$$

Note that if $\{d_u, d_v\} \neq \{1,2\}$, then $f(d_u,d_v) > f(2,2)$: it suffices to check that $f(1,3) > f(2,2)$, and this holds since $\sqrt{10} > 2\sqrt{2}$ implies

$$f(1,3) = \left(4\sqrt{10}\right)^\alpha > \left(8\sqrt{2}\right)^\alpha = f(2,2).$$

Consider any tree $T \in \mathcal{T}(n)$ that is not the path graph P_n, let $E_{1,2}$ be the set of edges in $E(T)$ with incident vertices of degrees 1 and 2, and let $m_{1,2}$ be the cardinality of the set $E_{1,2}$.

If $m_{1,2} \leq 2$, then

$$\begin{aligned}
\mathcal{ESO}_\alpha(T) &= \sum_{uv \in E(G) \setminus E_{1,2}} f(d_u, d_v) + \sum_{uv \in E_{1,2}} f(1,2) \\
&> (n-1-m_{1,2}) f(2,2) + m_{1,2} f(1,2) \\
&\geq (n-3)f(2,2) + 2f(1,2) = \mathcal{ESO}_\alpha(P_n).
\end{aligned}$$

Assume now that $m_{1,2} \geq 3$. For each $e \in E_{1,2}$, let us denote by e^* the closest edge to e with incident vertices of degrees 2 and $d(e) \geq 3$. Denote by $E^*_{1,2}$ the set $\{e^* \in E(T) : e \in E_{1,2}\}$. One can check that the map $M : E_{1,2} \to E^*_{1,2}$ defined by $M(e) = e^*$, is one to one.

Lemma 3 implies that

$$\left(\frac{5\sqrt{13}}{8\sqrt{2}}\right)^{\alpha}+\left(\frac{3\sqrt{5}}{8\sqrt{2}}\right)^{\alpha}\geq 2\,,$$
$$\left(3\sqrt{5}\right)^{\alpha}+\left(5\sqrt{13}\right)^{\alpha}\geq 2\left(8\sqrt{2}\right)^{\alpha},$$
$$f(1,2)+f(2,3)\geq 2f(2,2)\,.$$

Hence,

$$\begin{aligned}
\mathcal{ESO}_\alpha(T) &= \sum_{uv\in E(G)\setminus(E_{1,2}\cup E^*_{1,2})} f(d_u,d_v)+\sum_{uv\in E_{1,2}} f(1,2)+\sum_{uv\in E^*_{1,2}} f(d_u,d_v)\\
&\geq (n-1-2m_{1,2})f(2,2)+m_{1,2}f(1,2)+m_{1,2}f(2,3)\\
&= (n-1-2m_{1,2})f(2,2)+2f(1,2)+(m_{1,2}-2)f(1,2)\\
&\quad +2f(2,3)+(m_{1,2}-2)f(2,3)\\
&> (n+1-2m_{1,2})f(2,2)+2f(1,2)+(m_{1,2}-2)(f(1,2)+f(2,3))\\
&\geq (n+1-2m_{1,2})f(2,2)+2f(1,2)+2(m_{1,2}-2)f(2,2)\\
&= (n-3)f(2,2)+2f(1,2)=\mathcal{ESO}_\alpha(P_n).
\end{aligned}$$

■

Proposition 3 and Theorem 7 have the following consequence.

Theorem 8. *Consider an integer $n\geq 2$ and $\alpha\geq\alpha_0$. The only graph that minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{G}_c(n)$ is the path graph P_n, and every graph $G\in\mathcal{G}_c(n)$ satisfies*

$$\mathcal{ESO}_\alpha(G)\geq (n-3)\left(8\sqrt{2}\right)^{\alpha}+2\left(3\sqrt{5}\right)^{\alpha}.$$

Note that since $\alpha_0<1$ by Lemma 3, Theorem 7 holds for $\mathcal{ESO}$.

Theorem 9. *Consider an integer $n\geq 2$ and $\alpha\in\mathbb{R}$.*

(1) *If $\alpha>0$, then the only graph that maximizes $\mathcal{ESO}_\alpha$ in $\mathcal{T}(n)$ is the star graph S_n, and every tree $T\in\mathcal{T}(n)$ satisfies*

$$\mathcal{ESO}_\alpha(T)\leq (n-1)\,n^{\alpha}\left(n^2-2n+2\right)^{\alpha/2}.$$

(2) *If $\alpha<0$, then the only graph that minimizes $\mathcal{ESO}_\alpha$ in $\mathcal{T}(n)$ is the*

star graph S_n, and every tree $T \in \mathcal{T}(n)$ satisfies

$$\mathcal{ESO}_\alpha(T) \geq (n-1)\, n^\alpha \left(n^2 - 2n + 2\right)^{\alpha/2}.$$

Proof. Assume that $\alpha > 0$.

Consider any tree $T \in \mathcal{T}(n)$ and any $uv \in E(T)$. Since T is a tree, there is no vertex w with $wu, wv \in E(T)$ (T is triangle free). Hence, $d_u + d_v \leq n$ and so, $d_u^2 + d_v^2 \leq (n-1)^2 + 1$; also, $d_u^2 + d_v^2 = (n-1)^2 + 1$ if and only if $\{d_u, d_v\} = \{n-1, 1\}$. Hence,

$$(d_u + d_v)\sqrt{d_u^2 + d_v^2} \leq n\sqrt{(n-1)^2 + 1}\,,$$
$$(d_u + d_v)^\alpha (d_u^2 + d_v^2)^{\alpha/2} \leq n^\alpha ((n-1)^2 + 1)^{\alpha/2},$$

and the equality is attained if and only if $(d_u, d_v) = (n-1, 1)$ or viceversa. Consequently,

$$\begin{aligned}\mathcal{ESO}_\alpha(T) &= \sum_{uv \in E(G)} (d_u + d_v)^\alpha (d_u^2 + d_v^2)^{\alpha/2} \\ &\leq \sum_{uv \in E(G)} n^\alpha ((n-1)^2 + 1)^{\alpha/2} = \mathcal{ESO}_\alpha(S_n)\end{aligned}$$

and the equality is attained if and only if $(d_u, d_v) = (n-1, 1)$ or viceversa for every edge in $E(T)$, i.e., T is the star graph S_n.

If $\alpha < 0$, then the previous argument gives the converse inequality, and we also have the statement on the equality. ■

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